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Active Suspension Control based on a Full-Vehicle Model

Eng. Ahmad Alenezi
Public Authority for Applied Education and Training

Abstract: *Insuring a comfortable ride for passengers, in a vehicle, is based on external disturbances (e.g. road bumps or holes) that can affect the vehicle. The objective of this project is to design controllers to eliminate or reject the effect of the external disturbances (possibly in impulse form or infinite energy) on the performance of the full-vehicle suspension system in order also to reduce heave, pitch and roll acceleration. Three types of controllers were designed for this system. The design controllers are a state feedback controller, a* H_2 *and a* $H\infty$ *controller by using the Linear Matrix Inequality (LMI) technique. Simulation results are given to illustrate the performance of the proposed controllers.*

I. Introduction

Suspension systems have been used widely for many decades. From the beginning of carriages pulled by horses up until the present day automobiles, they can be clearly seen. In theory, a perfect suspension system should provide a comfortable ride for passengers and good handling for the driver. In reality, it seems that one of these characteristics must be compromised for the other. For example, in a luxury sedan, a suspension system will provide a comfortable ride for the passengers in exchange for poor road handling while on the other hand; in sport cars you can see a suspension system that provides optimal road handling yet providing a hard ride for the passengers.

From the standpoint of a system design, there are two important classes of disturbances in a vehicle. There are load and road disturbances. A load disturbance can be caused by acceleration, deceleration, or simply by changing direction of the vehicle. On the other hand, road disturbances can be caused by a bumpy road or even hills. Because of this, a beneficial suspension system should be designed to counter the effects of both load and road disturbances.

There are two main categories of suspension systems: passive and active suspensions. A passive system is a system in which energy is stored or dissipated only within the limitations of its specific design. An active system is a system that includes external energy sources to enhance a compromise between both disturbances so as to produce the most optimal ride possible.

Active systems are a hot topic of research and study in recent years. During my research, I came across many papers that discuss active and passive suspension systems. One paper talks about combining a filtered feedback control scheme and an input decoupling transformation for a full vehicle suspension system [4]. In another paper, the author derives a comprehensive analysis that was applied on a full-car model [3]. Author Fu-Cheng Wang discusses controller parameterization for disturbance response decoupling for a half and full car model in his PhD dissertation [2].

In this project, I will use the same model that was used in the research [3] to apply a new technique called the Linear Matrix Inequality (LMI) to design three controllers a state feedback controller, a H_2 and a $H\infty$ controller. The goal for designing these controllers are to control the pitch, heave, and roll motions of a vehicle and reject the effects of external disturbances (possibly in impulse form or infinite energy) on the performance of the system.

1.1 Passive Suspensions

Springs and dampers, used in passive systems, can only store or release energy. Although different kinds of springs and dampers may be used, most suspensions of this type are usually viewed as a damper and a spring placed together in parallel and called a “strut”. These struts can normally be located at the corners of a vehicle.

As a supplement to the strut, other essential items are also used to boost the performance of the passive system. Additional roll springs or anti-roll bars can be used to increase the stiffness-roll motion. Another item, trailing arms, can be added between the wheel hubs and the sprung mass to reduce the squat motion and the dive of the vehicle body during starting or stopping.

Because of the obvious limitations of a passive system, we can safely conclude that an active system would provide a greater ability, in a vehicle, to improve its performance requirements.

1.2 Active Suspensions

For many years, the use of active suspensions on road vehicles has been pondered. Several different arrangements have been examined from partially active to fully active proposals. Also, there has been interest in defining the degrees of limitation and freedom that are involved in the design. In terms of the transfer-function and energy passivity point of view, there have been previous investigations on the achievable response of the constraints. In active suspension design for full-car models it has been found advantageous to decompose the motion into bounce, pitch, and roll components for the vehicle body, and warp for the wheels in contact with the road.

1.3 Outline of the Project

This project contains 4 chapters and is organized as follows:

Chapter 2: This chapter explains and models the full car suspension system and shows the various possible motions that a vehicle can go through during movement. It also shows the various pitch, roll, and vertical equilibriums of the sprung and unsprung masses within the vehicle. These equations were taken from [3] where they had previously been derived. The system states were also assigned depending on the sprung/ unsprung masses located at each of the four wheels located in the vehicle and their results were shown. The parameters of the full vehicle model were also shown.

Chapter 3: This chapter shows the equations and the derivations of the design of the state feedback controllers, H_2 and $H\infty$ that will be used for designing the controllers.

Chapter 4: This chapter shows the Lyapunov stability test for the CTLTI and DTLTI systems and analyzes them. By checking for a result of greater than zero in the matrices and the equations are satisfied, we can determine if the system is stable and therefore feasible or not. Since stability was already achieved, a state feedback controller was designed to study the behavior of the system. Using *Matlab*, the gain was calculated and used to produce graphs using *Similink*.

1.4 Software Used in the Project

The following software was used in this project:

Matlab: It is used for numerical calculation and simulation of the linearized models.

Similink: It is used for designing the block diagram for the system and getting the graph for the input and output of the system.

