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Matrix-Norm Aggregation Operators

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Abstract: *Multiple set theory is a new mathematical approach to handle vagueness together with multiplicity. Multiple set is expert in formalising numerous properties of objects multifariously. This paper proposes a generalization for t-norms and t-conorms on multiple sets. Recently, a study on the concept of uninorm aggregation operators on bounded lattice has been done. This paper discusses uninorm aggregation operators on the bounded lattice* M*, called matrix-norm aggregation operators, where* $\mathrm{M} = \mathrm{M}_{n\times k}([0,1])$ *denotes the collection of all* $n \times k$ *matrices with entries from* $[0,1]$*. The paper also introduces matrix-norm aggregation operator induced by uninorm aggregation operators on* $[0,1]$*. Finally, the paper investigates some properties of matrix-norm aggregation operators.*
Keywords: *aggregation operator, multiple set, t-norm, t-conorm, uninorm.*

I. Introduction

Uncertainty, vagueness, inexactness, etc. are inevitable part of our daily life. Many theories have been introduced to represent these concepts mathematically, which includes fuzzy set [1], vague set [2], intuitionistic fuzzy set [3], multi fuzzy set [4], fuzzy multi set [5] and many more. Recently, Shijina, John and Thomas introduced multiple sets [6-7] to handle vagueness and multiplicity together. Multiple set is expert in formalising numerous properties of objects multifariously. Multiple set assigns a membership matrix for each object in the universal set and each raw in membership matrix characterizes its various properties where each entries in the raw corresponds object's multiplicity.

Triangular norms and conorms (*t*-norms and *t*-conorms) [8-10] are important notions in fuzzy set theory and are widely used in many contexts. Also various other set-based *t*-norms and *t*-conorms are developed, such as, on intuitionistic fuzzy set [11], multi fuzzy set [12] and has found many applications in various fields. As a continuation , Shijina and John introduced *t*-norms and *t*-conorms on multiple sets, called multiple *t*-norms and *t*-conorms [13-14] and it is proven that they generalizes the standard operations of union and intersection on multiple sets.

In 1996, Yager and Rybalov [15] introduced the concept of uninorm aggregation operators (simply, uninorms) as a generalization of fuzzy *t*-norms and *t*-conorms. Later, an exhaustive study of uninorm operators is established through many papers [16-21]. Uninorms have proven to be useful in many fields like fuzzy logic, expert systems, neural networks, aggregation and fuzzy systems [22-23]. In 2015, Karacal and Mesiar [24], generalized the concepts of uninorms by defining them on bounded lattices instead of $[0,1]$ and in 2017, Karacal, Ertugrul and Mesiar [25] proposed a characterization of uninorms on a general bounded lattice by means of sextuple of a *t*-norms, *t*-conorms and four aggregation function on the bounded lattice.

This paper provides a study on matrix-norm aggregation operators-uninorms on the bounded lattice $\mathrm{M} = \mathrm{M}_{n\times k}([0,1])$, the set of all $n \times k$ matrices with entries from $[0,1]$. The paper investigates that matrix-norms generalizes multiple t-norms and t-conorms. The paper also discusses various properties of matrix-norms.

II. Basic Concepts

This section describes some basic definitions and properties related to multiple sets and uninorms.

2.1 Multiple sets

In 2015, Shijina, John and Thomas, introduced the concept of multiple set to handle vagueness and multiplicity of objects simulaneously. The membership matrix assigned to each objects helps the multiple set to formalize numerous properties of objects at a time. Each raw in membership matrix characterizes its various properties where each entries in the raw corresponds object's multiplicity.

