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Marshall-Olkin New Pareto Distribution and Max-Min Processes

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Abstract*: In this paper we consider a new form of Pareto distribution introduced by Borguignon et al. (2016) and extend it to develop a new model called Marshall- Olkin New Pareto (MONP) distribution. Its elementary properties are explored and distributions of geometric minimum and maximum are derived. Different types of Autoregressive processes with Max-Min structure are developed. The innovation distributions are derived and sample path properties are studied. The models are extended to the k^{th} order also. The new models can be used for modelling time series data on income, wealth, survival times etc.*
Keywords*: Autoregressive models, New Pareto Distribution, Marshall-Olkin New Pareto (MONP) distribution, Time series modelling, Max-Min Autoregressive models.*

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I. Introduction

Pareto distribution is a good model for the distribution of income over a population. It is a power law probability distribution which can be is used in the description of social, economic, geographical, actuarial and many other types of observable phenomena. Originally Pareto distribution is used to describe the allocation of wealth among individuals. Now this distribution is used in many other situations in which equilibrium is found in the distribution of the small to the large. In hydrology Pareto distribution is applied to extreme events such as maximum one day rain falls and river discharges. Pareto type II distribution is a mixture of exponential distribution with Gamma mixing weights.

Both Pareto and Lognormal distributions are the distributions of exponentiated random variables distributed according to other common distributions such as exponential and Normal distribution. Also it is a special type of generalized Pareto distribution. Zeta distribution is a discrete counter part of the Pareto distribution. Pareto distribution has many applications in other areas like population studies, natural resource problems, size of farms, and distribution of income over population etc. Pareto distribution curves give a good fit to the data towards the extreme cases (Pusha (2009). Figueira et al (2010) showed that the Gomperz curve combined with Pareto’s law is a good descriptive model for income distribution.

In literature there exist a number of generalizations of this distribution. The Pareto distribution was first introduced by an Italian born Swiss Professor of Economics and Statistics named Vilfredo Pareto as model for the distribution of income over population. The Pareto distribution was proposed to model an unequal distribution of wealth in any society. Pareto distribution plays an important role to analyse a wide range of real world situations, in economics, actuarial science, reliability, finance, and climatology. It describes the occurrence of extreme weather. Several forms of Pareto distribution are found in the literature. Draglescu and Yakovenco (2001), Silva(2001), Yakovenco and Rosser (2009) advanced an exponential distribution of individual income similar to Boltzmann-Gibbs.

Recently there is an increasing interest in extending distributions to more general families for wider use and applications. Marshall and Olkin (1997) developed a new family which has been applied to various distributions for developing Marshall-Olkin extended distributions. Marshall-Olkin distributions with Weibull and Logistic marginals are introduced by Thomas and Jose (2005a, b). A detailed study of Marshall-Olkin Weibull distribution was given by Jose et al.(2011) and Githany et al.(2005). Sankaran and Jayakumar (2006) gave a physical interpretation of the Marshall-Olkin extended family of distributions using proportionate odds model in reliability theory. Parikh et al.(2008) discussed both estimation and testing problems along with numerical examples of Marshall-Olkin generalized exponential distribution. Jose et al.(2009 b) discussed

Marshall-Olkin beta distribution and its applications. Marshall-Olkin q-Weibull distribution and Max-Min processes are discussed by Jose et al.(2010). Krishna et al.(2011a) introduced Marshall-Olkin asymmetric Laplace distributions and processes. Krishna et al. (2013a,b) introduced and studied Marshall-Olkin Fre'chet distribution and its applications. Krishna et al. (2011b) introduced Marshall-Olkin Uniform distribution.

This paper is organized as follows. After the introduction in section 1, a brief review on NP distribution and its properties are given in section 2. In section 3, the MONP distribution is introduced and its properties are studied. AR(1) models with NP marginal distribution are introduced and studied in section 4. A more general Max-Min AR(1) process is introduced and its innovation distribution is derived in section 5. Conclusion is given in section 6.

