IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X.
PP 11-17
www.iosrjournals.org

A New Interval Convexity In Weighted Graphs

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Abstract: *Let G : (V, E, ω) be a finite, connected, weighted graph without loops and multiple edges. In a weighted graph each arc is assigned a weight by the weight function* $\omega: E \rightarrow \Re^{+}$. *A u−v path P in G is called a weighted u−v geodesic if the weighted distance between u and v is calculated along P. The strength of a path is the minimum weight of its arcs, and length of a path is the number of edges in the path. In this paper, we introduce the concept of weighted geodesic convexity in weighted graphs. A subset W of V (G) is called weighted geodetic convex if the weighted geodetic closure of W is W itself. The concept of weighted geodetic blocks is introduced and discussed some of their properties. The notion of weighted geodetic boundary and interior points are included.*
Keywords - *intervals in graph, geodesic, graph convexity,*

I. INTRODUCTION

Weighted graph theory has numerous applications in various fields like clustering analysis, operations research, database theory, network analysis, information theory, etc. During the last few decades, the fast development of a number of discrete and combinatorial mathematical structures has lead to the study of generalizations of a number of classical concepts from continuous mathematics. Among them, the concept of convex set of a metric space plays a key role. The reason is all connected graphs can be seen as metric spaces just by considering their shortest paths. This fact has lead to the study of the behaviour of these structures as convexity spaces. An ordinary graph is a weighted graph with unit weight assigned for all arcs. That means an ordinary graph is particular weighted graph. All the practical problematic situations which can be solved by using any graph theory technique can only be modelled by a weighted graph. This fact is the main motivation for this work. In this paper, we define the weighted geodesic convexity in weighted graphs.

Any subset of the vertex set of a weighted graph is convex with respect to a distance taken, means the induced weighted subgraph by that convex subset is alone sufficient to identify all the graph parameters of the subgraph. This fact is the relevance of the study in applied level.

Let G: (V, E) be a crisp ordinary graph. Then the distance between two nodes u and v of G is defined as the length of a u − v geodesic [1-3]. A shortest u – v path is called a u−v geodesic [1- 3]. The length of a path is the number of edges present in the path [1-3]. Strength of path $P = v_0e_1v_1e_1v_2...e_n v_n$, is denoted and defined by $S(P) = \min\{w(e_1), w(e_2),...,w(e_n)\}$[4-5]. The length of the path P is the number of arcs present in P. The weighted distance between two nodes u and v in G is defined and denoted by $d_\omega(u, v) = \min_{P} \{l(P) * S(P)/P$ is a u − v path, l(P) is the strength and S(P) is the strength of P} [6-7]. A u−v path P is called a weighted u − v geodesic if $d_\omega(u, v) = l(P) * S(P)$ [6-7]. Several authors have made remarkable contributions to weighted graph theory. They include Paul Erdos, Bondy and Fan [1-2], Broersma, Zhang and Li [8], Mathew and Sunitha [4-5], Sampathkumar[9], Soltan[10].

II. WEIGHTED GEODETIC CONVEX SET

For unweighted graphs, different types of convexities and other related parameters are introduced and studied by many authors in the literature. In this section, we define the concepts of weighted geodetic convex sets and discuss some of their elementary properties.

Definition 2.1: Let G: (V, E, ω) be a connected weighted graph without loops and multiple edges. Let u, v be any two vertices of G. A u − v path P is called a weighted u – v geodesic if $d_\omega(u, v) = S(P) * l(P)$. This means a u − v path P is called a weighted u – v geodesic if the weighted distance between u and v is calculated along the path P.

Definition 2.2: For any two vertices u and v of G, the weighted geodetic closed interval $I_\omega[u, v]$ is the set of all vertices in all weighted u − v geodesics including u and v.