Definition 2.1.1 [7] Let X be a non-empty crisp set called the universal set. A *multiple set* A of order (n,k) over X is an object of the form $\{(x,A(x)); x \in X\}$, where for each $x \in X$ its membership value is an $n \times k$ matrix

$$A(x) = \begin{bmatrix} A_1^1(x) & A_1^2(x) & \dots & A_1^k(x) \\ A_2^1(x) & A_2^2(x) & \dots & A_2^k(x) \\ \dots & \dots & \dots & \dots \\ A_n^1(x) & A_n^2(x) & \dots & A_n^k(x) \end{bmatrix}$$

where $A_1, A_2, ..., A_n$ are fuzzy membership functions and for each $i = 1, 2, ..., n$, $A_i^1(x), A_i^2(x), ..., A_i^k(x)$ are membership values of the fuzzy membership function A_i for the element $x \in X$, written in decreasing order. The matrix $A(x)$ is called the membership matrix. The universal set X can be viewed as a multiple set of order (n,k) over X for which the membership matrix for each $x \in X$ is an $n \times k$ matrix with all entries one. Also, the empty set Φ can be viewed as a multiple set of order (n,k) over X for which the membership matrix for each $x \in X$ is an $n \times k$ matrix with all entries zero. The set of all multiple sets of order (n,k) over X is denoted by $MS_{(n,k)}(X)$. Let $\mathrm{M} = \mathrm{M}_{n\times k}([0,1])$ be the set of all $n \times k$ matrices with entries from $[0,1]$. It is noticed that a multiple set A of order (n,k) over X can be viewed as a function $A: X \to \mathrm{M}$ which maps each $x \in X$ to its (n,k) membership matrix $A(x)$. From its structure, it is clear that multiple set generalizes the concepts of fuzzy sets [1], multi fuzzy sets [4], fuzzy multisets [5] and multisets [26].
The notions of subset and equality between multiple sets is defined as follows;

Definition 2.1.2 [7] Given two multiple sets A and B in $MS_{(n,k)}(X)$, then A is a subset of B, denoted as $A \subseteq B$, if and only if $A_i^j(x) \le B_i^j(x)$ for every $x \in X, i = 1,2,...,n$ and $j = 1,2,...,k$. Also, A is equal to B, denoted as $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$, that is, if and only if $A_i^j(x) = B_i^j(x)$ for every $x \in X, i = 1,2,...,n$ and $j = 1,2,...,k$.
Given two multiple sets A and B in $MS_{(n,k)}(X)$, standard multiple set operations are defined as follows;

Definition 2.1.3 [7] The *union* of A and B, denoted as $A \cup B$, is a multiple set whose membership matrix for each $x \in X$ is given by $(A \cup B)(x) = \left[(A \cup B)_i^j(x)\right]$, where $(A \cup B)_i^j(x) = max\left\{A_i^j(x), B_i^j(x)\right\}$ for every $i = 1,2,...,n$ and $j = 1,2,...,k$.

Definition 2.1.4 [7] The *intersection* of A and B, denoted as $A \cap B$, is a multiple set whose membership matrix for each $x \in X$ is given by $(A \cap B)(x) = \left[(A \cap B)_i^j(x)\right]$, where $(A \cap B)_i^j(x) = min\left\{A_i^j(x), B_i^j(x)\right\}$ for every $i = 1,2,...,n$ and $j = 1,2,...,k$.

Definition 2.1.5 [7] The *complement* of A, denoted as $\overline{A}$ is a multiple set whose membership matrix for each $x \in X$ is given by $(\overline{A})(x) = \left[(\overline{A})_i^j(x)\right]$ where $(\overline{A})_i^j(x) = 1 - A_i^{k-j+1}(x)$ for every $i = 1,2,...,n$ and $j = 1,2,...,k$.

There exist a broad class of functions whose members qualify as generalizations of standard operations of intersection and union on multiple sets. Each of these classes is characterized by properly defined axioms and they are called multiple *t*-norm and *t*-conorm, respectively.

Definition 2.1.6 [14] A multiple *t*-norm T is a binary operation on M that satisfies the following axioms, for all $A, B, D \in \mathrm{M}$;

(T1) Monotonicity: $T(A,B) \le T(A,D)$, whenever $B \le D$.