II. Modeling

The need to control the pitch, heave, and roll motions in a vehicle made it necessary to apply the force-balance analysis to the full-vehicle suspension system as shown in Figure 1. It will be represented as a system that is a linearized seven degree of freedom system (DOF). The system will consist of 4 unsprung masses (one at each wheel) connected to a single sprung mass that is the car body. The unsprung masses will be free to move vertically with respect to the sprung mass while the sprung mass will be free to heave (vertical motion), pitch, and roll. Linear viscous dampers and spring elements will model the suspensions between the sprung and the unsprung masses while the tires are simply modeled as linear springs without damping. All pitch and roll angles will be assumed as small for simplicity.

The force-balance analysis was applied on the model in Fig (1).

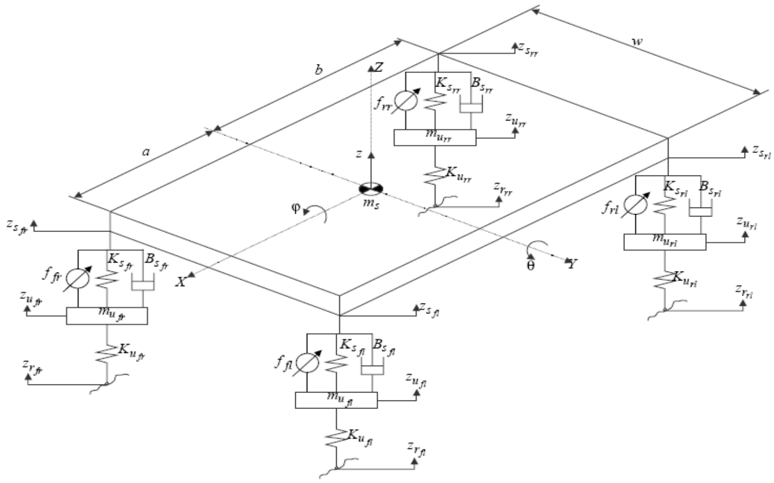

Figure 1. Model of full-vehicle system

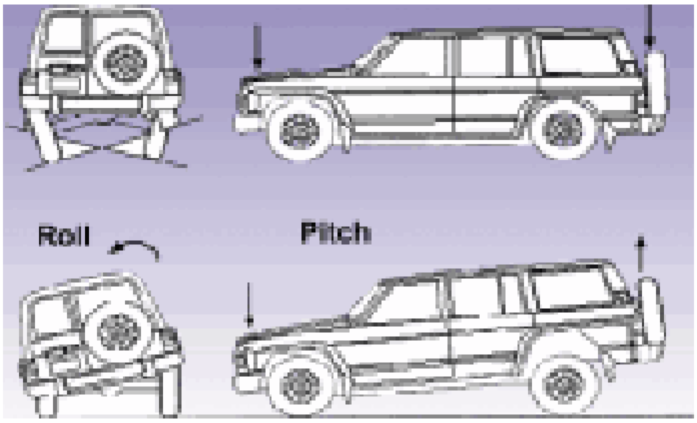

Figure 2. Various vehicle motions during driving

After applying a force-balance analysis to the model in Fig. 1, the equations of motion are given as:

1- Vertical equilibrium of sprung mass:

$$m_s \ddot{z} = -(2K_{sf} + 2K_{sr})z - (2B_{sf} + 2B_{sr})\dot{z} + (2aK_{sf} - 2bK_{sr})\theta - (2aB_{sf} - 2bB_{sr})\dot{\theta} + K_{sf} z_{u\,fl} +$$

$$+ B_{sf}\dot{z}_{u\,fl} + K_{sf} z_{u\,fr} + B_{sf}\dot{z}_{u\,fr} + K_{sr} z_{u\,rl} + B_{sr}\dot{z}_{u\,rl} + K_{sr} z_{u\,rr} + B_{sr}\dot{z}_{u\,rr} + f_{fl} + f_{fr} + f_{rl} + f_{rr}$$

2- Pitch equilibrium of sprung mass:

$$I_{yy}\ddot{\theta} = (2aK_{sf} - 2bK_{sr})z + (2aB_{sf} - 2aB_{sr})\dot{z} - (2a^2K_{sf} + 2b^2K_{sr})\theta - (2a^2B_{sf} + 2b^2B_{sr})\dot{\theta} - aK_{sf} z_{u\,fl} +$$

$$- aB_{sf}\dot{z}_{u\,fl} - aK_{sf} z_{u\,fr} - aB_{sf}\dot{z}_{u\,fr} + bK_{sr} z_{u\,rl} + bB_{sr}\dot{z}_{u\,rl} + bK_{sr} z_{u\,rr} + bB_{sr}\dot{z}_{u\,rr} - af_{fl} - af_{fr} + bf_{rl} + bf_{rr}$$