II. New Pareto distribution and properties

The cdf of NPdistribution is given by

$$F(x) = 1 - \left(\frac{2ß^{\alpha}}{x^{\alpha}+ß^{\alpha}}\right) \quad, x \geq ß > 0,\ \alpha > 0 \qquad (1)$$

The pdf is

$$f(x) = \left(\frac{2\alpha ß^{\alpha} x^{(\alpha-1)}}{(x^{\alpha}+ß^{\alpha})^2}\right), \quad x \geq ß > 0, \qquad \alpha > 0 \qquad (2)$$

The hazard function is given by,

$$r(t) = \frac{\alpha t^{(\alpha-1)}}{t^{\alpha}+ß^{\alpha}} \qquad (3)$$

III. Marshall-Olkin New Pareto Distribution

In this section we introduce a new probability model known as Marshall-Olkin New Pareto(MONP) distribution. Various properties of the distribution and hazard rate functions are considered. The corresponding time series models are developed to illustrate its application in time series modelling. Consider a survival function $\bar{F}(x)$. Marshall-Olkin (1997) introduced a new family with survival functions given by,

$$\bar{G}(x) = \frac{p\bar{F}(x)}{1-(1-p)\bar{F}(x)}, x\varepsilon R, p > 0$$

The pdf of MO distribution is given by,

$$g(x) = \frac{pf(x)}{\left(1-(1-p)\bar{F}(x)\right)^2}, x\varepsilon R, p > 0$$

The hazard rate of Marshall -Olkin distribution is given by ,

$$h(t) = \frac{r(t)}{1-(1-p)\bar{F}(t)},$$

where, $r(t) = \frac{f(t)}{\bar{F}(t)}$

The Survival function of the MONP distribution is given by

$$\bar{G}(x) = \frac{p\left(\frac{2ß^{\alpha}}{x^{\alpha}+ß^{\alpha}}\right)}{1-(1-p)\left(\frac{2ß^{\alpha}}{x^{\alpha}+ß^{\alpha}}\right)} \qquad (4)$$

The density of the MONP distribution is given by,

$$g(x) = \frac{\left[p\left(\frac{2\alpha ß^{\alpha} x^{(\alpha-1)}}{(x^{\alpha}+ß^{\alpha})^{2}}\right)\right]}{\left(1-(1-p)\left(\frac{2ß^{\alpha}}{x^{\alpha}+ß^{\alpha}}\right)\right)^{2}} \quad (5)$$

The hazard rate function of MONP distribution is given by,

$$h(t) = \frac{\frac{\alpha t^{(\alpha-1)}}{t^{\alpha}+ß^{\alpha}}}{\left[1-(1-p)\left(\frac{2ß^{\alpha}}{t^{\alpha}+ß^{\alpha}}\right)\right]}$$

The graphs of the pdf (probability density function) and HRF(Hazard Rate Function) of MONP distribution are given below.

P=5 α=1 ß=4 P=5 α=5 ß=1.5

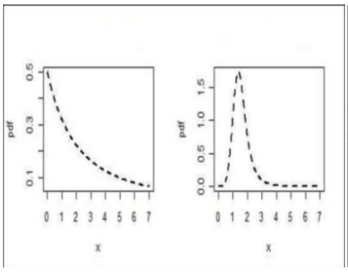

Figure1

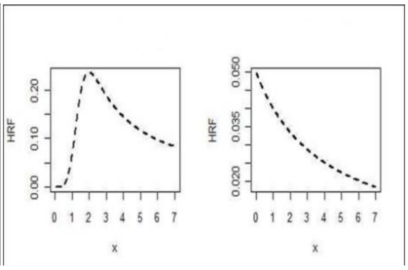

Figure2

Theorem 3.1 Let $\{X_i, i \geq 1\}$be a sequence of i.i.d.(independent and identically distributed) random variables with common survival function $\bar{F}(x)$and let N be a geometric random variable independently distributed of $\{X_i\}$such that P[N = n] = $\theta(1-\theta)^{n-1}$, $n = 1,2,\ldots 0 < \theta < 1$, which is for all $i \geq 1$. Let U_N = $min(X_1, X_2, \ldots X_n)$. Then $\{U_N\}$ is distributed as MONP iff $\{X_i\}$ follows NP distribution .

