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Generalized Multi-Fuzzy Rough Sets And The Induced Topology

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Abstract: *In this paper, we study the topological structure of multi-fuzzy rough sets based on fuzzy logical operators. We discuss the relationship between generalized multi-fuzzy rough sets and multi-fuzzy topological spaces. This paper examines the peculiarities of approximation operators when the associated binary relation R is reflexive and transitive. We obtain the conditions under which the collection of $\mathcal{I}$-lower approximation sets becomes a multi-fuzzy topology. It is seen that the reflexivity and transitivity of relation R and the properties of the implicator $\mathcal{I}$ play a role in describing the topological structure of $(\mathcal{J} - \mathcal{T})$ multi-fuzzy rough sets.*

I. Introduction

Uncertainty, imprecision are few among the certain aspects of real-world problems which classical set theory fails to address. A milestone in the evolution of modern concept of uncertainty is the publication of a seminal paper by Lotfi A. Zadeh in which he introduced the concept of Fuzzy sets (Zadeh, 1965). This paved way for many rigorous and extensive researches which resulted in many generalized set theories like L-fuzzy Sets (Goguen, 1967), type-2 fuzzy sets (Mizumoto & Tanaka, 1976), Intuitionistic Fuzzy Sets (Atanassov, 1986), interval valued fuzzy sets (Gorzałczany, 1987), Multi-fuzzy sets (Sebastian & Ramakrishnan, 2011) etc.

Rough Sets introduced by Zdzislaw Pawlak (Pawlak, 1982) also served the purpose of dealing uncertainty and imprecision but in a different way from the theory of fuzzy sets. The main idea of rough set theory is based on the indiscernability relation that every object is associated with a certain amount of information. We can directly obtain the information from the given data using
Rough sets.The basic structure of rough set theory is an approximation space consisting of a universe of discourse and a binary relation imposed on it.

Scholars like Dubois and Prade put forward the idea of hybrid structures namely fuzzy rough sets and rough fuzzy sets considering the complementary nature of fuzzy set theory and rough set theory. Generalization of rough set theory to various fuzzy environment is a new, promising direction of research in the recent years. Interval-valued type-2 fuzzy rough sets (Wu, Wu, & Luo, 2009; Zhang, 2012), intuitionistic fuzzy-rough sets (Zhou & Wu, 2008, 2011) , multi-fuzzy rough sets (Varma & John, 2014) are certain hybrid models in the literature.

The studies on the topological properties of newly formed structures led to the theory of fuzzy topological spaces(Chang, 1968; Lowen, 1976), intuitionistic fuzzy-topological spaces (Çoker, 1997) and so on. A detailed study on multi-fuzzy topological spaces can be found in (Sebastian & V Ramakrishnan, 2011). The theory of rough sets is combined with topology when the topology induced by binary relations is used to generalise basic rough set concepts (Lashin, Kozae, Abo Khadra, & Medhat, 2005). The discussion on various rough set models and their topological structures is given in (Qin & Pei, 2005; Raghavan & Tripathy, 2011; Yu & Zhan, 2014).In this paper, the authors examine the topological structure of generalized multi-fuzzy rough sets when the associated binary relation is reflexive and transitive. By generalised multi-fuzzy rough sets , we mean a broad family of multi-fuzzy rough sets, each one of which is called an $(\mathcal{I} - \mathcal{T})$ multi-fuzzy rough set where $\mathcal{I}$ is a fuzzy implication and $\mathcal{T}$ is a continuous t-norm (Varma & John). The section 2 contains some preliminaries needed in section 3. In section 3, it is established that when the associated multi-fuzzy relation is reflexive and transitive and when the fuzzy implication satisfies the law of importation, the collection of $\mathcal{J}$-lower approximations of multi-fuzzy sets forms a multi-fuzzy topology.

II. Basic Concepts

In this section, we review some basic concepts of logical operators and multi-fuzzy rough sets.

Definition 2.1 (Baczynski & Jayaram, 2008) *A function* $\mathcal{J}: [0,1] \times [0,1] \to [0,1]$ *is called a fuzzy implication if it satisfies, for all* $x, x_1, x_2, y, y_1, y_2 \in [0,1]$ *the following conditions:*

if $x_1 \le x_2$, then $\mathcal{I}(x_1, y) \ge \mathcal{I}(x_2, y)$ i.e., $\mathcal{I}\ , y)$ is decreasing, $(I1)$

if $y_1 \le y_2$, then $\mathcal{I}(x, y_1) \le \mathcal{I}(x, y_2)$ i.e., $\mathcal{I}(y, .)$ is increasing, $(I2)$

$$\mathcal{I}(1,0) = 0 \quad (I3)$$

$$\mathcal{I}(1,1) = 1 \quad (I4)$$

$\mathcal{I}(0,0) = 1 \quad (I5)$.

