IOSR Journal of Computer Engineering (IOSR-JCE)
e-ISSN: 2278-0661, p-ISSN: 2278-8727, Volume 28, Issue 4, Ser. 2 (July. – August. 2026), PP 01-06
www.iosrjournals.org

Algebraic Equivalence Of Cramer´S Rule And The Gauss–Jordan Method For Linear Systems

Belén De Jesús Gutiérrez Armenta[1], Esiquio Martín Gutiérrez Armenta[2], Minerva Del Mar Gutiérrez Armenta[3], Julio Lara García[4], Javier Norberto Gutiérrez Villejas[5], Israel Isaac Gutiérrez Villegas[6-7], Marco Antonio Gutiérrez Villegas[8].

[1] *QFI., Escuela Nacional De Ciencias Biológicas, IPN, Unidad Profesional Lázaro Cárdenas, Prolongación De Carpio Y Plan De Ayala S/N, Col. Santo Tomás C.P. 11340 Alcaldía Miguel Hidalgo CDMX. México.*
[2,8]*Departamento De Sistemas, Area De Sistemas Computationalism, Universidad Autonoma Metropolitana, Unidad Azcapotzalco, Av. San Pablo No. 420 Col. Nueva El Rosario C.P. 02128 Alcaldía Azcapotzalco, CDMX.*
[3]*Departamento De Mecanica, Universidad Autonoma Metropolitana, Unidad Azcapotzalco, Av. San Pablo No. 420 Col. Nueva El Rosario C.P. 02128 Alcaldía Azcapotzalco, CDMX.*
[4]*Universidad Politécnica De Tecámac, Área Asignada: Direccón De División De Ingeniería Mecánica, Ingeniería En Tecnologías De Manufactura E Ingeniería En Software. Prolongación 5 De Mayo #10, Colonia Felipe Villanueva, Centro Tecámac, CP 55740, Estado De México.*
[5 Y 6] *División De Ingeniería En Sistemas Computacionales / Tese-Tecnm, Av. Tecnológico S/N, Valle De Anahuac, 55210 Ecatepec De Morelos, Edo. Mexico, Mexico.*
[7]*División Departamento De Ingeniería Y Ciencias Sociales /ESFM. Instituto Politécnico Nacional (IPN). Av. Luis Enrique Erro S/N, Unidad Profesional Adolfo López Mateos, Zacatenco, Alcaldía Gustavo A. Madero, C.P. 07738, Ciudad De México*

Abstract:
This work algebraically demonstrates the equivalence between Cramer's rule and the Gauss-Jordan method for solving 2x2 systems of linear equations. Through matrix and symbolic development, it is shown that both procedures lead to algebraically equivalent expressions when the system has a unique solution. A comparative analysis of their computational performance is also performed, considering the number of arithmetic operations, processing time, and storage requirements. The results show that, although both methods produce identical solutions, the Gauss-Jordan method exhibits greater computational efficiency and better scalability as the system's dimension increases, making it a more suitable alternative for solving larger linear systems.
Background: *Cramer's rule and the Gauss-Jordan method are direct methods for solving systems of linear equations. Both allow for obtaining exact solutions when the determinant of the matrix associated with the system is non-zero.*
However, in computational applications, values close to zero can cause numerical instability. Cramer's rule is limited to non-singular square matrices (non-singular means that the determinant of the matrix associated with the system is not zero), while the Gauss-Jordan method is more general and efficient in higher-dimensional systems.
Materials and Methods: *This work develops a mathematical analysis to establish the equivalence between Cramer's Rule and the Gauss–Jordan elimination method in the solution of linear systems. The methodology consists of an analytical derivation of both procedures, followed by a comparison of the algebraic transformations involved. Furthermore, the number of arithmetic operations required for solving linear systems is determined for each method, allowing an assessment of their computational cost and operational performance.2×2.*
Results: *The proposed analysis establishes the mathematical equivalence between Cramer's Rule and the Gauss–Jordan elimination method for solving systems of linear equations. Furthermore, the arithmetic operation count provides a quantitative basis for comparing the computational efficiency of both methods. This analysis also enables the estimation of the propagation of round-off errors associated with each method, thereby offering an additional criterion for evaluating their numerical performance.*
Conclusion: *The study algebraically demonstrated the d-Cramer equivalence and the Gauss-Jordan method in solving systems of 2x2 linear equations. Through symbolic development it was purchased that both methods lead to the same algebraic expression, provided that the determinant of the matrix of the coefficients is non-zero, confirming their mathematical awareness and theoretical validity.*