Proof: The survival function of the random variable U_N is

$$\bar{H}(x) = P(U_N > x) = \theta\sum_{n=1}^{\infty}[\bar{F}(x)]^{n}(1-\theta)^{n-1} = \frac{\theta\left(\frac{2ß^{\alpha}}{x^{\alpha}+ß^{\alpha}}\right)}{1-\left((1-\theta)\left(\frac{2ß^{\alpha}}{x^{\alpha}+ß^{\alpha}}\right)\right)} \quad (6)$$

If X_i has the survival function of the Pareto distribution then U_N has the survival function of the MONP distribution. The converse easily follows from(6) that

$$\bar{F}(x) = \left(\frac{2ß^{\alpha}}{x^{\alpha}+ß^{\alpha}}\right)$$

Theorem 3.2 Let $\{X_i, i \geq 1\}$be a sequence of i.i.d. random variables with common survival function $\bar{F}(x)$and let N be a geometric random variable independently distributed of $\{X_i\}$ such that P[N = n] = $\theta(1-\theta)^{n-1}, n = 1,2,\ldots 0 < \theta < 1$, for all $i \geq 1$.. Let $V_N = max(X_1, X_2, \ldots X_n)$. Then $\{V_N\}$ is distributed as MONP iff $\{X_i\}$ follows NP distribution..

Proof: The distribution function of the random variable V_N is

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$$M(x) = P(V_N < x) = \theta \sum_{n=1}^{\infty} [F(x)]^n \,(1-\theta)^{n-1} = \frac{\theta F_X(x)}{1-(1-\theta)F_X(x)}$$

$$= \frac{\theta \left(\frac{2ß^{\alpha}}{x^{\alpha} + ß^{\alpha}} \right)}{1 - \left((1-\theta) \left(\frac{2ß^{\alpha}}{x^{\alpha} + ß^{\alpha}} \right) \right)} \qquad (7)$$

$$\bar{M}(x) = 1 - M(x) = \frac{\frac{1}{\theta} F_X(x)}{1 - \left(1 - \frac{1}{\theta} F_X(x) \right)}$$

If X_i has the survival function of the NP distribution then V_N has the survival function of the MONP distribution. The converse easily follows from (7) that

$$F(x) = 1 - \left(\frac{2ß^{\alpha}}{x^{\alpha} + ß^{\alpha}} \right)$$

IV. AR(1) minification models with MONP marginal distribution

Two stationary Markov processes with similar structural forms which had found useful in hydrological applications was introduced by Tavares(1977,1980).The various aspects on first order auto regressive minification processes was discussed by Lewis and Mc Kenze(1991).

In this section we develop autoregressive minification processes of order one and order k with minification structures where MONP distribution is the stationary marginal distribution. We call the process as MONP AR(1) process. Now we have the following theorem.

Theorem 4.1 Consider an AR(1) structure given by

$$X_n = \begin{cases} \varepsilon_n, & w.p \quad p_1 \\ min(X_{n-1}, \varepsilon_n) & w.p \quad 1-p_1 \end{cases}$$

Where w.p. denotes 'with probability', $0 < p_1 < 1$ and $\{\varepsilon_n\}$ is a sequence of i.i.d. random variables independently distributed of X_n. Then $\{X_n\}$ is a stationary Markovian AR(1) process with MONP marginal if and only if $\{\varepsilon_n\}$ is distributed as NP distribution.