Definition 2.3: Let G: (V, E, ω) be a connected weighted graph without loops and multiple edges and let S $\subset$ V. The union of all geodetic closed intervals $I_\omega[u, v]$ over all pairs u, v ϵ S is called the weighted geodetic closure of S. It is denoted by $I_\omega[S]$.

Definition 2.4: Let G: (V, E, ω) be a connected weighted graph without loops and multiple edges. Any subset S of V (G) is called weighted geodetic convex if $I_\omega[S]$ = S.

In the following example [Example 2.5], we illustrate all the above definitions.

Example 2.5

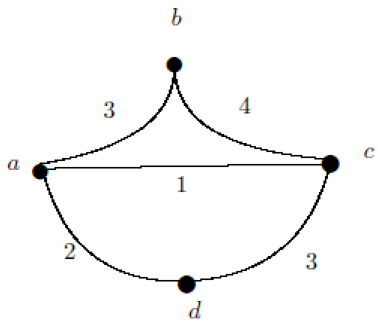

Figure 1: Weighted geodetic convex sets

The weighted distance matrix for the above graph [Figure1] is given below.

$$\begin{pmatrix} 0 & 2 & 1 & 2 \\ 2 & 0 & 2 & 3 \\ 1 & 2 & 0 & 2 \\ 2 & 3 & 2 & 0 \end{pmatrix}$$

In the above weighted graph [Figure 1], the path a−c−b is a weighted a−b geodesic. Note that it is unique. The direct edge (a, c) is a weighted a−c geodesic. It is also unique. Again $I_\omega[a, d]$ = {a, c, d}, $I_\omega[a, c]$ = {a, c}, $I_\omega[a, b]$ = {a, c, b}. If S = {a, b}, $I_\omega[S]$ = {a, b, c} 6≠ S. Thus S is not weighted geodetic convex. But if S = {a, c}, then $I_\omega[S]$ = S, which proves S is weighted geodetic convex.

The following proposition [Proposition 2.5] is obvious. The proof is omitted.

Proposition 2.6: Let G: (V, E, ω) be a connected weighted graph. Then the empty set Φ the full vertex set V and all singletons V (G) are weighted geodetic convex.

Remark 2.7: By a nontrivial weighted geodetic convex set, we mean a weighted geodetic convex set S with 2 ≤ | S |< | V (G) |.

In the next theorem [Theorem 2.8], we show that the arbitrary intersection of weighted geodetic convex sets is again weighted geodetic convex.

Theorem 2.8: The intersection of two weighted geodetic convex sets is again weighted geodetic convex.

Proof. Let G: (V, E, ω) be a connected weighted graph. Let S and T be any two weighted geodetic convex sets of V (G). We have to prove that S ∩ T is weighted geodetic convex. Let u and v be any two vertices of S∩T. This means u and v are two vertices in both S and T, which are weighted geodetic convex. Therefore all vertices in all weighted u – v geodesics are both in S and T, and hence in S ∩T. This shows that S∩T is weighted geodetic convex.

in example 2.5 [Figure 1], it is clear that union of two weighted geodetic convex sets is not weighted geodetic convex. Consider S = {a} and T = {b}. S U T = {a, b}, which is not weighted geodetic convex.

III. WEIGHTED GEODETIC BLOCKS AND THEIR CHARACTERIZATION

In this section, we introduce the concept of weighted geodetic blocks. Some necessary conditions and a characterization is also included. By a complete weighted graph G, we mean that the underlying unweighted graph G^* of G is complete.

Definition 3.1: Let G: (V, E, ω) be a connected weighted graph and let H be a complete weighted subgraph of G. Then H is called a weighted geodetic block of G if there exists no edge e = (u, v) in H such that $\omega(e) \geq d_\omega(u, v)$.

Remark 3.2: From the definition of weighted geodetic blocks of a weighted graph G, it is clear that all edges e = (u, v) in a weighted geodetic block are the only weighted u−v geodesics in the weighted graph.