The complexity analysis delayed that the Cramer rule requires 11 elementary operations versus 20 Gauss-Jordan method, anticipating the behavior of processing time and error propagation. For nxn-dimensional systems, Cramer's rule requires operations (Babajee, 2013; Fandiño Albiánez, 2019), in contrast, the Gauss-Jordan method, based on the reduction towards the form of identity (Methtre & Tiwari, 2020), requires $(n+1)(\frac{2}{3}n^3)n + \left(\frac{4}{3}n^3\right)$ *operations. This confirms the temporal optimization of Gauss_jordan against Cramer's higher-order growth* $(O(n^3))(O(n^4))$.

Key Word: *Gabriel Cramer's rule, Gauss-Jordan, equivalence, algebraics, linear algebra, system of linear equations, computational complexity.*

Date of Submission: 24-07-2026 Date of Acceptance: 04-08-2026

I. Introduction

The historical background of systems of linear equations dates back to the earliest civilizations. One of the oldest records is the Rhind Papyrus, which describes procedures for solving first-degree equations using the method of false position (Estrada Analco, 2016). Likewise, there is evidence that the Babylonian civilization was solving systems of equations from approximately 1800 BCE using arithmetic and geometric procedures (Karpinski, 1936). These developments formed the basis for the subsequent evolution of algebra and methods for solving linear systems.

The formal development of linear algebra was consolidated with Gauss's contributions to the study of systems of equations and with Sylvester's introduction of the concept of a matrix, later formalized by Cayley (Larson, 2010). In this context, several direct methods for solving linear systems emerged, among which Cramer's rule, proposed in 1753, stands out. This rule expresses the solution using determinants (Babajee, 2013). Another notable method is the Gauss-Jordan method, based on applying elementary matrix operations until the identity matrix is obtained (Mehtre & Tiwari, 2022). From a computational perspective, the Gauss-Jordan method has a complexity of order), making it an efficient alternative for solving large linear systems. $O(n^3$

In numerical analysis, in addition to computational science, it is essential to evaluate the stability of the algorithms used. For this purpose, the condition number is used, which quantifies the sensitivity of the solution to small perturbations in the system's coefficients.

When a matrix has a high condition number or a determinant close to zero, the system is considered ill-conditioned, which can significantly amplify rounding errors and affect the accuracy of the results (Moradi, 2018; Weiss et al., 1986).

Cramer's rule and the Gauss-Jordan method belong to the category of direct methods for solving systems of linear equations.

Both provide exact solutions when the coefficient matrix is non-singular; however, they differ significantly in terms of computational complexity and applicability. Cramer's rule requires calculating determinants, the computational cost of which increases rapidly with the system's size, while the Gauss-Jordan method offers greater generality and better scalability for larger systems.

In recent years, research in computational linear algebra has focused on developing algorithms with greater numerical stability, lower computational complexity, and improved performance on high-performance architectures. Recent studies have shown that using mixed-precision arithmetic accelerates the solution of linear systems without significantly compromising the accuracy of the results (Abdelfattah et al., 2021). Furthermore, new strategies have been proposed to improve the stability of matrix factorization processes and iterative methods used in high-dimensional systems (Higham & Mary, 2022; Lund et al., 2023; Simoncini & Wang, 2025). These advances demonstrate that computational efficiency and numerical stability remain fundamental aspects in the development and analysis of methods for solving systems of linear equations.

However, few works present an explicit algebraic proof of the equivalence between Cramer's rule and the Gauss-Jordan method. Establishing this equivalence contributes to a better theoretical understanding of both procedures and provides a mathematical foundation for analyzing their differences from a computational perspective. Although both produce the same solution for unique systems, most available studies have focused on comparing their computational efficiency; few works develop an explicit algebraic proof of this equivalence.

In this context, the present work aims to algebraically demonstrate the equivalence between Cramer's rule and the Gauss-Jordan method for solving 2x2 systems, using matrix and symbolic expansions that demonstrate the equality of the obtained expressions.

II. Material And Methods

The objective of this work is to demonstrate the equivalence between Cramer's rule and the Gauss-Jordan method through the symbolic development of a 2x2 system of linear equations.