Definition 2.2 (Baczynski & Jayaram, 2008)*A t-norm (triangular norm) is an increasing, associative and commutative mapping $\mathcal{T}: [0,1] \times [0,1] \to [0,1]$ that satisfies the boundary condition:*
for all $\alpha \in [0,1], \mathcal{T}(\alpha, 1) = \alpha$

Definition 2.3 *A fuzzy implication $\mathcal{I}$ is said to satisfy*

• the exchange principle, if

$\mathcal{I}(x, \mathcal{I}(y, z)) = \mathcal{I}(y, \mathcal{I}(x, z)) x, y, z \in [0,1]$ (EP)

• the ordering property, if

$x \le y \Leftrightarrow \mathcal{I}(x, y) = 1, \quad x, y \in [0,1]$ (OP)

• the left neutrality property, if

$\mathcal{I}(1, x) = x, \quad x \in [0,1]$ (NP)

• the law of importation , if

$\mathcal{I}(x, \mathcal{I}(y, z)) = \mathcal{I}(\mathcal{T}(x, y), z) x, y, z \in [0,1]$ (LI)

where $\mathcal{T}$ is a t-norm.

Remark 2.4 (Baczynski & Jayaram, 2008) *If a fuzzy implication $\mathcal{I}$ satisfies (LI) with respect to any t-norm $\mathcal{T}$, by the commutativity of the t-norm $\mathcal{T}$, $\mathcal{I}$ satisfies the exchange principle (EP).*

Definition 2.5 (Baczynski & Jayaram, 2008) *For a fuzzy implication $\mathcal{I}$, the following statements are equivalent:*
(i) $\mathcal{I}$ is continuous.
(ii) $\mathcal{I}$ is continuous in each variable.

Multi-fuzzy Sets and Multi-fuzzy Topology

Definition 2.6 *(Sebastian & Ramakrishnan, 2011) Let X be a non-empty set, $\mathbb{N}$ the set of all natural numbers and $\{L_i : i \in \mathrm{N}\}$ a family of complete lattices. A* ***multi-fuzzy set*** *A in X is a set of ordered sequences*

$$A = \{< x, \mu_A^1(x), \mu_A^2(x), ..., \mu_A^i(x), ... >: x \in X\}$$

where $\mu_A^i \in L_i^X$ (i.e., $\mu_A^i : X \to L_i$) for $i \in \mathrm{N}$

Remark 2.7 *If the sequences of the membership functions have only k-terms (finite number of terms), k is called the dimension of A. The set of all multi-fuzzy sets in X with the value domain $\prod_{j \in J} M_j$ where each M_j is a complete lattice, is denoted by $\prod_{j \in J} M_j^X$ and is called multi-fuzzy space. Let $L_i = [0,1]$ for $i \in \mathrm{N}$, then the set of all multi-fuzzy sets in X, is denoted by $MFS(X)$.*

Definition 2.8 (Sebastian & V Ramakrishnan, 2011a) *A subset δ of $\prod_{j \in J} M_j^X$ where each M_j is a complete lattice, is called a multi-fuzzy topology on X if it satisfies the following conditions:*

• $0_X, 1_X \in \delta$

• $A \cap B \in \delta$ for every $A, B \in \delta$

• $\vee \sigma \in \delta$ for every $\sigma \subset \delta$, that is arbitrary union of multi-fuzzy sets in δ s in δ

The multi-fuzzy topology δ is called a **strong multi-fuzzy topology** if $\alpha \in \delta$ for all constant multi-fuzzy sets, α in X.

The ordered triple $(X, \prod_{j \in J} M_j^X, \delta)$ is called a multi-fuzzy topological space. Multi-fuzzy sets in δ are called δ −open multi-fuzzy sets in X, simply open multi-fuzzy sets in X.