(T2) Commutativity: $T(A,B) = T(B,A)$

(T3) Associativity: $T(A,T(B,D)) = T(T(A,B),D)$

(T4) Boundary condition: $T(A,\mathbf{1}) = A$

where $\mathbf{1}$ denotes the $n \times k$ matrix with all entries 1.
A special type of multiple *t*-norms are introduced with the aid of fuzzy *t*-norms as follows;

Definition 2.1.7 [14] Let t be fuzzy *t*-norm. Define the binary operation T on M as follows: $A = [a_{ij}]$ and $B = [b_{ij}]$ in M are mapped to $C = [c_{ij}]$ in M by $c_{ij} = t(a_{ij}, b_{ij})$ for all $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$. Then T is a multiple *t*-norm on M , called the multiple *t*-norm induced by fuzzy *t*-norm t .
Some examples for multiple *t*-norms induced by fuzzy *t*-norms are given as follows;

(1) Standard multiple intersection: Multiple *t*-norm MIN induced by fuzzy *t*-norm $t(a,b) = min(a,b)$.

(2) Algebraic product: Multiple *t*-norm AP induced by fuzzy *t*-norm $t(a,b) = ab$.

(3) Drastic intersection: Multiple *t*-norm DI induced by fuzzy *t*-norm $t(a,b) = \begin{cases} a & \text{when } b = 1 \\ b & \text{when } a = 1 \\ 0 & \text{otherwise} \end{cases}$

Definition 2.1.8 [14] A multiple *t*-conorm S is a binary operation on M that satisfies the following axioms, for all $A, B, D \in \mathrm{M}$;

(S1) Monotonicity: $S(A,B) \le S(A,D)$, whenever $B \le D$.

(S2) Commutativity: $S(A,B) = S(B,A)$

(S3) Associativity: $S(A,S(B,D)) = S(S(A,B),D)$

(S4) Boundary condition: $S(A,\mathbf{0}) = A$

where $\mathbf{0}$ denotes the $n \times k$ matrix with all entries 0.
A special type of multiple *t*-conorms are introduced with the aid of fuzzy *t*-conorms as follows;

Definition 2.1.7 [14] Let s be fuzzy *t*-conorm. Define the binary operation S on M as follows: $A = [a_{ij}]$ and $B = [b_{ij}]$ in M are mapped to $C = [c_{ij}]$ in M by $c_{ij} = s(a_{ij}, b_{ij})$ for all $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$. Then S is a multiple *t*-conorm on M , called the multiple *t*-conorm induced by fuzzy *t*-conorm s .
Some examples for multiple *t*-conorms induced by fuzzy *t*-conorms are given as follows;

(1) Standard multiple union: Multiple *t*-conorm MAX induced by fuzzy *t*-conorm $s(a,b) = max(a,b)$.

(2) Algebraic sum: Multiple *t*-conorm AS induced by fuzzy *t*-conorm $s(a,b) = a + b - ab$.

(3) Drastic union: Multiple *t*-conorm DU induced by fuzzy *t*-conorm $s(a,b) = \begin{cases} a & \text{when } b = 0 \\ b & \text{when } a = 0 \\ 1 & \text{otherwise} \end{cases}$

2.2 Uninorm Aggregation Operator

In 1996, Yager and Rybalov introduced uninorm aggregation operators as a generalization of the fuzzy *t*-norm and *t*-conorm. Uninorms allow for an identity element lying anywhere in the unit interval rather than at one or zero as in the case of *t*-norms and *t*-conorms.

Definition 2.2.1 [15] A uninorm is a mapping $u : [0,1] \times [0,1] \to [0,1]$ having the following properties;

(u1) Commutativity: $u(a,b) = u(b,a)$

(u2) Monotonicity: $u(a,b) \ge u(c,d)$ if $a \ge c$ and $b \ge d$

(u3) Associativity: $u(a,u(b,c)) = u(u(a,b),c)$

(u4) There exist some elements $e \in [0,1]$ called the identity element or neutral element such that for all $a \in [0,1]$, $u(a,e) = a$.

