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On the Dimensional Sensitivity of Black Hole Entropy

Aasmaan Patil Hashmi

Abstract
This investigation began with an aim to understand the behaviour of entropy of black holes across various spacetime dimensions. Unexpectedly, the process led to the derivation of a generalized entropy formula that not only conforms to classical results in 4D but also follows consistent principles from quantum field theory and higher-dimensional gravity. We explore the thermodynamic structure of black holes in N - dimensional spacetimes, propose a new entropy law sensitive to dimensional parity (even vs. odd) based on intuition and mathematical reasoning, and examine its implications. This model draws inspiration from Bekenstein and Hawking's work, while introducing speculative yet mathematically grounded corrections potentially tied to quantum gravitational effects. Furthermore, this work has used ChatGPT for some assistance on the mathematics involved and for understanding of quantum corrections to entropy, yet the idea remains completely unconventional and orignal.

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I. Introduction

The black hole information paradox has long challenged the reconciliation of general relativity and quantum mechanics. Hawking's discovery that black holes emit radiation led to the insight that black holes must have entropy proportional to the cross-sectional area of their horizon[1]. The Bekenstein-Hawking entropy formula, central to this realization, is:

$$S = \frac{kc^3A}{4Gh} \tag{1}$$

where A is the horizon area in 4-dimensional spacetime.
In this paper, we extend this concept to N - dimensional spacetimes, we propose a general entropy formula that reflects the geometric and thermodynamic changes caused by dimensionality. Moreover, we propose that entropy corrections differ based on dimensional parity which means whether the spacetime dimension number is even or odd, potentially pointing to a new class of microphysical behavior.

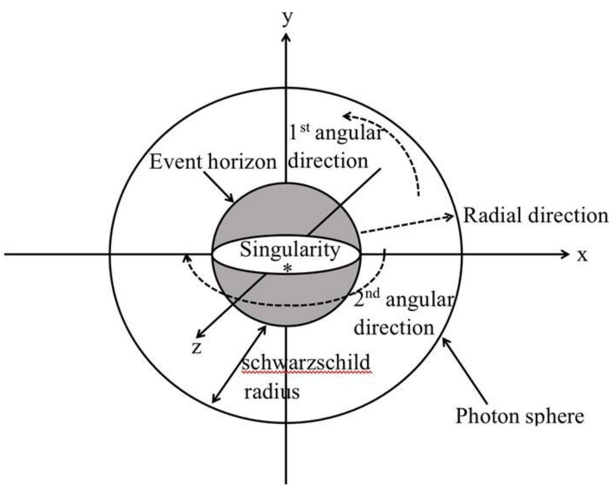

Figure 1: Anatomy-of-a-Schwarzschild-black-hole
[2]

2 Mathematical Preliminaries

2.1 The Schwarzschild Black Hole

Black holes have been visualized as 3 - dimensional objects till now and in order to understand them in higher dimensions, it is important to understand the anatomy of a black hole first since some elements like the event horizon and the Schwarzschild radius are very important variables.

2.2 Area of the Event Horizon

As this research explores spherical black holes in higher or lower dimensions, it would be better to first understand spheres in n - dimensions first. A n - dimensional sphere is embedded in a $(n + 1)$ - dimensional space. This means that an n-sphere exists as a subset of the $(n + 1)$ - dimensional Euclidean space. The derivation is very conventional yet it is shown to aid the understanding of the aim of this investigation

We begin with a well-known Gaussian integral in $\mathbb{R}^{n+1}$:

$$\int_{\mathbb{R}^{n+1}} e^{-\|x\|^2}\, d^{n+1}x = \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^{n+1} = \pi^{(n+1)/2} \tag{2}$$

We now compute the same integral in spherical coordinates, where:

$$d^{n+1}x = S_n(r)\, dr \tag{3}$$

$$\Rightarrow \int_{\mathbb{R}^{n+1}} e^{-\|x\|^2} d^{n+1}x = \int_0^{\infty} e^{-r^2} \cdot S_n(r)\, dr \tag{4}$$

where $S_n(r)$ is the surface area of the n-sphere of radius r.