Example 3.3:

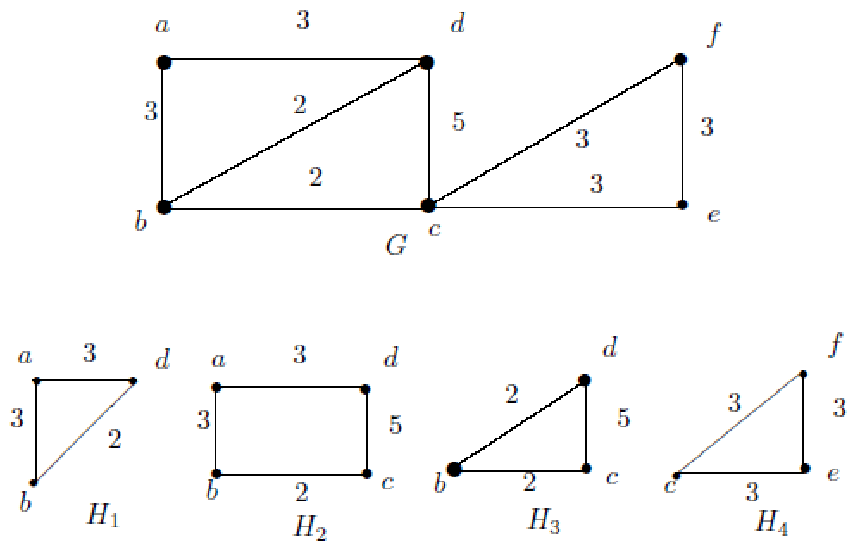

Figure 2: A weighted graph and its weighted geodetic blocks

In the above figure [Figure 2], H_1 and H_4 are weighted geodetic blocks of G, but H_2 and H_3 are not. Even though all the edges in H_2 are weighted geodesics between their end vertices, its underlying graph H_2^* is not complete. In H_3, the edge (c, d) is not a weighted c − d geodesic.

The following theorem [Theorem 3.4, is a necessary condition for a weighted geodetic block.

Theorem 3.4: Let G: (V, E, ω) be a connected weighted graph. Let H be a complete weighted subgraph of G. If H is a weighted geodetic block of G, then V (H) is a weighted geodetic convex set of G.

Proof. Let G: (V, E, ω) be a connected weighted graph and H be a complete weighted subgraph of G. Suppose that H is a weighted geodetic block of G. We have to prove that V (H) is a weighted geodetic convex set of V. Since H is a weighted geodetic block of G, there exists no edge e = (u, v) in H such that w(e) ≥ d_ω(u, v). This means every edge e = (u, v) in H is the only weighted u − v geodesic in H. Therefore the vertices in all weighted u−v geodesics are exactly u and v. This is true for any pair of vertices in V (H). Hence V (H) is a weighted geodetic convex set of V

.

This theorem [Theorem 3.4], is not sufficient. That means, if H is a complete weighted subgraph of G such that V (H) is a weighted geodetic convex subset of V , then H need not be a weighted geodetic block of G. This is explained in the following example [Example 3.5]

Example 3.5:

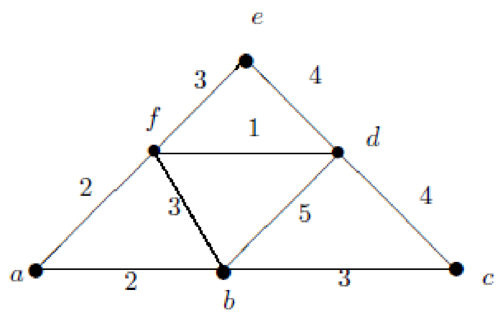

Figure 3: A weighted graph

In the above weighted graph [Figure 3], the vertex induced subgraph induced by the vertices d, e and f is not a weighted geodetic block, even though {d, e, f} is a weighted geodetic convex set of V . Note that the edges (e, f) and (d, f) are not weighted geodesics between their end points.