Following are some examples for uninorms [19];

(1) $u_1(a,b) = min(a,b)$ with the neutral element 1.

(2) $u_2(a,b) = max(a,b)$ with the neutral element 0.

(3) $u_3(a,b) = \frac{ab}{e}$ with the neutral element $e \in [0,1]$.

(4) $u_4(a,b) = \frac{a+b-ab-e}{1-e}$ with the neutral element $e \in [0,1]$.

(5) $u_5(a,b) = max\{0, a+b-e\}$ with the neutral element $e \in [0,1]$.

(6) $u_6(a,b) = min\{1, a+b-e\}$ with the neutral element $e \in [0,1]$.

(7) $u_7(a,b) = \frac{ab}{\bar{a}\bar{b} + ab}$, where $\bar{a} = 1-a$, with the neutral element $e = 0.5$.

In 2015, Kamacal and Mesiar generalized the concepts of uninorms by defining them on bounded lattice instead of $[0,1]$.

Definition 2.2.2 [24] Let $\langle L, \le, 0, 1 \rangle$ be a bounded lattice. An operation $u : L^2 \to L$ is called a uninorm on L if it is an associative symmetric aggregation function which has a neutral element $e \in L$.

III. Matrix-Norm Aggregation Operators

Let $\mathrm{M} = \mathrm{M}_{n \times k}([0,1])$ be the set of all $n \times k$ matrices with entries from $[0,1]$. Define the binary relation $\le$ on M as, for $A = [a_{ij}]$ and $B = [b_{ij}]$ in M, $A \le B$ if and only if $a_{ij} \le b_{ij}$ for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$. Let $\mathbf{0}$ denotes an $n \times k$ matrix with all entries zero and $\mathbf{1}$ denotes an $n \times k$ matrix with all entries one. Then $\langle \mathrm{M}, \le, \mathbf{0}, \mathbf{1} \rangle$ is a bounded lattice with the least element $\mathbf{0}$ and the greatest element $\mathbf{1}$.

Definition 3.1 Consider the bounded lattice $\langle \mathrm{M}, \le, \mathbf{0}, \mathbf{1} \rangle$. The binary operation $U : \mathrm{M}^2 \to \mathrm{M}$ is called a matrix-norm aggrtegation operators(simply, matrix-norm) on M if it satisfies the following axioms; for all $A, B, C, D \in \mathrm{M}$,

(U1) Commutativity: $U(A,B) = U(B,A)$

(U2) Monotonicity: $U(A,B) \ge U(C,D)$ if $A \ge C$ and $B \ge D$

(U3) Associativity: $U(A, U(B,C)) = U(U(A,B), C)$

(U4) There exist some elements $E \in \mathrm{M}$ called the neutral matrix such that for all $A \in \mathrm{M}$, $U(A,E) = A$.

Note that matrix-norms aggregation operators are uninorm aggregation operators on the bounded lattice $\langle \mathrm{M}, \le, \mathbf{0}, \mathbf{1} \rangle$. An element $A \in \mathrm{M}$ is said to be idempotent if $U(A,A) = A$ and a matrix norm U is called idempotent if $U(A,A) = A$ for all $A \in \mathrm{M}$.

Theorem 3.1 For any matrix-norm U, its neutral matrix is unique.

Proof. Suppose that E_1 and E_2 are two neutral matrices of the matrix-norm U. Then, from the property of neutral matrix, $U(E_1, E_2) = E_1$ and $U(E_2, E_1) = E_2$. Then, from commutativity, $E_1 = E_2$ and hence neutral matrix is unique.

Theorem 3.2 Neutral matrix E of a matrix-norm U is idempotent.

Proof. From the property of neutral matrix, $U(E,E) = E$ and hence neutral matrix is idempotent.

