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Measure space on Weak Structure

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Abstract: *Császár in [4] introduce a weak structure as generalization of general topology. The aim of this paper is to give basic concepts of the measure theory in weak structure.*
Keywords: *weak structure, σ - algebra, σ - additive function, Measures*

I. Notation and Preliminaries

In mathematical analysis. Measurement theory plays a vital role in the expression completely for some mathematical concepts. In our research, we introduced some of the concepts of measurement in a weak structure. And we study their properties and some applications it. So we shall denote by X nonempty set, by ω a weak structure [1] and by $P(X)$ the set of all parts (i.e., subsets) of X, and by ϕ the empty set. For any subset λ of X we shall denote by λ^c its complements, i.e., $\lambda^c = \{x \in X | x \notin \lambda\}$. For any $\lambda, \mu \in P(X)$ we set $\lambda | \mu = \lambda \cap \mu^c$. Let (λ_n) be a sequence in $P(X)$.

The following Demorgan identity holds $(\bigcup_{n=1}^{\infty} \lambda_n) = \bigcap_{n=1}^{\infty} \lambda_n^c$. we define $\lim_{n\to\infty}(\vee \lambda_n) = \bigcap_{n=1}^{\infty}(\bigcup_{m=n}^{\infty} \lambda_m)$, $\lim_{n\to\infty}(\wedge \lambda_n) = \bigcup_{n=1}^{\infty}(\bigcap_{m=n}^{\infty} \lambda_m)$. If $L = \lim(\bigvee_{n\to\infty} \lambda_n) = \lim(\bigwedge_{n\to\infty} \lambda_n)$, then we set $L = \lim_{n\to\infty}(\lambda_n)$, and we say that (λ_n) converges to L. As easily checked, $\lim(\bigvee_{n\to\infty} \lambda_n)$ (resp., $\lim(\bigwedge_{n\to\infty} \lambda_n)$ consists of those elements of X that belong to infinte elements of (λ_n) (resp., that belong to infinte elements of (λ_n) expect perhaps a finite number. Therefore, $\lim(\bigwedge_{n\to\infty} \lambda_n) \subset \lim(\bigvee_{n\to\infty} \lambda_n)$. And it easy also to check that, if (λ_n) is increasing ($\lambda_n \subset \lambda_{n+1}, n \in N$), then $\lim_{n\to\infty} \lambda_n = \bigcup_{n=1}^{\infty} \lambda_n$ where, if (λ_n) is decreasing ($\lambda_n \supset \lambda_{n+1}, n \in N$), then $\lim_{n\to\infty} \lambda_n = \bigcap_{m=n}^{\infty} \lambda_n$. In the first case we shall write $\lambda_n \uparrow L$, and in the second $\lambda_n \downarrow L$.

II. Algebra and σ - algebra on a weake structure ω

Let A be a nonempty subset of ω

Definition 1.1 A is said to be an algebra in ω if

$a)$ $\phi \in \mathsf{A}$

$b)$ $\lambda, \mu \in \mathsf{A} \Rightarrow \lambda \cup \mu \in \mathsf{A}$

$c)$ $\lambda \in \mathsf{A} \Rightarrow \lambda^c \in \mathsf{A}$

Remark 1.1 It easy to see that, if A is an algebra and $\lambda, \mu \in \mathsf{A}$, then $\lambda \cap \mu$ and $\lambda | \mu$ belong to A. Therfore, the symmetric difference $\lambda \Delta \mu = (\lambda|\mu) \cup (\mu|\lambda)$ also belong to A. Moreover, A is stable under finite union and intersection,

that is $\lambda_1, \ldots\ldots\ldots, \lambda_n \in \mathsf{A} \Rightarrow \begin{cases} \lambda_1 \cup \ldots\ldots\ldots \cup \lambda_n \in \mathsf{A} \\ \lambda_1 \cap \ldots\ldots\ldots \cap \lambda_n \in \mathsf{A} \end{cases}$.