3- Roll equilibrium of sprung mass:

$$I_{xx}\ddot{\phi} = -0.25w^2(2K_{sf} + 2K_{sr})\phi - 0.25w^2(2B_{sf} + 2B_{sr})\dot{\phi} + 0.5wK_{sf} z_{u\,fl} + 0.5wB_{sf}\dot{z}_{u\,fl}$$

$$-0.5wK_{sf} z_{u\,fr} - 0.5wB_{sf}\dot{z}_{u\,fr} + 0.5wK_{sr} z_{u\,rl} + 0.5wB_{sr}\dot{z}_{u\,rl} - 0.5wK_{sr} z_{u\,rr} - 0.5wB_{sr}\dot{z}_{u\,rr}$$

$$+0.5wf_{fl} - 0.5wf_{fr} + 0.5wf_{rl} - 0.5wf_{rr}$$

4- Vertical equilibrium of unsprung mass 1:

$$m_u \ddot{z}_{u\,fl} = K_{sf} z + B_{sf} \dot{z} - aK_{sf}\theta - aB_{sf}\dot{\theta} + 0.5wK_{sf}\phi + 0.5wB_{sf}\dot{\phi} +$$
$$-(K_{sf} + K_u) z_{u\,fl} - B_{sf}\dot{z}_{u\,fl} + K_u z_{rfl} - f_{fl}$$

5- Vertical equilibrium of unsprung mass 2:

$$m_u \ddot{z}_{u\,fr} = K_{sf} z + B_{sf} \dot{z} - aK_{sf}\theta - aB_{sf}\dot{\theta} - 0.5wK_{sf}\phi - 0.5wB_{sf}\dot{\phi} +$$
$$-(K_{sf} + K_u) z_{u\,fl} - B_{sf}\dot{z}_{u\,fr} + K_u z_{rfr} - f_{fr}$$

6- Vertical equilibrium of unsprung mass 3:

$$m_u \ddot{z}_{u\,rl} = K_{sr} z + B_{sr} \dot{z} + bK_{sr}\theta + bB_{sr}\dot{\theta} + 0.5wK_{sr}\phi + 0.5wB_{sr}\dot{\phi} +$$
$$-(K_{sr} + K_u) z_{u\,rl} - B_{sr}\dot{z}_{u\,rl} + K_u z_{rrl} - f_{rl}$$

7- Vertical equilibrium of unsprung mass 4:

$$m_u \ddot{z}_{u\,rr} = K_{sr} z + B_{sr} \dot{z} + bK_{sr}\theta + bB_{sr}\dot{\theta} - 0.5wK_{sr}\phi - 0.5wB_{sr}\dot{\phi} +$$
$$-(K_{sr} + K_u) z_{u\,rl} - B_{sr}\dot{z}_{u\,rl} + K_u z_{rrl} - f_{rr}$$

The system states are assigned a:

$x_1 = z$ heave position (ride height of sprung mass)

$x_2 = \dot{z}$ heave velocity (payload velocity of sprung mass)

$x_3 = \theta$ pitch angle

$x_4 = \dot{\theta}$ pitch angular velocity

$x_5 = \phi$ roll angle

$x_6 = \dot{\phi}$ roll angular velocity

$x_7 = z_{ufl}$ front left wheel unsprung mass height

$x_8 = \dot{z}_{ufl}$ front left wheel unsprung mass velocity

$x_9 = z_{ufr}$ front right wheel unsprung mass height

$x_{10} = \dot{z}_{ufr}$ front right wheel unsprung mass velocity

$x_{11} = z_{url}$ rear left wheel unsprung mass height

$x_{12} = \dot{z}_{rl}$ rear left wheel unsprung mass velocity

$x_{13} = z_{urr}$ rear right wheel unsprung mass height

$x_{14} = \dot{z}_{urr}$ rear right wheel unsprung mass velocity

The results in the system state equations are shown as:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{-(2K_{sf}+2K_{sr})}{m_s}x_1 - \frac{(2B_{sf}+2B_{sr})}{m_s}x_2 + \frac{(2aK_{sf}-2bK_{sr})}{m_s}x_3 - \frac{(2aB_{sf}-2bB_{sr})}{m_s}x_4 + \frac{K_{sf}}{m_s}x_7 +$$

$$+\frac{B_{sf}}{m_s}x_8 + \frac{K_{sf}}{m_s}x_9 + \frac{B_{sf}}{m_s}x_{10} + \frac{K_{sr}}{m_s}x_{11} + \frac{B_{sr}}{m_s}x_{12} + \frac{K_{sr}}{m_s}x_{13} + \frac{B_{sr}}{m_s}x_{14} + \frac{1}{ms}f_{fl} + \frac{1}{m_s}f_{fr}$$

$$+\frac{1}{m_s}f_{rl} + \frac{1}{m_s}f_{rr}$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = \frac{(2aK_{sf}-2bK_{sr})}{I_{yy}}x_1 + \frac{(2aB_{sf}-2aB_{sr})}{I_{yy}}x_2 - \frac{(2a^2K_{sf}+2b^2K_{sr})}{I_{yy}}x_3 - \frac{(2a^2B_{sf}+2b^2B_{sr})}{I_{yy}}x_4$$

$$-\frac{aK_{sf}}{I_{yy}}x_7 - \frac{aB_{sf}}{I_{yy}}x_8 - \frac{aK_{sf}}{I_{yy}}x_9 - \frac{aB_{sf}}{I_{yy}}x_{10} + \frac{bK_{sr}}{I_{yy}}x_{11} + \frac{bB_{sr}}{I_{yy}}x_{12} + \frac{bK_{sr}}{I_{yy}}x_{13} + \frac{bB_{sr}}{I_{yy}}x_{14} - \frac{a}{I_{yy}}f_{fl}$$