Proof: From the given structure it follows that

$$\bar{F}_{X_n}(x) = p_1 \bar{F}_{\epsilon_n}(x) + (1-p_1) \bar{F}_{X_{n-1}}(x) \bar{F}_{\epsilon_n}(x)$$

Under stationary equilibrium, it reduces to

$$F(x) = \frac{p_1 \left(\frac{2ß^{\alpha}}{x^{\alpha} + ß^{\alpha}} \right)}{1-(1-p_1)\left(\frac{2ß^{\alpha}}{x^{\alpha} + ß^{\alpha}} \right)} \qquad (8)$$

This is the MONP distribution. Conversely on substituting the survival function of the innovations ε_n, we get

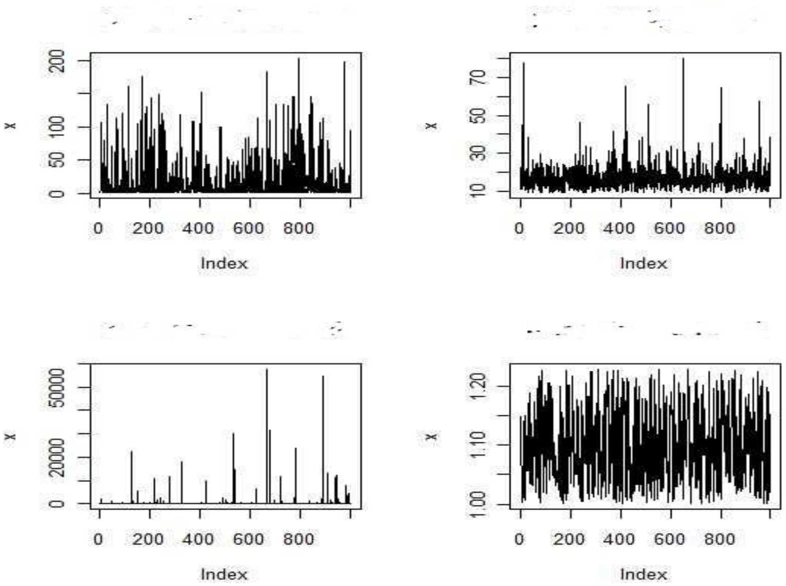
$$\bar{F}_{\varepsilon_n}(x) = \left(\frac{2ß^{\alpha}}{x^{\alpha} + ß^{\alpha}} \right) \qquad (9)$$

This is the survival function of NP distribution.

The following are the sample paths for the above structure.

ß=3,α=0.5,p1=0.7 ß=9, α=5, p1=7

Figure3



ß=1, p1=2, α=1 Figure4 ß=0.1,p1=0.6,α=5

Now we consider another AR (1) structure having three components.

Theorem 4.2 Consider an AR(1) structure given by

$$X_n = \begin{cases} X_{n-1}, & w.p. \quad p_2 \\ \varepsilon_n, & w.p. \quad p_1(1-p_2) \\ min(X_{n-1}, \varepsilon_n), & w.p. \quad (1-p_1)(1-p_2) \end{cases}$$

Where $\{\varepsilon_n\}$ is a sequence of i.i.d. random variables independently distributed of X_n. Then $\{X_n\}$ is a stationary Markovian AR (1) process with MONP marginal if and only if $\{\varepsilon_n\}$ is distributed as NP distribution.

Proof: From the given structure it follows that

$$\bar{F}_{X_n}(x) = p_2\bar{F}_{X_{n-1}}(x) + p_1(1-p_2)\bar{F}_{\varepsilon_n}(x) + (1-p_1)(1-p_2)\bar{F}_{X_{n-1}}(x)\bar{F}_{\varepsilon_n}(x).$$

On simplification we get, the same expression as in equation (8) stationarity. Then the result is obvious.
The following theorem generalizes the results to a k^{th} order autoregressive structure.