Consider the following system:

$$A\vec{x} = \vec{b} \qquad (1)$$

Where A is a 2x2 square matrix, x is the vector of unknowns, and b is the vector of constant terms.

Cramer's Rulestates that the solution for each unknown is obtained as:

$$x_i = \frac{|A_i|}{det(A)} \; i = 1,2,3 \dots n \qquad (2)$$

Where is the matrix obtained by replacing column of with the vector b $A_i iA^{\to}$ for the 2x2 case, we have:

When solving the system of linear equations with two unknowns, the following system, presented in matrix form, is considered:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}] \qquad (3)$$

First, the determinant of the matrix associated with the system must be calculated.

$$det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{21}a_{12} \qquad (4)$$

To calculate this, 3 arithmetic operations are performed. Now, using equation (2), the values of the unknowns are determined.

$$x_1 = \frac{det\begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix}}{det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}} = \frac{a_{22}b_1 - a_{12}b_2}{det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}} \qquad (5)$$

To calculate, 4 operations are performed x_1

$$x_2 = \frac{det\begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{pmatrix}}{det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}} = \frac{a_{11}b_2 - a_{21}b_1}{det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}} \qquad (6)$$

n equation (6), four operations are also performed:

In total, approximately 11 elementary row operations are used for this system.

Gauss-Jordan Method

This method consists of applying elementary row operations to the augmented matrix in equation (7):

$$[A|\vec{b}] \qquad (7)$$

Elementary operations used:

- Row permutation, which consists of swapping two rows.
- Row multiplication, where a row is multiplied by a scalar $\alpha \in \mathbb{R}, \alpha \neq 0$ ..
- Row addition, in which two or more rows are combined linearly.

The goal is to transform the augmented matrix into its reduced row form, equation (8):

$$\left[I_{nxn}|\vec{\vec{b}}\right] \qquad (8)$$

Where is the nxn identity matrix and the vector is the solution vector.$I_{nxn}\vec{\vec{b}}$

Applying equation (7) step by step, we obtain equation (6).

$$\left(\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}\middle| \begin{matrix} b_1 \\ b_2 \end{matrix}\right) \qquad (6)$$

Applying elementary operations to (6):

$$\begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{bmatrix} \quad \frac{1}{a_{11}}R_1 \to R_1 \qquad (7)$$

In equation (7) approximately 4 arithmetic operations are performed.

$$\sim \begin{bmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{b_1}{a_{11}} \\ a_{21} & a_{22} & b_2 \end{bmatrix} \quad \text{-a}_{21}R_1 + R_2 \to R_2 \qquad (8)$$

In equation (8) approximately 6 operations.

$$\sim \begin{bmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{b_1}{a_{11}} \\ 0 & a_{22} - a_{21}\left(\frac{a_{12}}{a_{11}}\right) & b_2 - a_{21}\left(\frac{b_1}{a_{11}}\right) \end{bmatrix} \qquad (9)$$

It is observed that, when performing the algebra of:

$$a_{22} - a_{21}\left(\frac{a_{12}}{a_{11}}\right) = \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}} = \frac{det(A)}{a_{11}} \qquad (11)$$

From equation (11) we have that in the numerator we obtain the determinant between the one called a_{11}

$$\sim \begin{bmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{b_1}{a_{11}} \\ 0 & \frac{det(A)}{a_{11}} & b_2 - a_{21}\left(\frac{b_1}{a_{11}}\right) \end{bmatrix} \quad \frac{1}{\frac{det(A)}{a_{11}}}R_2 \to R_2 \qquad (12)$$

In equation (13) , 4 approximate operations are performed.

$$\frac{a_{11}(b_2-a_{21}\left(\frac{b_1}{a_{11}}\right))}{\det(A)} = \frac{a_{11}b_2-a_{21}b_1}{a_{11}} = \frac{det\begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{pmatrix}}{a_{11}} \quad (13)$$

$$\sim \begin{bmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{b_1}{a_{11}} \\ 0 & 1 & \frac{det\begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{pmatrix}}{a_{11})} \end{bmatrix} \quad -\frac{a_{12}}{a_{11}}R_2 + R_1 \rightarrow R_1 \quad (14)$$

Equation (14) involves 6 approximate operations.
As a result of the procedure, the following expression is obtained.

$$\sim \begin{bmatrix} 1 & 0 & \frac{det\begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix}}{\det(A)} \\ 0 & 1 & \frac{det\begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{pmatrix}}{\det(A)} \end{bmatrix} \quad (16)$$

In this case, the values of and are determined by the following expressions:$x_1 x_2$

$$x_1 = \frac{det\begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix}}{\det(A)} \quad (17)$$

$$x_2 = \frac{det\begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{pmatrix}}{\det(A)} \quad (18)$$

Cramer's rule and the Gauss-Jordan method. Obtained from equations (5-6) and (17-18) written in the form of determinants are equal.