$$-\frac{a}{I_{yy}}f_{fr} + \frac{b}{I_{yy}}f_{rl} + \frac{b}{I_{yy}}f_{rr}$$

$$\dot{x}_5 = x_6$$

$$\dot{x}_6 = -\frac{0.25w^2(2K_{sf}+2K_{sr})}{I_{xx}}x_5 - \frac{0.25w^2(2B_{sf}+2B_{sr})}{I_{xx}}x_6 + \frac{0.5wK_{sf}}{I_{xx}}x_7 + \frac{0.5wB_{sf}}{I_{xx}}x_8 - \frac{0.5wK_{sf}}{I_{xx}}x_9$$

$$-\frac{0.5wB_{sf}}{I_{xx}}x_{10} + \frac{0.5wK_{sr}}{I_{xx}}x_{11} + \frac{0.5wB_{sr}}{I_{xx}}x_{12} - \frac{0.5wK_{sr}}{I_{xx}}x_{13} - \frac{0.5wB_{sr}}{I_{xx}}x_{14} + \frac{0.5w}{I_{xx}}f_{fl} - \frac{0.5w}{I_{xx}}f_{fr}$$

$$+\frac{0.5w}{I_{xx}}f_{rl} - \frac{0.5w}{I_{xx}}f_{rr}$$

$$\dot{x}_7 = x_8$$

$$\dot{x}_8 = \frac{K_{sf}}{m_u}x_1 + \frac{B_{sf}}{m_u}x_2 - \frac{aK_{sf}}{m_u}x_3 - \frac{aB_{sf}}{m_u}x_4 + \frac{0.5wK_{sf}}{m_u}x_5 + \frac{0.5wB_{sf}}{m_u}x_6 +$$

$$-\frac{(K_{sf}+K_u)}{m_u}x_7 - \frac{B_{sf}}{m_u}x_8 + \frac{K_u}{m_u}z_{r\,fl} - \frac{1}{m_u}f_{fl}$$

$$\dot{x}_9 = x_{10}$$

$$\dot{x}_{10} = \frac{K_{sf}}{m_u}x_1 + \frac{B_{sf}}{m_u}x_2 - \frac{aK_{sf}}{m_u}x_3 - \frac{aB_{sf}}{m_u}x_4 - \frac{0.5wK_{sf}}{m_u}x_5 - \frac{0.5wB_{sf}}{m_u}x_6 + \\ -\frac{(K_{sf}+K_u)}{m_u}x_9 - \frac{B_{sf}}{m_u}x_{10} + \frac{K_u}{m_u}z_{r\,fr} - \frac{1}{m_u}f_{fr}$$

$$\dot{x}_{11} = x_{12}$$

$$\dot{x}_{12} = \frac{K_{sr}}{m_u}x_1 + \frac{B_{sr}}{m_u}x_2 + \frac{bK_{sr}}{m_u}x_3 + \frac{bB_{sr}}{m_u}x_4 + \frac{0.5wK_{sr}}{m_u}x_5 + \frac{0.5wB_{sr}}{m_u}x_6 + \\ -\frac{(K_{sr}+K_u)}{m_u}x_{11} - \frac{B_{sr}}{m_u}x_{12} + \frac{K_u}{m_u}z_{r\,rl} - \frac{1}{m_u}f_{rl}$$

$$\dot{x}_{13} = x_{14}$$

$$\dot{x}_{14} = \frac{K_{sr}}{m_u}x_1 + \frac{B_{sr}}{m_u}x_2 + \frac{bK_{sr}}{m_u}x_3 + \frac{bB_{sr}}{m_u}x_4 - \frac{0.5wK_{sr}}{m_u}x_5 - \frac{0.5wB_{sr}}{m_u}x_6 + \\ -\frac{(K_{sr}+K_u)}{m_u}x_{13} - \frac{B_{sr}}{m_u}x_{14} + \frac{K_u}{m_u}z_{r\,rr} - \frac{1}{m_u}f_{rr}$$

The full-vehicle model parameters selected for this project are given as:
sprung mass, m_s =1500 kg
unsprung mass, m_u =59 kg
front suspension spring stiffness, $K_{sf} = K_{s\,fl} = K_{s\,fr}$ = 35000 N/m
rear suspension spring stiffness, $K_{sr} = K_{s\,rl} = K_{s\,rr}$ = 38000 N/m
tire spring stiffness, $K_u = K_{u\,fl} = K_{u\,fr} = K_{url} = K_{urr} =$ 190000 N/m
front suspension damping, $B_{sf} = B_{sfl} = B_{sfr}$ = 1000 N/m/s
rear suspension damping, $B_{sr} = B_{srl} = B_{srr}$ = 1100 N/m/s
roll axis moment of inertia, I_{xx} = 460 kg-m^2
pitch axis moment of inertia, I_{yy} = 2160 kg-m^2
length between front of vehicle and center of gravity of sprung mass, a = 1.4 m
length between rear of vehicle and center of gravity of sprung mass, b = 1.7 m
width of sprung mass, w = 3m