Theorem 3.3 Assume that U is a matrix-norm with neutral matrix E. Then
(a) For any $A \in \mathrm{M}$ and all $B \geq E$, we get $U(A, B) \geq A$.
(b) For any $A \in \mathrm{M}$ and all $B \leq E$, we get $U(A, B) \leq A$.
Proof. (a) From the property of neutral matrix $U(A, E) = A$. If $B \geq E$ then, from monotonicity, $U(A, B) \geq U(A, E)$. Therefore $U(A, B) \geq A$.
(b) From the property of neutral matrix $U(A, E) = A$. If $B \leq E$ then from monotonicity, $U(A, B) \leq U(A, E)$. Therefore $U(A, B) \leq A$.

Theorem 3.4 Assume that U is a matrix-norm with neutral matrix E. Then
(a) $U(A, \mathbf{0}) = \mathbf{0}$ for all $A \leq E$
(b) $U(A, \mathbf{1}) = \mathbf{1}$ for all $A \geq E$
Proof. (a) For $A \leq E$, $U(A, \mathbf{0}) \leq U(E, \mathbf{0}) = \mathbf{0}$. Therefore $U(A, \mathbf{0}) = \mathbf{0}$.
(b) For $A \geq E$, $U(A, \mathbf{1}) \geq U(E, \mathbf{1}) = \mathbf{1}$. Therefore $U(A, \mathbf{1}) = \mathbf{1}$.
The following two corollaries are easy to prove.
Corrollary 3.5 If $E = \mathbf{0}$, $U(A, \mathbf{1}) = \mathbf{1}$ for all $A \in \mathrm{M}$. This is the case of multiple *t*-conorm.
Corrollary 3.6 If $E = \mathbf{1}$, $U(A, \mathbf{0}) = \mathbf{0}$ for all $A \in \mathrm{M}$. This is the case of multiple *t*-norm.

Theorem 3.7 If U is a matrix-norm then $U(\mathbf{0}, \mathbf{0}) = \mathbf{0}$ and $U(\mathbf{1}, \mathbf{1}) = \mathbf{1}$.
Proof. Since $\mathbf{0} \leq E$, from monotonicity, $U(\mathbf{0}, \mathbf{0}) \leq U(E, \mathbf{0})$. From the property of neutral matrix, $U(E, \mathbf{0}) = \mathbf{0}$ and therefore $U(\mathbf{0}, \mathbf{0}) \leq \mathbf{0}$. Since $U(\mathbf{0}, \mathbf{0}) \geq \mathbf{0}$, we have $U(\mathbf{0}, \mathbf{0}) = \mathbf{0}$. Similarly, since $E \leq \mathbf{1}$, from monotonicity, $U(\mathbf{1}, \mathbf{1}) \geq U(E, \mathbf{1})$. From the property of neutral matrix $U(E, \mathbf{1}) = \mathbf{1}$ and therefore $U(\mathbf{1}, \mathbf{1}) \geq \mathbf{1}$. Since $U(\mathbf{1}, \mathbf{1}) \leq \mathbf{1}$, we have $U(\mathbf{1}, \mathbf{1}) = \mathbf{1}$.

Definition 3.2 Let u be uninorm aggregation operator. Define the binary operation U on M as follows; $A = [a_{ij}]$ and $B = [b_{ij}]$ in M are mapped to $C = [c_{ij}]$ in M by $c_{ij} = u(a_{ij}, b_{ij})$ for every $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$. Then U is a matrix norm on M, called the matrix norm induced by uninorm u.

Matrix-norm induced by the uninorm $u_1(a, b) = min(a, b)$ is the multiple *t*-norm with neutral matrix $\mathbf{1}$ and matrix-norm induced by the uninorm $u_2(a, b) = max(a, b)$ is the multiple *t*-conorm with neutral matrix $\mathbf{0}$.

Definition 3.3 Consider the matrix-norms U_1 and U_2. Then U_1 is distributive over U_2 if it satisfies $U_1(A, U_2(B, C)) = U_2(U_1(A, B), U_1(A, C))$ for all $A, B, C \in \mathrm{M}$.