Theorem 4.3 Consider an AR (k) structure given by

$$X_n = \begin{cases} \varepsilon_n, & w.p. \quad p_0 \\ min(X_{n-1}, \varepsilon_n), & w.p. \quad p_1 \\ min(X_{n-2}, \varepsilon_n), & w.p. \quad p_2 \\ \vdots & \vdots \quad \vdots \\ min(X_{n-k}, \varepsilon_n), & w.p. \quad p_k \end{cases}$$

where$\{\varepsilon_n\}$ is a sequence of i.i.d. random variables independently distributed of X_n, $0 < p_i < 1, p_1 + p_2 + \cdots + p_k = 1 - p_0$. Then the stationary marginal distribution of$\{X_n\}$ is MO Lomax if and only if $\{\varepsilon_n\}$ is distributed as Lomax distribution.

Proof: From the given structure the survival function is given as follows:

$\bar{F}_{X_n}(x) = p_0\bar{F}_{\varepsilon_n}(x) + p_1\bar{F}_{X_{n-1}}(x)\bar{F}_{\varepsilon_n}(x) + \cdots + p_k\bar{F}_{X_{n-1}}(x)\bar{F}_{\varepsilon_n}(x).$
Under stationary equilibrium, this yields

$\bar{F}_X(x) = p_0\bar{F}_\varepsilon(x) + p_1\bar{F}_X(x)\bar{F}_\varepsilon(x) + \cdots + p_k\bar{F}_X(x)\bar{F}_\varepsilon(x).$

This reduces to

$$\bar{F}_X(x) = \frac{p_0\left(\frac{2ß^\alpha}{x^\alpha + ß^\alpha}\right)}{1 - (1 - p_0)\left(\frac{2ß^\alpha}{x^\alpha + ß^\alpha}\right)}.$$

Then the theorem easily follows by similar arguments as in Theorem 4.2.

V. The Max-Min AR(1) processes

Next we introduce a new model called the max-min process which incorporates both maximum and minimum values of the process. This has wide applications in atmospheric and oceanographic studies. The structure is given as follows.

Theorem 5.1 Consider an AR(1) structure given by

$$X_n = \begin{cases} max(X_{n-1}, \varepsilon_n), & w.p. \quad p_1 \\ min(X_{n-1}, \varepsilon_n), & w.p. \quad p_1 \\ X_{n-1}, & w.p. \quad 1 - p_1 - p_2 \end{cases}$$

subject to the conditions $0 < p_1, p_2 < 1, p_2 < p_1$ and $p_1 + p_2 < 1$, where$\{\varepsilon_n\}$ is a sequence of i.i.d. random variables independently distributed of X_n. Then$\{X_n\}$ is a stationary Markovian AR(1) max-min process with stationary marginal distribution $\bar{F}_X(x)$ if and only if $\{\varepsilon_n\}$ follows Marshall-Olkin distribution.

Proof: From the given structure it follows that

$$P(X_n > x) = p_1P(max(X_{n-1}, \varepsilon_n) > x) + p_2P(min(X_{n-1}, \varepsilon_n) > x)$$
$$+(1 - p_1 - p_2)P(X_{n-1} > x)$$
$$= p_1\left[1 - \left(1 - \bar{F}_{X_{n-1}}(x)\right)\left(1 - \bar{F}_{\varepsilon_n}(x)\right)\right] + p_2\bar{F}_{X_{n-1}}(x)\bar{F}_{\varepsilon_n}(x)$$
$$+(1 - p_1 - p_2)\bar{F}_{X_{n-1}}(x).$$

Under stationary equilibrium, we get

$$\bar{F}_\varepsilon(x) = \frac{p_2\bar{F}_X(x)}{p_1 + (p_2 - p_1)\bar{F}_X(x)} = \frac{p'\bar{F}_X(x)}{1 - (1 - p')\bar{F}_X(x)} \quad (10)$$

Where $p' = \frac{p_2}{p_1}$. This has same functional form of Marshall-Olkin survival function. The converse can be proved by mathematical induction, assuming that$\bar{F}_{X_{n-1}}(x) = \bar{F}_X(x)$.