The state space equation in matrix form are given by

$$\dot{x}(t) = Ax(t) + Bf(t) + B_w w(t)$$
$$y(t) = Cx(t),$$

with the control input $f(t)$ defined as the force generated at the front-left, front-right, rear-left and rear-right suspensions respectively as $f(t)$=[$f_{fl}(t)\ f_{fr}(t)\ f_{rl}(t)\ f_{rr}(t)]^T$, and the disturbance input $w(t)$ defined as $w(t)$ =[z_{rfl} (t) z_{rfr} (t) z_{rrl} (t) z_{rrr} (t)]^T where g is the acceleration due to gravity. Signals z_{rfl} (t) ,z_{rfr} (t), z_{rrl} (t), z_{rrr} (t) are the terrain disturbance heights at the front-left, front-right, rear-left and rear-right wheels respectively. The output $y(t)$ will change; it is selected for specific performance-analysis objectives.

III. Methodology

3.1 Linear Systems Stability

It is sufficient to consider quadratic forms as Lyapunov functions in the case of linear systems. This is because, for linear systems in general, stability signifies global stability and the parabolic shape of a quadratic function fulfills all of the requirements of the previously stated theorems. We will begin by considering the special case of linear systems with the time-invariant case. When examined with Lyapunov's direct method as described here, LTI systems result in some tests and equations. Consider the LTI system:

$$\dot{x} = Ax \quad ; x(0) = x_0; x(t) \in R^n \qquad \text{-----} \quad (1)$$

and the candidate Lyapunov function $V(x) = x^T Px$ where the P matrix is positive definite.
Then testing the LTI system above, we can compute:

$$\begin{aligned} \dot{V} &= x^T P\dot{x} + \dot{x}^T P x \\ &= (Ax)^T Px + x^T P(Ax) \\ &= x^T A^T Px + x^T PAx \\ &= x^T (A^T P + PA)x \end{aligned} \qquad \text{-----} (2)$$

Therefore, for asymptotic stability it is necessary for the matrix $A^T P + PA$ to be negative definite, to satisfy the theorem (The origin of the system in (1) described by matrix A is asymptotically stable if and only if, given a positive definite matrix Q, the matrix Lyapunov equation (3) has a solution P that is positive definite.) and also negative semi definite for Lyapunov stability. To be exact, for some positive (semi)definite matrix, Q, it is enough to show that

$$A^T P + PA = -Q \qquad \text{-----} \quad (3)$$

to show that the homogeneous system in (1) has stability. Equation (3) is also known as the Lyapunov equation.

Nonetheless, caution must be exercised when applying this test because the quantity of $A^T P + PA$ must be negative; if it isn't negative (semi)definite then nothing can be concluded about the stability. Remember that the direct method of Lyapunov stresses that a system is stable if a Lyapunov function is found. The direct method does not state that if a function such as $x^T Px$ fails then the system is not stable. Therefore, if positive-definite matrix P is chosen and we compute that Q is indefinite, then we have shown nothing. But if Q is negative definite then we may be able to show instability. Instead, we turn to the reverse process. A positive (semi)definite Q is selected and the solution to P is calculated to the Lyapunov equation (2).

Lyapunov stability test for Continuous-Time Linear Time Invariance (CTLTI) system:

The matrix P Є S^n such that the LMI
1- P > 0
2- $A^T P + PA < 0$, is feasible

Lyapunov stability test for Discrete- Time Linear Time Invariance (DTLTI) system:
1-P >0
2- [P A'P;
PA P]>0, is feasible

3.2 Stabilizability Feedback Control:

The objective in designing a stabilizing controller is so that we can stabilize the system by combining the dynamics of the system

$$\dot{x} = Ax + B_u u$$

$$\dot{x} = (A + B_u K)x$$

with the expression of the controller for the CTLTI system which is stabilizable by the state feedback control **,** $u = Kx$**.** Based on our previous results in stability, $(A + B_u K)'P + P(A + B_u K) < 0$
,if and only if, $\exists X \in S^n \; and \; L \in R^{m\times n}$ such that

$$X > 0\ ,\ AX + XA^T + B_u L + L^T B_u^T < 0$$

if so, a stabilizing control gain is $K = LX^{-1}$ where $X = P^{-1} and\ \ L = KP^{-1}$.

3.3 State Feedback H_2 control:

If we have impulse external disturbances, we can use the H_2 control.