Definition 2.9 *(Varma and John (a)) Let $\mathcal{T}$ be a continuous t-norm on* $[0,1]$ *and $\mathcal{I}$ be an implicator on* $[0,1]$. *For a multi-fuzzy approximation space (U, R) and any multi-fuzzy set $A \in MF(U)$, the $\mathcal{T}$-upper and $\mathcal{I}$-lower multi-fuzzy rough approximation of A, denoted as $\overline{R}^{\mathcal{T}}(A)$ and $\underline{R}_{\mathcal{I}}(A)$ respectively, with respect to the approximation space (U, R) are multi-fuzzy sets of U whose membership functions are defined respectively*

$$\mu^i_{\overline{R}^{\mathcal{T}}(A)}(x) = \bigvee_{y \in U} \mathcal{T}(\mu_R^i(x, y), \mu_A^i(y)) \quad \forall\ x \in U, \forall\ i \in \mathbb{N} \quad (2.1)$$

$$\mu^i_{\underline{R}_{\mathcal{I}}(A)}(x) = \bigwedge_{y \in U} \mathcal{I}(\mu_R^i(x, y), \mu_A^i(y)) \quad \forall\ x \in U, \forall\ i \in \mathbb{N} \quad (2.2)$$

The operators $\overline{R}^{\mathcal{T}}$ and $\underline{R}_{\mathcal{I}}$ on $MF(U)$ are referred to as $\mathcal{T}$-upper and $\mathcal{I}$ -lower multi-fuzzy rough approximation operators of (U, R) respectively and the pair $(\overline{R}^{\mathcal{T}}(A), \underline{R}_{\mathcal{I}}(A))$ is called the $(\mathcal{I}, \mathcal{T})$ multi-fuzzy rough set of A.

Remark 2.10 *Given a t-norm $\mathcal{T}$, an implicator $\mathcal{I}$, and two multi-fuzzy sets A and B, we can define the corresponding multi-fuzzy sets as*

$\mu^i_{(A \cap_{\mathcal{T}} B)}(x) = \mathcal{T}(\mu^i_A(x), \mu^i_B(x))$
$\mu^i_{(A \Rightarrow_{\mathcal{J}} B)}(x) = \mathcal{J}\,(\mu^i_A(x), \mu^i_B(x))$

Properties of $\mathcal{J}$-lower multi-fuzzy rough approximation operators

In this section, a continuous fuzzy implicator $\mathcal{J}$ on [0,1] is considered for the study. The following theorem gives some basic properties of $\mathcal{J}$ -lower multi-fuzzy rough approximation operators.

Theorem 2.11 *(Varma and John (a))* Let (U, R) be a multi-fuzzy approximation space. Then the $\mathcal{J}$ -lower multi-fuzzy rough approximation operator $\underline{R_{\mathcal{J}}}$ has the following properties:

For all $A, B \in MF(U)$, $A_j \in MF(U)\ where\ j \in J\ \ (J\ is\ any\ index\ set)$, $M \subseteq U,\ (x, y) \in U \times U$ and all $\alpha \in [0,1], \forall\ i\ \in \mathbb{N}$

(MFL1) $\underline{R_{\mathcal{J}}}\,(\hat{\alpha} \Rightarrow_{\mathcal{J}} A) = \hat{\alpha} \Rightarrow_{\mathcal{J}} \underline{R_{\mathcal{J}}}\,(A)$ provided that $\mathcal{J}$ is an EP implicator

(MFL2) $\underline{R_{\mathcal{J}}}\,(\bigcap_{j \in J} A_j) = \bigcap_{j \in J} \underline{R_{\mathcal{J}}}\,(A_j)$

(MFL3) $\underline{R_{\mathcal{J}}}\,(\hat{\alpha}) \supseteq \hat{\alpha}$ provided that $\mathcal{J}$ is an NP implicator.

(MFL4) $\underline{R_{\mathcal{J}}}\,(W) = U$

(MFL5) $\underline{R_{\mathcal{J}}}\,(\hat{\alpha} \Rightarrow_{\mathcal{J}} \phi) = \hat{\alpha} \Rightarrow_{\mathcal{J}} \phi \Leftrightarrow \underline{R_{\mathcal{J}}}\,(\phi) = \phi$ provided that $\mathcal{J}$ is an EP implicator

(MFL6) $\underline{R_{\mathcal{J}}}\,(\bigcup_{j \in J} A_j) \supseteq \bigcup_{j \in J} \underline{R_{\mathcal{J}}}\,(A_j)$

(MFL7) $A \subseteq B \Rightarrow \underline{R_{\mathcal{J}}}\,(A) \subseteq \underline{R_{\mathcal{J}}}\,(B)$

Theorem 2.12 *(Varma and John (a))* Let (U, R) be a multi-fuzzy approximation space and $\mathcal{J}$ be a NP and OP implicator. Then R is reflexive if and only if (MFLR) holds:

$(MFLR)\ \underline{R_{\mathcal{J}}}\,(A) \subseteq A \quad \forall\ A \in MF(U)$

Theorem 2.13 *(Varma and John (a))* Let (U, R) be a multi-fuzzy approximation space and let $\mathcal{J}$ be an implicator satisfying the law of importation. Then for the $\mathcal{J} -$ lower multi-fuzzy approximation operator $\underline{R_{\mathcal{J}}}$ of (U, R),

R is $\mathcal{T} -$ transitive $\Rightarrow (MFLT)\quad \underline{R_{\mathcal{J}}}\,(A) \subseteq \underline{R_{\mathcal{J}}}\,(\underline{R_{\mathcal{J}}}\,(A)) \quad \forall\ A \in MF(U)$

Conversely, if $\mathcal{J}$ is an OP and NP implicator, then

$(MFLT) \Rightarrow R$ is $\mathcal{T} -$ transitive.