The sequential deletion of vertices from a weighted geodetic block results in a nested sequence of weighted geodetic blocks. This fact is explained in the following theorem [Theorem 3.6].

Theorem 3.6: Every vertex deleted subgraph of a weighted geodetic block is again a weighted geodetic block.

Proof. Let G: (V, E, ω) be a connected weighted graph and H be a weighted geodetic block of G. Let u be any vertex of H. Consider the vertex deleted subgraph H – {u}. We have to prove that H − {u} is again a weighted geodetic block of G. Let e = (x, y) be any edge of H − {u}. Clearly x ≠u ≠y, and since (x, y) is an edge of H, which is weighted geodetic convex, e is the only weighted x − y geodesic in G. This is true for any edge in H − {u}. Thus H − {u} is a weighted geodetic block of G. By the above theorem [Theorem 3.4], it is clear that, if we have a weighted geodetic block with order p, then there exists at least one weighted geodetic blocks of orders 1, 2, 3,..., (p − 1) each.

In the next theorem [Theorem 3.7], we characterize weighted geodetic blocks of a connected weighted graph G.

Theorem 3.7: Let G: (V, E, ω) be a connected weighted graph and let H be a complete weighted subgraph of G. Then H is a weighted geodetic block of G if and only if {u, v} is weighted geodetic convex for every two vertices u, v in H.

Proof. Let G: (V, E, ω) be a connected weighted graph and let H be a complete weighted subgraph of G. Suppose that H is a weighted geodetic block of G. Let u, v be any two vertices of H. We have to prove that {u, v} is weighted geodetic convex. Since H is a weighted geodetic block, it is complete. Therefore every pair of vertices of H are adjacent and end every edge is the only weighted geodesic between its end vertices. In particular, the vertices u and v are adjacent and (u, v) is the only weighted u−v geodesic. This proves that {u, v} weighted geodetic convex.

Conversely suppose that for any two vertices u, v in H, {u, v} is weighted geodetic convex. We have to prove that H is a weighted geodetic block of G. That is to prove every edge in H is the only weighted geodesic between its end vertices.

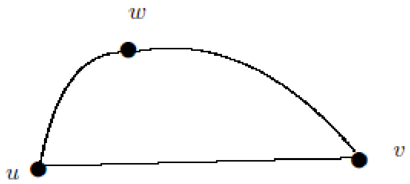

Figure 4: Uniqueness of the weighted geodesic

By the assumption and since H is complete, each edge e = (u, v) is a weighted geodesic between its end vertices. Now it remains to prove the uniqueness of the geodesic. Suppose the contrary. Let there be another weighted u−v geodesic in H, say, P. Since we consider only simple weighted graphs throughout this thesis, the path P has at least one vertex, say, w other than u and v. Since P is a weighted u−v geodesic, we have $I_{\omega}[u, v] = \{u, v, w\} \neq \{u, v\}$. This means {u, v} is not weighted geodetic convex, a contradiction. Hence our assumption is wrong. So all the edges in H are the only weighted geodesics between their end vertices. Thus H is a weighted geodetic block.

Corollary 3.8: A complete connected weighted graph G: (V,E, ω is a weighted geodetic block if and only if it has $(2^{|V|} - (|V| + 2))$ nontrivial weighted geodetic convex sets.

Proof. Let G : (V,E, ω) is a complete connected weighted graph. Suppose that G is a weighted geodetic block. Then by the above theorem [Theorem 3.7], for any two vertices u and v, {u, v} is weighted geodetic convex. If we extend this set by adding any number of vertices, the resultant set is again a weighted geodetic convex set. Thus there are $2^{|V|}$ weighted geodetic convex. Therefore the number of nontrivial weighted geodetic convex sets is $(2^{|V|} - (|V| + 2))$. Conversely if there are $(2^{|V|} - (|V| + 2))$ number of nontrivial weighted geodetic convex sets of G, all the two elemented subsets of V (G) are weighted geodetic convex. By the above theorem [Theorem 3.7], G is weighted geodetic convex.