Definition 1.2 An algebra A in ω is said to be a σ - algebra if, for any sequence (λ_n) of elements of A, we have $\bigcup_{n=1}^{\infty}\lambda_n \in \mathsf{A}$. We note that, if A is σ - algebra and $(\lambda_n) \subset \mathsf{A}$, then $\bigcap_{n=1}^{\infty}\lambda_n \in \mathsf{A}$ owing to the De morgan identity.
Moreover, $\lim_{n\to\infty}(\bigwedge \lambda_n) \in \mathsf{A}$, $\lim_{n\to\infty}(\bigvee \lambda_n) \in \mathsf{A}$.

The following examples explain the difference between algebras and σ - algebras.

Example 1.1 Obviously, $P(X)$ and $\varepsilon = \{\phi\}$ are σ - algebras in X. Morover, ω is the largest σ - algebras in X , and ε is the smallest.

Example 1.2 In $[0,1)$, the class ρ consisting of ϕ , and of all finite unions
$\beta = \bigcup_{i=1}^{n}[a_i, b_i)$ with $0 \le a_i \le b_i \le a_{i+1} \le 1$ is an algebra.

Example 1.3 In an infinite set X consider the class $\rho = \{\theta \in \omega | \ \theta$ is finite, or θ^c is finite $\}$. Then ρ is an algebra.

Example 1.4 In an uncountable set X consider the class $\rho = \{\theta \in \omega | \ \theta$ iscountable, or θ^c iscountable $\}$. Then ρ is a σ - algebra.

Definition 1.3 The intersection of all σ - algebras including $\tau \subseteq \omega$ is called the σ - algebra generated by τ, and will be denoted by $\sigma(\tau)$.

Example 1.5 Let E be a metric space. The σ - algebra generated by all open subsets of E is called the Borel σ - algebra of E, and denoted by $B(E)$.

2. **Measure**

2.1 Additive and σ - additive functions

Let $\mathsf{A} \subset \omega$ be an algebra

Definition 2.1 Let $\mathsf{F} : \mathsf{A} \to 0, +\infty]$ be such that $\mu(\phi) = 0$.

(1) We say that F is additive if, for any family $A_1, \ldots\ldots, A_n \in \mathsf{A}$ of mutually disjoint sets, we have $\mathsf{F}(\bigcup_{k=1}^{n} A_k) = \sum_{k=1}^{n}\mathsf{F}(A_k)$.

(2) We say that F is $\sigma-$ additive if, for any sequence $(A_n) \in \mathsf{A}$ of mutually disjoint sets such that $\bigcup_{k=1}^{\infty} A_k \in \mathsf{A}$, we have $\mathsf{F}(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty}\mathsf{F}(A_k)$.

Remarke 2.1 Let $\mathsf{A} \subset \omega$ be an algebra

(1) Any $\sigma-$ additive function on A is also additive.

(2) If F is additive, $\lambda, \mu \in \mathsf{A}$, and $\lambda \supset \mu$, then $\mathsf{F}(\lambda) = \mathsf{F}(\mu) + \mathsf{F}(\lambda | \mu)$.
Therefore, $\mathsf{F}(\lambda) \ge \mathsf{F}(\mu)$.

(3) Let $\digamma$ is additive on A, and let $(A_n)\in\mathsf{A}$ be mutually disjoint sets such that $\bigcup_{k=1}^{\infty}A_k\in\mathsf{A}$. Then, $\digamma(\bigcup_{k=1}^{\infty}A_k)\geq\sum_{k=1}^{n}\digamma(A_k)$ for all n $\in$ N.

Therefore, $\digamma(\bigcup_{k=1}^{\infty}A_k)\geq\sum_{k=1}^{\infty}\digamma(A_k)$

(4) Any $\sigma-$additive function $\digamma$ on A is also countably subadditive, that is, for any sequence $(A_n)\subset\mathsf{A}$ such that $\bigcup_{k=1}^{\infty}A_k\in\mathsf{A}$, $\digamma(\bigcup_{k=1}^{\infty}A_k)\leq\sum_{k=1}^{\infty}\digamma(A_k)$.