III. Multi-Fuzzy Topology Generated By Multi-Fuzzy Relations

Throughout this section, $\mathcal{J}$ is a continuous fuzzy implicator that satisfies the law of importation and R is a reflexive and transitive multi-fuzzy relation.

Lemma 3.1 For all $A_j \in MF(U), j \in J\ \ (J\ is\ an\ index\ set)$,

$\underline{R_{\mathcal{J}}}\,(\bigcup_{j \in I} \underline{R_{\mathcal{J}}}\,(A_j)) = \bigcup_{j \in I} \underline{R_{\mathcal{J}}}\,(A_j)$ (3.1)

Proof *Since* $\underline{R_{\mathcal{J}}}\,(\bigcup_{j \in I} \underline{R_{\mathcal{J}}}\,(A_j)) \in MFS(X)$ *and* R *is reflexive,*

$\underline{R_{\mathcal{J}}}\,(\bigcup_{j \in I} \underline{R_{\mathcal{J}}}\,(A_j)) \subseteq \bigcup_{j \in I} \underline{R_{\mathcal{J}}}\,(A_j)$ (3.2)

For each $j \in J,\ \underline{R_{\mathcal{J}}}\,(A_j) \subseteq \bigcup_{j \in I} \underline{R_{\mathcal{J}}}\,(A_j)$

$$\underline{R_{\mathcal{J}}}\,(\underline{R_{\mathcal{J}}}\,(A_j)) \subseteq \underline{R_{\mathcal{J}}}\,(\bigcup_{j \in I} \underline{R_{\mathcal{J}}}\,(A_j))$$
$$\underline{R_{\mathcal{J}}}\,(A_j) \subseteq \underline{R_{\mathcal{J}}}\,(\bigcup_{j \in I} \underline{R_{\mathcal{J}}}\,(A_j))$$
$$\bigcup_{j \in I} \underline{R_{\mathcal{J}}}\,(A_j) \subseteq \underline{R_{\mathcal{J}}}\,(\bigcup_{j \in I} \underline{R_{\mathcal{J}}}\,(A_j))$$

Theorem 3.2 The set of all $\mathcal{J}$ -lower approximations of multi-fuzzy sets on U forms a multi-fuzzy topology. i.e,

$T_{MFR} = \{\underline{R_{\mathcal{J}}}\,(A) / A \in MFS(U)\} is\ a\ multi - fuzzy\ topology\ on\ U.$

Proof

• To prove $\phi, U \in T_{MFR}$.

Since $\underline{R_{\mathcal{J}}}\,(U) = 1_U = U$ by (Varma and John, 2014), $U \in T_{MFR}$.

Also, since R is reflexive, $\underline{R_{\mathcal{J}}}\,(\phi) \subseteq \phi$ and hence $\phi \in T_{MFR}$.

• If $A, B \in T_{MFR}$, there exists $A_1, B_1 \in MFS(U)\ such\ that A = \underline{R_{\mathcal{J}}}\,(A_1), B = \underline{R_{\mathcal{J}}}\,(B_1)$.

$$A \cap B = \underline{R}(A_1) \cap \underline{R}(B_1) = \underline{R}(A_1 \cap B_1).$$

Thus $A \cap B \in T_{MFR}$ for $A, B \in T_{MFR}$.
• For all $A_j \in T_{MFR}$,

$$\bigcup_{j\in I} A_j = \bigcup_{j\in I} \underline{R_J}(B_j)$$
$$= \underline{R_J}\left(\bigcup_{j\in I} \underline{R_J}(B_j)\right) \quad by\ lemma\ 3.1$$
$$\Rightarrow \bigcup_{j\in I} A_j \in T_{MFR}$$

Thus the approximation space (U, R) generates the topology T_{MFR}

Remark 3.3 *By (MFL3), it is seen that* T_{MFR} *is a strong-multi-fuzzy topology.*

IV. Conclusion

The paper studied the topological structure of generalized multi-fuzzy rough sets induced by special multi-fuzzy relations. The further study aims on the relationship between the topological operators and the approximation operators.

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