For the Plant (CTLTI system):

$$\dot{x} = Ax + B_w w + B_u u,\ \ x(0) = 0,$$
$$z = C_z x + D_{zw} w + D_{zu} u.$$

The controller (state-feedback controller), $u = Kx$ and the Closed loop system:

$$\dot{x} = \underbrace{(A + B_u K)}_{Acl} x + \underbrace{B_w}_{Bcl} w,\quad x(0) = 0,$$
$$z = \underbrace{C_z + D_{zu} K}_{Ccl} x + \underbrace{D_{zw}}_{Dcl} w.$$

the analysis conditions (H_2 norm) $\exists K \in R^{m\times n}\ such\ that\ \|H_{cl}(K,s)\|_2^2 < \mu$

If, and only if , $\exists K \in R^{m\times n}\ and\ X \in S^n\ such\ that$

$$X > 0,\ \underbrace{(A + B_u K)}_{Acl} X + X \underbrace{(A^T + K^T B_u^T)}_{A_{cl}^T} + \underbrace{B_w}_{Bcl}\ \underbrace{B_w^T}_{B_{cl}^T} < 0$$

$$tr\left[\underbrace{(C_z + D_{zu} K)}_{Ccl} X \underbrace{(C_z^T + K^T D_{zu}^T)}_{C_{cl}^T}\right] < \mu,\qquad \underbrace{D_{zw} = 0.}_{Dcl}$$

Method I (congruence + change-of-variables). Since we are using the dual expressions, there is no need to apply congruence in this case. The first two inequalities provide:

$$X > 0,\ \ AX + XA^T + B_u KX + XK^T B_u^T + B_w B_w^T < 0.$$

which can be transformed into the LMI.

$$X > 0,\ \ AX + XA^T + B_u L + L^T B_u^T + B_w B_w^T < 0.$$

Using the change-of-variables $L = KX$, the "cost inequality" can be manipulated by introducing the auxiliary matrix W as:

$$W > (C_z + D_{zu} K) X (C_z^T + K^T D_{zu}^T),\ so\ that\ if\ \ tr[W] < \mu\ then$$
$$tr[(C_z + D_{zu} K) X (C_z^T + K^T D_{zu}^T)] < tr[W] < \mu.$$

Notice that a Schur complement can be used to rewrite

$$W > (C_z + D_{zu} K) X X^{-1} X (C_z^T + K^T D_{zu}^T),$$
$$\Updownarrow$$
$$\begin{bmatrix} W & (C_z + D_{zu} K) X \\ X (C_z^T + K^T D_{zu}^T) & X \end{bmatrix} > 0.$$

This last form is also "linearized" by the change-of-variables L = KX yielding the LMI

$$\begin{bmatrix} W & C_z X + D_{zu} L \\ X C_z^T + L^T D_{zu}^T & X \end{bmatrix} > 0$$

Then for the CTLTI system:

$$\dot{x} = Ax + B_w w + B_u u, \quad x(0) = 0,$$
$$z = C_z x + D_{zw} w + D_{zu} u.$$

is stabilizable by the state feedback control $u = Kx$ such that $\|H_{wz}(s)\|_2^2 \prec \mu$

if, and only if, D_{zw}=0 and $\exists X \in S^n \; and \; L \in R^{m\times n}$ such that

$$AX + XA^T + B_u L + L^T B_u^T + B_w B_w^T < 0,$$
$$\begin{bmatrix} W & C_z X + D_{zu} L \\ XC_z^T + L^T D_{zu}^T & X \end{bmatrix} .. > 0, \quad tr[W] < \mu.$$

If so, a stabilizing control gain is $K = LX^{-1}$ and we will get optimal H_2 control if we can minimize μ.

3.4 State Feedback H_2 control (primal):

Plant (CTLTI system):

$$\dot{x} = Ax + B_w w + B_u u, \quad x(0) = 0,$$
$$z = C_z x + D_{zw} w + D_{zu} u.$$

The controller (state-feedback controller) $u = Kw$ and the Closed loop system:

$$\dot{x} = \underbrace{(A + B_u K)}_{Acl} x + \underbrace{B_w}_{Bcl} w, \quad x(0) = 0,$$
$$z = \underbrace{C_z + D_{zu} K}_{Ccl} x + \underbrace{D_{zw}}_{Dcl} w.$$

The analysis conditions (H_2 norm) $\exists K \in R^{m\times n} \; such \; that \; \|H_{cl}(K,s)\|_2^2 < \mu$

If, and only if , $\exists K \in R^{m\times n} \; and \; X \in S^n \; such \; that$

$$P > 0, \; \underbrace{(A^T + K^T B_u^T}_{A_{cl}^T} P + P\underbrace{(A + B_u K)}_{Acl} + \underbrace{(C_z^T + K^T D_{zu}^T)}_{C_{cl}^T}\underbrace{(C_z + D_{zu} K)}_{C_{cl}} < 0$$

$$tr[\underbrace{B_w^T}_{B_{cl}^T} \; P \; \underbrace{B_w}_{B_{cl}}] < \mu, \; \underbrace{D_{zw}}_{D_{cl}} = 0$$

Method I (congruence + change-of-variables): In primal form, we need to apply the

$$AP^{-1} + P^{-1}A^T + B_u KP^{-1} + P^{-1}K^T B_u^T + (P^{-1}C_z^T + P^{-1}K^T D_{zu}^T) + (C_z P^{-1} + D_{zu} KP^{-1}) < 0,$$

which can be transformed into

$$X > 0, \; AX + XA^T + B_u KL + L^T B_u^T + (XC_z^T + L^T D_{zu}^T) + (C_z X + D_{zu} L) < 0$$

using the change-of-variables $X := P^{-1}, \; L := KP^{-1}$*.This inequality can be transformed into an LMI by Schur complement*

$$\begin{bmatrix} AX + XA^T + B_u KL + L^T B_u^T & XC_z^T + L^T D_{zu}^T \\ C_z X + D_{zu} L & -I \end{bmatrix} < 0$$

the "cost inequality" can be manipulated by introducing the auxiliary matrix W as