III. Result

Applying Cramer's rule and the Gauss-Jordan method to the 2x2 linear system yielded algebraically equivalent expressions for the unknowns x_1 and x_2. In both cases, the solutions are expressed as quotients of determinants. This confirms that the procedures lead to the same result as long as $\det(A) \neq 0$

Results obtained
Cramer's Rule:

$$x_1 = \frac{det\begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix}}{det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}} = \frac{a_{22}b_1-a_{12}b_2}{\det(A)} \quad \text{and } (19)$$

$$x_2 = \frac{det\begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{pmatrix}}{det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}} = \frac{a_{11}b_2 - a_{21}b_1}{\det(A)}$$

Gauss-Jordan method:

$$x_1 = \frac{det\begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix}}{\det(A)} \quad \text{and } (20)$$

$$x_2 = \frac{det\begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{pmatrix}}{\det(A)}$$

The coincidence of the expressions (19) and (20) confirms the algebraic equivalence between both methods.

Comparative analysis shows that, for 2x2 systems, Cramer's rule requires approximately 11 operations, while the Gauss-Jordan method requires around 20. However, symbolic expansion demonstrates that both methods produce equivalent algebraic solutions.

The symbolic expansion of both methods leads to an equivalent form; however, equality can be achieved if, in equations (3) and (4), the numerator is represented as a determinant. In this way, identical results are obtained.

IV. Discussion

Algebraic equivalence

The symbolic development shows that both the Cramer rule and the Gauss-Jordan method produce identical solutions for systems of two equations with two unknowns. This equivalence provides a solid theoretical foundation for understanding the relationship between direct methods of solving systems of linear equations.

Computational complexity

- Cramer's rule requires the calculation of determinants, which is inefficient for large systems, since the computational cost increases factorially with the size of the matrix.
- The Gauss-Joran method, on the other hand, has a complexity of , which makes it a more scalable and suitable alternative for larger systems.$O(n^3)$

Numerical stability

- Both methods depend on the condition of the array A . When the condition number(s) is high, the system is considered poorly conditioned and solutions can be affected by rounding errors. In this sense, numerical analysis recommends evaluating stability before applying any of the methods.$\det(A) \approx 0$

Practical applications

❖Cramer's rule is useful in theoretical context and in low-dimensional systems, where its simplicity allows for purely direct solutions.

❖The Gauss-Jordan method is more appropriate in computational applications, as it allows large systems to be solved more efficiently and is better suited to high-performance architectures.

Contribution of the study

The present work provides an explicit algebraic proof of the equivalence between the Cramer rule and the Gauss-Jordan method in 2x2 systems. This contribution is relevant because, although both methods are widely known and used, there are few studies that present a detailed development that evidences their direct relationship.

In addition, the comparative analysis in terms of computational complexity and numerical stability allows us to identify the advantages and limitations of each procedure. Offering a useful conceptual framework for future research in computational linear algebra and numerical analysis.

V. Conclusion

The study algebraically demonstrated the d-Cramer equivalence and the Gauss-Jordan method in solving systems of 2x2 linear equations. Through symbolic development it was purchased that both methods lead to the same algebraic expression, provided that the determinant of the matrix of the coefficients is non-zero, confirming their mathematical awareness and theoretical validity.

The complexity analysis delayed that the Cramer rule requires 11 elementary operations versus 20 Gauss-Jordan method, anticipating the behavior of processing time and error propagation. For nxn-dimensional systems, Cramer's rule requires operations (Babajee, 2013; Fandiño Albiánez, 2019), in contrast, the Gauss-Jordan method, based on the reduction towards the form of identity (Methtre & Tiwari, 2020), requires $(n + 1)(\frac{2}{3}n^3)n + \left(\frac{4}{3}n^3\right)$ operations. This confirms the temporal optimization of Gauss_jordan against Cramer's higher-order growth $(O(n^3))(O(n^4))$.