Theorem 3.8 Let U_1 and U_2 be two matrix-norms with neutral matrices E_1 and E_2, respectively. Suppose that U_1 is distributive over U_2. Then
(a) $U_1(E_2, E_2) = E_2$
(b) If $U_2(E_1, E_1) = E_1$, then U_2 is idempotent.
(c) If $E_2 \leq E_1$ and $U_1(\mathbf{0}, \mathbf{1}) = \mathbf{1}$, then $U_2(\mathbf{0}, \mathbf{1}) = \mathbf{1}$.
(d) If $E_1 \leq E_2$ and $U_1(\mathbf{0}, \mathbf{1}) = \mathbf{0}$, then $U_2(\mathbf{0}, \mathbf{1}) = \mathbf{0}$.
Proof. (a) Taking $A = C = E_2$ and $B = E_1$ in equation (1), we get
$U_2(U_1(E_2, E_1), U_1(E_2, E_2)) = U_1(E_2, U_2(E_1, E_2))$, which gives $U_2(E_2, U_1(E_2, E_2)) = U_1(E_2, E_1)$. Therefore $U_1(E_2, E_2) = E_2$.

(b) Let $A \in \mathrm{M}$. Taking $B = C = E_1$ in equation (1), we get $U_2(U_1(A, E_1), U_1(A, E_1)) = U_1(A, U_2(E_1, E_1))$, which gives $U_2(A, A) = U_1(A, E_1)$. Therefore $U_2(A, A) = A$ and hence U_2 is idempotent.

(c) Taking $A = \mathbf{0}, B = E_1, C = \mathbf{1}$ in equation (1), we get $U_2(U_1(\mathbf{0}, E_1), U_1(\mathbf{0}, \mathbf{1})) = U_1(\mathbf{0}, U_2(E_1, \mathbf{1}))$. Since $E_2 \le E_1$, we have $U_2(E_2, \mathbf{1}) \le U_2(E_1, \mathbf{1})$. Therefore

$U_2(U_1(\mathbf{0}, E_1), U_1(\mathbf{0}, \mathbf{1})) = U_1(\mathbf{0}, U_2(E_1, \mathbf{1})) \ge U_1(\mathbf{0}, U_2(E_2, \mathbf{1})) = U_1(\mathbf{0}, \mathbf{1}) = \mathbf{1}$, which gives $U_2(\mathbf{0}, \mathbf{1}) \ge \mathbf{1}$.

Therefore $U_2(\mathbf{0}, \mathbf{1}) = \mathbf{1}$.

(c) Taking $A = \mathbf{0}, B = E_1, C = \mathbf{1}$ in equation (1), we get $U_2(U_1(\mathbf{0}, E_1), U_1(\mathbf{0}, \mathbf{1})) = U_1(\mathbf{0}, U_2(E_1, \mathbf{1}))$. Since $E_1 \le E_2$, we have $U_2(E_1, \mathbf{1}) \le U_2(E_2, \mathbf{1})$. Therefore

$U_2(U_1(\mathbf{0}, E_1), U_1(\mathbf{0}, \mathbf{1})) = U_1(\mathbf{0}, U_2(E_1, \mathbf{1})) \le U_1(\mathbf{0}, U_2(E_2, \mathbf{1})) = U_1(\mathbf{0}, \mathbf{1}) = \mathbf{0}$, which gives $U_2(\mathbf{0}, \mathbf{1}) \le \mathbf{0}$.

Therefore $U_2(\mathbf{0}, \mathbf{1}) = \mathbf{0}$.

IV. Conclusion

Uninorms are important generalizations of *t*-norms and *t*-conorms, having a neutral element lying anywhere in the unit interval. It has found many applicatrions in various fields. After uninorms on unit interval, uninorms on bounded lattices have recently become a challenging study object. This paper extended this study of uninorms on the bounded lattice M and discussed some of its properties. Uninorms on the bounded lattice M are called matrix-norm aggregation operators and they are the generalizations of multiple *t*-norms and *t*-conorms. Also, matrix-norms induced by uninorms are developed and their properties are investigated.

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