The surface area of an n-sphere of radius r is:

$$S_n(r) = \Omega_n \cdot r^n \tag{5}$$

Here, Ω_n is the surface area of the unit n-sphere. Then:

$$\int_{\mathbb{R}^{n+1}} e^{-\|x\|^2} d^{n+1}x = \Omega_n \int_0^{\infty} e^{-r^2} r^n\, dr \tag{6}$$

Use the substitution $u = r^2 \Rightarrow du = 2r\, dr \Rightarrow dr = \frac{du}{2\sqrt{u}}$

So:

$$\int_0^{\infty} e^{-r^2} r^n\, dr = \frac{1}{2}\int_0^{\infty} u^{(n-1)/2} e^{-u}\, du = \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right) \tag{7}$$

We now equate both expressions for the Gaussian integral:

$$\pi^{(n+1)/2} = \Omega_n \cdot \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right) \tag{8}$$

Solving for Ω_n:

$$\Omega_n = \frac{2\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} \tag{9}$$

The event horizon is a spatial surface in a N - dimensional spacetime, hence the surface area of the event horizon of a N - dimensional sphere is given by:

$$\Omega_{N-2} = \frac{2\pi^{(N-1)/2}}{\Gamma\left(\frac{N-1}{2}\right)} \tag{10}$$

Thus, the horizon "area" in dimensions is:

$$A_N = \Omega_{N-2} r_H^{N-2} \tag{11}$$

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Where r_H is the radius of the event horizon, N is the number of spacetime dimensions and A_N is the surface area of the event horizon

3 First Law of Black Hole Thermodynamics and N - Dimensional Schwarzschild Black Hole

Hawking's radiation derives from quantum field theory in curved spacetime. A simplified expression for Hawking temperature in N - dimensional Schwarzschild spacetime[3] is:

$$T_H = \frac{\hbar c(N-3)}{4\pi k r_H} \tag{12}$$

This temperature scales with the inverse of the horizon radius, with dimension-dependent prefactors.

The first law of black hole thermodynamics in general dimensions is:

$$dM = T_H dS \tag{13}$$

which motivates expressing entropy in terms of horizon area and dimensional constants.

The N - dimensional Schwarzschild Radius for a black hole of mass M is:

$$r_H^{N-3} = \frac{16\pi G_N M}{(N-2)\Omega_{N-2}c^2} \tag{14}$$

Rewriting the equation to express M as a function of r_H gives:

$$M = \frac{(N-2)\Omega_{N-2}c^2}{16\pi G_N} r_H^{N-3} \tag{15}$$

Now differentiating M with respect to r_H gives:

$$D_{r_H} M = \frac{(N-2)\Omega_{N-2}c^2}{16\pi G_N} \cdot \frac{d(r_H^{N-3})}{dr_H} \tag{16}$$

$$D_{r_H} M = \frac{(N-2)\Omega_{N-2}c^2}{16\pi G_N} \cdot (N-3) r_H^{N-4} \tag{17}$$

$$dM = \frac{(N-2)\Omega_{N-2}c^2}{16\pi G_N M} \cdot (N-3) r_H^{N-4}\, dr_H \tag{18}$$

$$dM = \frac{(N-2)(N-3)\Omega_{N-2}c^2 r_H^{N-4}}{16\pi G_N} dr_H \tag{19}$$

Now substituting the Bekenstein-Hawking entropy formula and dM in the first law of black hole thermodynamics gives:

$$dM = T_H dS \tag{20}$$

$$\frac{dM}{T_H} = dS \tag{21}$$

$$dS = \frac{\frac{(N-2)(N-3)\Omega_{N-2}c^2 r_H^{N-4}}{16\pi G_N} dr_H}{\frac{\hbar c(N-3)}{4\pi k r_H}} \tag{22}$$