In the following theorem [Theorem 3.9], we see that the vertex set of any weighted tree can be considered as the nested union of subsets, all of which are weighted geodetic convex.

Theorem 3.9: Let G: (V, E, ω) be a weighted tree, then there exists a sequence of sets $V = V_n \supset V_{n-1} \supset, ..., \supset V_1$, where for each i, V_i is weighted geodetic convex and $|V_i| = n_i$

Proof. Let G: (V, E, ω) be a weighted tree with n vertices. Then between any two vertices of G, there exists only one path, and hence it is the unique weighted geodesic between them. So all vertices in all weighted geodesics between any two vertices are again in V (G). Therefore $V(G) = V_n$ is weighted geodetic convex and $|V_n| = n$. Let v_1 be a pendent vertex of G and set $V_{n-1} = V_n - \{v_1\}$. Then clearly the vertex induced subgraph of G by the set V_{n-1} is again a weighted tree. By the above same argument, we see that V_{n-1} is weighted geodetic convex and $|V_{n-1}| = (n - 1)$. Let v_2 be a pendent vertex of $G - v_1$, set $V_{n-2} = V_{n-1} - \{v_2\}$. We can easily prove V_{n-2} is weighted geodetic convex, and $|V_{n-2}| = (n - 2)$. Continue the above procedure of deleting pendent vertices until we get a singleton set. So finally we get the nested sequence $V = V_n \supset V_{n-1} \supset, ..., \supset V_1$.

Remark 3.10: The converse of the above theorem [Theorem 3.9] is not true. This means even though we can find a nested sequence of vertex sets satisfying the property in the condition, the graph may not be a weighted tree. It can be a weighted geodetic block also.

IV. Weighted Geodetic Boundary And Interior Vertices

In this section, we define the boundary and interior vertices of a weighted geodetic convex set. Some of their properties are discussed.

Definition 4.1: Let G: (V, E, ω) be a connected weighted graph and S be a weighted geodetic convex set of G. A vertex u ϵ S is called a weighted geodetic boundary vertex of S if and only if S − {u} is weighted geodetic convex.

Definition 4.2: Let G: (V, E, ω) be a connected weighted graph and S be a weighted geodetic convex set of G. A vertex u ϵ S is called a weighted geodetic interior vertex of S if and only if S − {u} is not weighted geodetic convex.

Example 4.3:

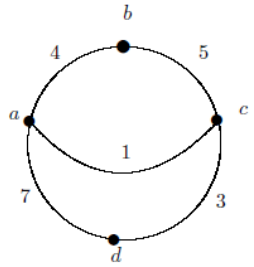

Figure 5: Boundary and Interior vertices

In the above example [Figure 5], S = {a, b, c} is a weighted geodetic convex set. Vertex c is a weighted geodetic interior vertex of S, since S − {c} is not weighted geodetic convex. But vertex b is a weighted geodetic boundary vertex of S, as S − {b} is weighted geodetic convex.

In the following two theorems [Theorem 4.4, Theorem 4.5], we characterize the boundary and interior vertices of a weighted geodetic convex set.

Theorem 4.4: Let G: (V, E, ω) be a connected weighted graph, and S be a weighted geodetic convex set of G. Then a vertex u ϵ S is a weighted geodetic boundary vertex of S if and only if u does not lie on any weighted v − w geodesic for all v, w ϵ S − {u}.