$W > B_w^T PB_w$, *so that if* $tr[W] < \mu$ *then* $tr[B_w^T PB_w] < tr[W] < \mu$

Then Schur complement can be used to convert it into and LMI in the form

$$\begin{bmatrix} W & B_w^T \\ B_w & P^{-1} \end{bmatrix} = \begin{bmatrix} W & B_w^T \\ B_w & X \end{bmatrix} > 0.$$

Then the state feedback $\mathbf{H}_2$ control for the CTLTI system:

$$\dot{x} = Ax + B_w w + B_u u, \quad x(0) = 0,$$
$$z = C_z x + D_{zw} w + D_{zu} u.$$

is stabilizable by the state feedback control $u = Kx$ such that $\|H_{wz}(s)\|_2^2 \prec \mu$
if, and only if, D_{zw}=0 and $\exists X \in S^n$ *and* $L \in R^{m\times n}$ and $W \in S^r$ such that $tr[W]$<μ and

$$\begin{bmatrix} AX + XA^T + B_u KL + L^T B_u^T & XC_z^T + L^T D_{zu}^T \\ C_z X + D_{zu} L & -I \end{bmatrix} < 0, \quad \begin{bmatrix} W & B_w^T \\ B_w & X \end{bmatrix} > 0$$

If so, a stabilizing control gain is $K = LX^{-1}$ and the optimal H_2 control will be stable if we minimize μ.

3.5 State Feedback H∞ control:

Plant (CTLTI system):

$$\dot{x} = Ax + B_w w + B_u u, \quad x(0) = 0,$$
$$z = C_z x + D_{zw} w + D_{zu} u.$$

The controller (state-feedback controller) $u = Kx$ and the Closed loop system:

$$\dot{x} = \underbrace{(A + B_u K)}_{Acl} x + \underbrace{B_w}_{Bcl} w, \quad x(0) = 0,$$
$$z = \underbrace{C_z + D_{zu} K}_{Ccl} x + \underbrace{D_{zw}}_{Dcl} w.$$

The analysis conditions (**H∞ norm**) $\exists K \in R^{m\times n}$ *such that* $\|H_{cl}(K,s)\|_\infty < \mu$
If, and only if, $\exists K \in R^{m\times n}$ *and* $P \in S^n$ *such that* (BRL)

$$P > 0 \quad \begin{bmatrix} A_{cl}{}^T P + PA_{cl} & PB_{cl} & C_{cl}{}^T \\ B^T P & -\mu I & D_{cl}{}^T \\ C_{cl} & D_{cl} & -\mu I \end{bmatrix} \prec 0$$

Method I (congruence + change-of-variables): We need to apply the congruence transformation on the BRL.

$$\begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_{cl}{}^T P + PA_{cl} & PB_{cl} & C_{cl}{}^T \\ B^T P & -\mu I & D_{cl}{}^T \\ C_{cl} & D_{cl} & -\mu I \end{bmatrix} \begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \prec 0,$$

$$\Updownarrow$$

$$\begin{bmatrix} A_{cl}{}^T P^{-1} + P^{-1} A_{cl}^T & B_{cl} & P^{-1} C_{cl}{}^T \\ B^T & -\mu I & D_{cl}{}^T \\ C_{cl} P^{-1} & D_{cl} & -\mu I \end{bmatrix} < 0$$

The matrices $B_{cl} = B_w$ and $D_{cl} = D_{zw}$ are constant matrices and the products

$$A_{cl} P^{-1} = AP^{-1} + B_u KP^{-1}, \qquad C_{cl} P^{-1} = C_z P^{-1} + D_{zu} KP^{-1},$$

can by transformed into LMI using the change-of-variables $X=P^{-1}$, L= KX.

Then state feedback **H∞** control for the CTLTI system:

$$\dot{x} = Ax + B_w w + B_u u, \quad x(0) = 0,$$
$$z = C_z x + D_{zw} w + D_{zu} u.$$

Is stabilizable by the state feedback control $u = Kx$ such that $\|H_{wz}(s)\|_\infty \prec \mu$

If, and only if, $\exists X \in S^n$ *and* $L \in R^{m\times n}$ such that

$$X > 0, \quad \begin{bmatrix} AX + XA^T + B_u L + L^T B_u^T & B_w & XC^T + L^T D_{zu}^T \\ B_w^T & -\mu I & D_{zw}{}^T \\ C_z X + D_{zu} L & D_{zw} & -\mu I \end{bmatrix} < 0$$

If so, a stabilizing control gain is $K = LX^{-1}$ and the optimal H∞ control will be stable if we minimize μ.