(5) Inview of parts 3 and 4 an additive function is $\sigma-$additive if and only if it is countably subadditive.

Definition 2.2 A $\sigma-$additive function $\digamma$ on an algebra $\mathsf{A}\subset\omega$ is said to be

(1) finite if $\digamma(X)<\infty$,

(2) $\sigma-$finite if there exists a sequence sequence $(A_n)\subset\mathsf{A}$ such that $\bigcup_{n=1}^{\infty}A_n=X$, and $\digamma(A_n)<\infty$ for all n $\in$ N.

Example 2.1 In $X=N$, consider the algebra $\mathsf{A}=\{A\in\omega \mid A\text{isfinite, or}A^c\text{ finite}\}$. The function $\digamma:\mathsf{A}\to 0,\infty]$ defined as $\digamma(A)=\begin{cases} n(A) & if \quad A \quad finite \\ \infty & if \quad A^c \quad finite\end{cases}$ (where $n(A)$) stands for the number of elements of A is $\sigma-$additive. On the other hand.

The function $\digamma:\mathsf{A}\to 0,\infty]$ defined as $\digamma(A)=\begin{cases}\sum_{n\in A}\frac{1}{2^n} & if \quad A \quad finite \\ \infty & if \quad A^c \quad finite\end{cases}$ is additive but not $\sigma-$additive.

Theorem 2.1 Let μ be additive on A. Then $(i)\Leftrightarrow(ii)$ where:
(i) μ is $\sigma-$additive,
(ii) (A_n) and $A\subset\mathsf{A}$, $A_n\uparrow A\Rightarrow\mu(A_n)\uparrow\mu(A)$.

Proof $(i)\Rightarrow(ii)$ Let (A_n), $A\subset\mathsf{A}$, $A_n\uparrow A$. Then, $A=A_1\cup\bigcup_{n=1}^{\infty}(A_{n+1}\setminus A_n)$, the a bove being disjoint union. Since μ is $\sigma-$ additive, we deduce that $\mu(A)=\mu(A_1)+\sum_{n=1}^{\infty}(\mu(A_{n+1})-\mu(A_n))=\lim_{n\to\infty}\mu(A_n)$, and (ii) follows.

$(ii)\Rightarrow(i)$ Let $(A_n)\subset\mathsf{A}$ be a sequence of mutually disjoint sets such that $A=\bigcup_{k=1}^{\infty}A_k\in\mathsf{A}$. Define $B_n=\bigcup_{k=1}^{\infty}A_k$. Then $B_n\uparrow A$. So, in view of (ii), $\mu(B_n)=\sum_{n=1}^{\infty}\mu(A_k)\uparrow\mu(A_n)$.
This implies (i)

Definition 2.2 let $\varepsilon = \{\phi\}$ are σ - algebras in X.

(1) We say that the pair (X,ε) is a measurable space.
(2) A $\sigma -$ additive function $\mu : \varepsilon \to 0,+\infty]$ is called a measure on (X,ε)
(3) The triple (X,ε,μ), where μ is a measure on a measurable space (X,ε) is called a measurable space
(4) A measure μ is said to be complete if $A \in \varepsilon,\ B \subset A,\ \mu(A) = 0 \Rightarrow$
$B \in \varepsilon$ (and so $\mu(B) = 0$).
(5) A measure μ is said to be concentrated on a set $A \in \varepsilon$ if $\mu(A^c) = 0$.
In this case we say that A is a support of μ

Example 2.2 Let X be a nonempty set and $x \in X$. Define for every $A \in P(X)$

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ \\ 0 & \text{if } x \notin A \end{cases}$$

. Then δ_x is a measure in X.

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