Proof. Let G: (V, E, ω) be a connected weighted graph, and S be a weighted geodetic convex set of G. Let u ϵ S. Suppose that u is a weighted geodetic boundary vertex of S. We have to prove that u does not lie on any weighted v − w geodesic for all v, w ϵ S − {u}. Since u is a weighted geodetic boundary vertex of S, S − {u} is weighted geodetic convex. Let v, w ϵ S − {u}. Since S − {u} is weighted geodetic convex, all the vertices in all weighted v − w geodesics are in S − {u} itself. This means all the weighted v − w geodesics are independent of u. Hence u does not lie in any weighted v − w geodesic for all v, w ϵ S − {u}. Conversely suppose that u ϵ S and u does not lie in any weighted v − w geodesic for all v, w ϵ S − {u}. We have to prove that u is a weighted geodetic boundary vertex of S. It is enough if we prove that S − {u} weighted geodetic convex. Let v, w ϵ S − {u}. By the assumption, all the weighted v − w geodesics are free from u. Also (S − {u}) U {u} = S, which is weighted geodetic convex. Hence all the vertices in all weighted geodesics between any two vertices in S − {u} lie in S − {u} itself, which proves S − {u} is weighted geodetic convex. So u is a weighted geodetic boundary vertex of S.

Theorem 4.5. Let G: (V, E, ω) be a connected weighted graph, and S be a weighted geodetic convex set of G. Then a vertex u ϵ S is a weighted geodetic interior vertex of S if and only if u lies in at least one weighted v − w geodesic for any v, w 2 S − {u}

Proof. Let G: (V, E, ω) be a connected weighted graph, and S be a weighted geodetic convex set of G. Let u ϵ S. Suppose that u is a weighted geodetic interior vertex of S. We have to prove that u lies in at least one weighted v − w geodesic for any v, w ϵ S − {u}. Since u is a weighted geodetic interior vertex of S, S − {u} is not weighted geodetic convex. Let v, w ϵ S − {u}. Since S − {u} is not weighted geodetic convex, at least one vertex in any one of the weighted v−w geodesic does not belong to S − {u}. At the same time (S − {u}) U {u} = S, which is weighed geodetic convex. This means at least one of the weighted v − w geodesic contains the vertex u. Conversely suppose that u ϵ S and u lies in any weighted v − w geodesic for some v, w ϵ S − {u}. We have to prove that u is a weighted geodetic interior vertex of S. It is enough, if we prove that S − {u} is not weighted geodetic convex. Let v, w ϵ S − {u}. By the assumption, there exists at least one weighted v − w geodesic which contains u. Also (S − {u}) U {u} = S, which is weighted geodetic convex. Hence S − {u} is not weighted geodetic convex. So u is a weighted geodetic interior vertex of S.

In the following theorem [Theorem 4.6], we characterize the boundary and interior vertices of a weighted geodetic block.

Theorem 4.6: Let G: (V, E, ω) be a connected weighted graph, and let H be a weighted geodetic block of G. Then every vertex of H is a weighted geodetic boundary vertex of V (H).

Proof. Let G: (V, E, ω) be a connected weighted graph and let H be a weighted geodetic block of G. Let u be any vertex of H. We have to prove that u is a weighted geodetic boundary vertex of V (H). It is enough, if we prove V (H) − {u} is weighted geodetic convex. Since a weighted geodetic block is a complete weighted graph structure and each of its edges are unique weighted geodesics between their end vertices, by theorems 4.4 and 4.6, V (H) − {u} is weighted geodetic convex. This proves that u is a weighted geodetic boundary vertex of V (H). Since u is arbitrary, the proof is completed.

Corollary 4.7: Let G: (V, E, ω) be a connected weighted graph and let H be a weighted geodetic block of G. Then no vertex of H is a weighted geodetic interior vertex of V (H).

V. CONCLUSION

In this article, the authors made an attempt to generalize the concept of convexity. This generalization is done by using the help of weighted distance in weighted graphs. The concepts of weighted geodesics, weighted geodetic convexity, weighted geodetic blocks, boundary and interior nodes of weighted geodetic convex sets are introduced. Characterization for weighted geodetic blocks and boundary and interior nodes are also presented. We have proved that, for a weighted geodetic block, all nodes are boundary nodes and no node is an interior node

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