IV. Controllers Design

4.1 Stability Test

Lyapunov stability test for continuous-time linear time invariance (CTLTI) system:

The matrix P Є S^n such that the LMI

3- P > 0

4- $A^T P + PA < 0$, is feasible

Lyapunov stability test for Discrete-time linear time invariance (DTLTI) system:

1-P >0

2- [P A'P;

PA P]>0, is feasible

After applying the Lyapunov stability test to the full car model for the CTLTI and DTLTI system, we concluded from the Matlab program (StabilityCTcar.m & StabilityDTcar.m) that the system is stable because it satisfies the Lyapunov equations and we checked the eigenvalues for the P matrix and found that all of them are greater than zero.

This means that the system is stable and feasible according to Lyapunov.

4.2 Design Stabilizability Feedback Controller for the Full-Car model:

The objective in designing a stabilizing controller is so that we can stabilize the system by combining the dynamics of the system,

$$\dot{x} = Ax + B_u u$$

if and only if, $\exists X \in S^n$ *and* $L \in R^{m\times n}$ such that

$$X > 0 \, , \, AX + XA^T + B_u L + L^T B_u^T < 0$$

if so, a stabilizing control gain is $K=LX^{-1}$ where $X = P^{-1}$ *and* $L = KP^{-1}$.

Since my system was already stable, we designed the state feedback controller for the system to study its behavior. After running the Matlab program (Kcar.m), we took the data and used it in the Similink program, after designing the block diagram of the system, to produce two graphs (1 and 2) that show that the vehicle is more stable with a load than without.

4.3 Design Feedback H_2 Controller for Full Car Model:

If we have impulse external disturbances, we can use the H_2 control.

Then for the CTLTI system:

$$\dot{x} = Ax + B_w w + B_u u, \quad x(0) = 0,$$
$$z = C_z x + D_{zw} w + D_{zu} u.$$

Is stabilizable by the state feedback control u=Kx such that $\|H_{wz}(s)\|_2^2 \prec \mu$

If, and only if, D_{zw}=0 and $\exists X \in S^n$ *and* $L \in R^{m\times n}$ such that

$$AX + XA^T + B_u L + L^T B_u^T + B_w B_w^T < 0,$$
$$\begin{bmatrix} W & C_z X + D_{zu} L \\ XC_z^T + L^T D_{zu}^T & X \end{bmatrix} .. > 0, \quad tr[W] < \mu.$$

If so, a stabilizing control gain is $K=LX^{-1}$ and we will get optimal H_2 control if we can minimize μ.

In this design, we considered the external disturbance of the force on the front two wheels. After getting the gain from the Matlab program (K1H2Car.m), we designed the block diagram of the system and used the Similink program to get the graphs (3 and 4) for the vehicle with a disturbance and without. In our results we discovered that the disturbance will affect only the vertical motion and pitch.

4.4 Design Feedback H_2 Controller (primal):

State feedback $\mathbf{H}_2$ control for the CTLTI system can be stated as:

$$\dot{x} = Ax + B_w w + B_u u, \quad x(0) = 0,$$
$$z = C_z x + D_{zw} w + D_{zu} u.$$

and is stabilizable by the state feedback control u=Kx such that $\|H_{wz}(s)\|_2^2 \prec \mu$

If, and only if, D_{zw}=0 and $\exists X \in S^n$ *and* $L \in R^{m\times n}$ and **W**Є S^r such that **tr[w]** <μ and

$$\begin{bmatrix} AX + XA^T + B_u KL + L^T B_u^T & XC_z^T + L^T D_{zu}^T \\ C_z X + D_{zu} L & -I \end{bmatrix} < 0, \quad \begin{bmatrix} W & B_w^T \\ B_w & X \end{bmatrix} > 0$$

If so, a stabilizing control gain is $K=LX^{-1}$ and the optimal H_2 control will be stable if we minimize μ.

4.5 Design State Feedback H∞ Control:

Our goal is to design a state feedback controller that minimizes the H∞ norm of the system's closed loop transfer function between the controlled output z and the external disturbance w. In this part of the project, we will consider the external disturbance for the system as a stone hitting the front right part of the vehicle.

Then state feedback **H**∞ control for the CTLTI system:

$$\dot{x} = Ax + B_w w + B_u u, \quad x(0) = 0,$$
$$z = C_z x + D_{zw} w + D_{zu} u.$$

Is stabilizable by the state feedback control u=Kx such that $\|H_{wz}(s)\|_\infty \prec \mu$

If, and only if, $\exists X \in S^n$ *and* $L \in R^{m\times n}$ such that

$$X > 0, \quad \begin{bmatrix} AX + XA^T + B_u L + L^T B_u^T & B_w & XC^T + L^T D_{zu}^T \\ B_w^T & -\mu I & D_{zw}{}^T \\ C_z X + D_{zu} L & D_{zw} & -\mu I \end{bmatrix} < 0$$

If so, a stabilizing control gain is $K=LX^{-1}$ and the optimal **H**∞ control will be stable if we minimize **μ.** After finding the gain from the Matlab program (KHinfCar.m), we designed the block diagram by Similink and we got the graphs (5 and 6) that show the behavior of the vertical motion of the car with and without external disturbance.

V. Conclusion

The LMI technique is an excellent tool for designing state feedback controllers. The state feedback, H2 and H∞ controllers were applied to control the heave, pitch and roll acceleration for the vehicle. By using these controllers, the vehicle is stabilizable even while taking into consideration road disturbances.

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