This can be written in a simpler way as:

$$dS = \frac{(N-2)(N-3)\Omega_{N-2}c^2 r_H^{N-4}}{16\pi G_N} dr_H \cdot \frac{4\pi k r_H}{\hbar c(N-3)} \tag{23}$$

$$dS = \frac{(N-2)\Omega_{N-2} c r_H^{N-3} k}{4 G_N \hbar} dr_H \tag{24}$$

Integrating both sides gives the equation for entropy as:

$$\int dS = \int \frac{(N-2)\Omega_{N-2} c r_H^{N-3} k}{4 G_N \hbar} dr_H \tag{25}$$

$$S = \frac{(N-2)\Omega_{N-2} ck}{4 G_N \hbar} \cdot \int r_H^{N-3} dr_H \tag{26}$$

$$S = \frac{(N-2)\Omega_{N-2} ck}{4 G_N \hbar} \cdot \frac{r_H^{N-2}}{(N-2)} \tag{27}$$

$$S = \frac{(N-2)\Omega_{N-2} ck r_H^{N-2}}{(N-2) 4 G_N \hbar} \tag{28}$$

$$S = \frac{\Omega_{N-2} ck r_H^{N-2}}{4 G_N \hbar} \tag{29}$$

Recall from the previous derivation in section 2, equation 3 that

$$A_N = \Omega_{N-2} r_H^{N-2} \tag{30}$$

$$S = \frac{A_N ck}{4 G_N \hbar} \tag{31}$$

In natural units $G = k = c = 1$, hence this simplifies the equation to:

$$S = \frac{A_N}{4} \tag{32}$$

This tells us that Entropy in higher dimensions in proportional to the area of the event horizon and not the volume of the black hole, and it supports the holographic principle, also showing that higher dimensions hold more entropy for the same radius.

4 Even-Odd Dimensional Corrections (Speculative)

As seen in the generalized formula for N - dimensions, Entropy in higher dimensions depends on the Event Horizon area which is defined as:

$$A_N = \Omega_{N-2} r_H^{N-2} \tag{33}$$

$$\Omega_{N-2} = \frac{2\pi^{(N-1)/2}}{\Gamma\left(\frac{N-1}{2}\right)} \tag{34}$$

$$A_N = \frac{2\pi^{(N-1)/2}}{\Gamma\left(\frac{N-1}{2}\right)} \cdot r_H^{N-2} \tag{35}$$

One key observation that can be made here based on the Entropy depending on the horizon area which involves the Gamma function is that Entropy is highly dimension sensitive due to the features of the gamma function and quantum corrections to entropy.

The gamma function which is valid for all $z > 0$ is defined as:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \tag{36}$$

Hence for $n \in \mathbb{N}$ it is :

$$\Gamma(n) = \prod_{n-1} i = (n-1)! \tag{37}$$

Hence, it is always a rational number as factorials are products of integers. However when the gamma function has a half-integer argument, irrationality arises as:

$$\Gamma(n + \frac{1}{2}) = \frac{(2n)!}{n! \cdot 2^{2n}} \sqrt{\pi} \tag{38}$$

this introduces $\sqrt{\pi}$ which is irrational explicitly.

Moreover as the event horizon area depends upon the gamma function with the argument $\frac{N-1}{2}$, hence when the argument is even the result is a rational multiple of π^n which can simplify the numerator $\pi^{\frac{N-1}{2}}$ and then then the entropy achieves the following relation:

$$S \propto \pi^n \cdot r_H^{N-2} \tag{39}$$

However when the argument is odd the result of the gamma function is irrational as it involves $\sqrt{\pi}$ and hence the entropy involves irrational coefficients making the resulting entropy irrational in its mathematical nature.

This axiomatic relation evidently shows that for the argument of the gamma function to be even or odd it would depend on the dimension as:

For $\frac{N-1}{2}$ to be even, N must be odd, hence an odd numbered dimension(3,5,7,9,11,. . .)

For $\frac{N-1}{2}$ to be odd, N must be even, hence an even numbered dimension(2,4,6,8,10,. . .)

4.1 Entropy Quantization and Rationality

When N is even,

$$\frac{N-1}{2} \notin \mathbb{N} \tag{40}$$

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!}{n!2^{2n}}\sqrt{\pi}.$$

Hence Ω_{N-2} becomes irrational, and entropy picks up an irrational prefactor:

$$S \propto \text{irrational} \cdot r_H^{D-2} \tag{41}$$

When N is odd, $\frac{N-1}{2} \in \mathbb{N}$, so:

$$\Gamma\left(\frac{N-1}{2}\right) = (n-1)! \in \mathbb{Q} \tag{42}$$

, and Ω_{N-2} becomes a rational multiple of π^m. Thus, the entropy coefficient is algebraically clean:

$$S \propto \pi^m \cdot r_H^{N-2} \tag{43}$$

If area is quantized as:

$$A = n \cdot \ell_P^{N-2} \tag{44}$$

Where ℓ_P represents planck length, hence ℓ_p^{N-2} represents the fundamental unit of area in the N - dimensional then:

$$S = \frac{kA}{4G_N\hbar} = n \cdot (\text{prefactor}) \tag{45}$$

- In **odd** N, this prefactor is rational, so $S \in \mathbb{Q}$, and microstate count:

$$\mathcal{N} = e^{S} \in \text{algebraic integers} \tag{46}$$

- In **even** N, S contains irrational numbers like $\sqrt{\pi}$, so:

$$\mathcal{N} = e^S \notin \mathbb{Z} \tag{47}$$

which contradicts the idea of discrete, countable microstates[4]

If entropy or area is quantized as:

$$A_{N-2} = n \cdot \ell_P^{N-2} \quad \Rightarrow \quad S \propto \Omega_{N-2} \cdot n \tag{48}$$

Case A: Odd N

When N is odd, $\frac{N-1}{2} \in \mathbb{N} \Rightarrow \Gamma \frac{N-1}{2} \in \mathbb{Q}$

Hence:

$$\Omega_{D-2} \in \mathbb{Q} \cdot \pi^m, \quad S \propto \text{rational} \cdot n$$

Microstate count:

$$\mathsf{N} = e^S = e^{\text{rational}} \in \text{algebraic} \Rightarrow \text{possibly integer}$$

Case B: Even N

When N is even, $\frac{N-1}{2} \notin \mathbb{N} \Rightarrow \Gamma \frac{D-1}{2}$ is irrational

So:

$$\Omega_{D-2} \in \mathbb{R} \setminus \mathbb{Q}, \quad S \propto \text{irrational} \cdot n \Rightarrow \mathsf{N} = e^S \notin \mathbb{Z}$$

This breaks the assumption of discrete, countable microstates.

5 Quantum Corrections to Entropy

Quantum Corrections are small terms added to improve the accuracy of entropy predictions when quantum effects become significant. Quantum effects include very small black holes and the end of black hole evaporation or in loop quantum gravity models. The full entropy expression with quantum corrections is[5]:

$$S = \frac{A_{D-2}}{4G_D h} + \alpha \ln \frac{A_{D-2}}{\ell_P^{D-2}} + \frac{\beta}{A_{D-2}} + \cdots \tag{49}$$

where:

- α, β depend on the theory (Loop Quantum Gravity, string theory)
- $\alpha < 0$ in many cases, e.g., $\alpha = -\frac{3}{2}$ for LQG in 4D

Parity Effects:

- **Even** D: Log correction includes conformal anomaly; Euler terms from Wald entropy are non-zero.
- **Odd** D: No conformal anomaly; log term cleaner; Euler term vanishes.

Dimension D	Ω_{D-2}	Implication
Odd	Rational $\cdot \pi^n$	Clean quantization, $e^S \in \mathbb{Z}$ possible
Even	Irrational (involves $\sqrt{\pi}$)	Entropy not integer-multiples; $e^S \notin \mathbb{Z}$

Thus, entropy quantization and microstate interpretation are far more consistent in odd dimensions.

6. Implications and Interpretation

This formula implies:

- Dimensionality directly influences information retention or loss
- Even-dimensional black holes may be more stable thermodynamically due to smoother corrections
- Odd-dimensional black holes might exhibit more unstable and inconsistent entropy

7. Conclusion and Future Work

We have proposed a general entropy formula and its dependence on dimensional parity suggesting that black hole entropy behaves differently in even and odd dimensions. This research explores how black hole entropy behaves across different spacetime dimensions, revealing a surprising sensitivity to whether the number of dimensions is even or odd. By generalizing the Bekenstein-Hawking formula and analyzing the role of the Gamma function mathematically, we find that entropy quantization may break down in even dimensions due to irrational coefficients from the gamma function. Quantum corrections further highlight distinct behaviors in even vs. odd dimensions, affecting the physical interpretation of microstates. These findings suggest that dimensional parity could play a deeper role in quantum gravity, potentially influencing black hole thermodynamics, evaporation, and the structure of spacetime itself.

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