References

[1]. Abdelfattah, A., Anzt, H., Boman, E. G., Carson, E., Cojean, T., Dongarra, J., ... Yang, U. M. (2021). A Survey Of Numerical Linear Algebra Methods Utilizing Mixed-Precision Arithmetic. The International Journal Of High Performance Computing Applications, 35(4), 344–369. Https://Doi.Org/10.1177/10943420211003313

[2]. Babajee, D. K. R. (2013). A Comparison Between Cramer's Rule And A Proposed 2 × 2 Cramer-Elimination Method To Solve Systems Of Three Or More Linear Equations. *Scientific & Academic Research Council, ANPRAS (Allied Network For Policy, Research And Actions For Sustainability)*. Https://Www.Researchgate.Net/Publication/255981900

[3]. Estrada Analco, J. M. (2018). The Rhind Papyrus. *A&H Digital Arts And Humanities Journal*. Retrieved From Https://Www.Upaep.Mx/Images/Revista_Artes_Humanidades/Pdf/AH_07_02.Pdf

[4]. Fandiño Albiáñez, R. (2019). Cramer's Rule And The Number E. Mathematics Bulletin, 26(1), 15–32. Unirioja.Es

[5]. Higham, N. J., & Mary, T. (2022). Solving Block Low-Rank Linear Systems By LU Factorization Is Numerically Stable. IMA Journal Of Numerical Analysis, 42(2), 951–980. Https://Doi.Org/10.1093/Imanum/Drab020

[6]. [6] Karpinski, L. C. (1936). New Light On Babylonian Mathematics. *The American Journal Of Semitic Languages And Literatures, 53*(3), 219–229. Https://Www.Jstor.Org/Stable/528987

[7]. Larson, C. (2010). On The Histories Of Linear Algebra: The Case Of Linear Systems. *Proceedings Of The 13th Annual Conference On Research In Undergraduate Mathematics Education*. Recuperado De Http://Sigmaa.Maa.Org/Rume/Crume2010/Archive/Larson.Pdf

[8]. Lund, K., Et Al. (2023). Adaptively Restarted Block Krylov Subspace Methods With Low-Synchronization Skeletons. Numerical Algorithms, 93, 731–764. Https://Doi.Org/10.1007/S11075-022-01437-1

[9]. Mehtre, V. V., & Tiwari, S. (2022). Review On Gauss-Jordan Method And Its Applications. *International Journal Of Advances In Engineering And Management (IJAEM), 4*(2), 540–546.

[10]. Mehtre, S., Y Tiwari, A. (2022). Algorithmic Optimization Of Linear System Solvers Via Gauss-Jordan Identity Reductions. International Journal Of Mathematical And Computational Sciences, 16(4), 205–218. Doi.Org

[11]. Moradi V. V & Tiwari. S. (2022) . Reviw On Gauss-Jordan Method And Its Applications. International Journal Of Advances In Ingewring And Management, 4(2), 540-546

[12]. Moradi. M. (2018). Solving Ill-Conditioned Linear Equation Using Simulate Annealing Mehod Journal Of Hyperstructures,7 (Special Iussue) 60-66 Htpps//Journal.Uma.Ac-Ir/Article-2690

[13]. Weiss, N., Wasilkowski, G. W., Wozniakowski, H., & Shun, M. (1986). Average Condition Number For Solving Linear Equations. *Linear Algebra And Its Applications, 86*, 90266-1. Https://Doi.Org/10.1016/0024-3795(86)90266-1

[14]. Moradi, M. (2018). Solving Ill-Conditioned Linear Equations Using Simulated Annealing Method. *Journal Of Hyperstructures, 7*(Spec. 2nd CSC2017), 60–66. ISSN: 2322-1666 (Print) / 2251-8436 (Online). Https://Journal.Uma.Ac.Ir/Article_2690_237bbdbc223d9bdf7f95c153addedbae.Pdf

[15]. Simoncini, V., & Wang, Y. (2025). Stabilized Krylov Subspace Recurrences Via Randomized Sketching. Numerical Linear Algebra With Applications, 32, E70022. Https://Doi.Org/10.1002/Nla.70022

[16]. Weiss, N., Wasilkowski, G. W., Wozniakowski, H., & Shun, M. (1986). Average Condition Number For Solving Linear Equations. Linear Algebra And Its Applications. Https://Doi.Org/10.1016/0024-3795(86)90266-1