5.1. The Max-Min process with NP marginal distribution

Now consider the cumulative distribution function $F(x) = 1 - \left(\frac{2ß^\alpha}{x^\alpha + ß^\alpha}\right)$. To obtain the NP max-min process, consider the above structure and substitute the survival function of NP distribution in equation (10). Then we get

$$\bar{F}_{\varepsilon}(x) = \frac{p'}{\left(\frac{2ß^{\alpha}}{x^{\alpha} + ß^{\alpha}}\right) - (1 - p')}$$

which is the survival function of the MONP distribution, where $p' = \frac{p_2}{p_1}$, $p_2 < p_1$ and $p_1 + p_2 < 1$. When $\alpha = 1$ it gives a max-min process with NP marginal distribution. Where $p' = \frac{p_2}{p_1}$, $p_2 < p_1$ and $p_1 + p_2 < 1$.

Now consider a more general autoregressive structure which includes maximum, minimum as well as the innovations and the process values.

Theorem 5.2: Consider an AR(1) structure given by

$$X_n = \begin{cases} max(X_{n-1}, \varepsilon_n), & w.p \quad p_1 \\ min(X_{n-1}, \varepsilon_n), & w.p \quad p_2 \\ \varepsilon_n, & w.p \quad p_3 \\ X_{n-1}, & w.p \quad 1 - p_1 - p_2 - p_3 \end{cases}$$

With the condition that $0 < p_1, p_2, p_3 < 1, p_2 < p_1$ and $0 < p_1 + p_2 + p_3 < 1$, where $\{\varepsilon_n\}$ is a sequence of i.i.d. random variables independently distributed of X_n. Then $\{X_n\}$ is a stationary Markovian AR (1) max-min process with stationary marginal distribution $\bar{F}_X(x)$ if and only if $\{\varepsilon_n\}$ follows Marshall-Olkin distribution.
Proof: From the given structure it follows that

$$P(X_n > x) = p_1 P(max(X_{n-1}, \varepsilon_n) > x) + p_2 P(min(X_{n-1}, \varepsilon_n) > x)$$

$$+ p_3 P(\varepsilon_n > x) + (1 - p_1 - p_2 - p_3) P(X_{n-1} > x).$$

This simplifies to

$$\bar{F}_{X_n}(x) = p_1 \left[1 - \left(1 - \bar{F}_{X_{n-1}}(x)\right)\left(1 - \bar{F}_{\varepsilon_n}(x)\right)\right] + p_2 \bar{F}_{X_{n-1}}(x) \bar{F}_{\varepsilon_n}(x)$$
$$+ p_3 \bar{F}_{\varepsilon_n}(x) + (1 - p_1 - p_2 - p_3) \bar{F}_{X_{n-1}}(x).$$

Under stationary equilibrium, this reduces to

$$\bar{F}_{\varepsilon}(x) = \frac{(p_2 + p_3)\bar{F}_X(x)}{p_1 + p_3 + (p_2 - p_1)\bar{F}_X(x)} = \frac{\beta \bar{F}_X(x)}{1 - (1 - \beta)\bar{F}_X(x)} \tag{11}$$

where $\beta = \frac{p_2 + p_3}{p_1 + p_3}$. This has the same functional form of the Marshall-Olkin survival function. The converse follows as before.

Remark: By substituting the survival function of the NP distribution in equation (10) we obtain

$$\bar{F}_{\varepsilon}(x) = \frac{\beta\left(\frac{2ß^{\alpha}}{x^{\alpha} + ß^{\alpha}}\right)}{1 - (1 - \beta)\left(\frac{2ß^{\alpha}}{x^{\alpha} + ß^{\alpha}}\right)}$$

The above model is a more generalized form having four components. Hence it can be used to model a variety of situations.

VI. Conclusion

In this paper we have developed MONP distribution as generalisation of NP distribution by Bourguignon et al (2016). The distribution of extreme values like geometric minimum and geometric maximum are obtained and derived. The new distributions have applications in modelling reliability and income data. We also develop autoregressive processes with minification and Max-Min structures. These can be used for modelling time series data, financial and bio statistical contexts.

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