Weak Convergence Theorem of Khan Iterative Scheme for Nonself I-Nonexpansive Mapping

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Abstract: In this paper, we prove the weak convergence of a modified Khan iteration for nonself I-nonexpansive mapping in a Banach space which satisfies Opial’s condition. Our result extends and improves these announced by S. Chornphrom and S. Phonin, Weak Converges Theorem of Noor iterative Sch.

Keywords: Khan iterative scheme; weak convergence; nonself nonexpansive mapping; fixed point; Banach space.

2000 Mathematics Subject Classification: 47H09, 47H10 (2000 MSC)

I. Introduction

Let $E := (E, \| \cdot \|)$ be a real Banach space, $K$ be a nonempty convex subset of $E$, and $T$ be a self mapping of $K$. The Mann iteration [9] is defined as $x_1 \in K$ and

$$ x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n, \quad n \geq 1. \quad (1.1) $$

The Ishikawa iteration [5] is defined as $x_1 \in K$ and $y_n = (1 - \beta_n)x_n + \beta_nTx_n$

$$ x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTy_n, \quad n \geq 1. \quad (1.2) $$

The Noor iteration [8] is defined as $x_1 \in K$ and $z_n = (1 - \gamma_n)x_n + \gamma_nTy_n$

$$ x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTz_n, \quad n \geq 1. \quad (1.3) $$

Where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1]$.

In the above taking $\beta_n = 0$ in (1.2) and taking $\beta_n = 0, \gamma_n = 0$ in (1.3) we obtain iteration (1.1).

In 1975, Baillon [1] first introduced nonlinear ergodic theorem for general non-expansive mapping in a Hilbert space $H$: if $K$ is a closed and convex subset of $H$ and $T$ has a fixed point, then for every $x \in K, \{T^n x\}$ is a weakly almost convergent, as $n \to \infty$, to a fixed point of $T$. It was also shown by Pazy[11] that if $H$ is a real Hilbert space and \( \frac{1}{n} \sum_{i=0}^{n-1} T^i x \) converges weakly, as $n \to \infty$, to $y \in K, y \in F(T)$.

In 1941, Tricomi introduced the concept of a quasi-nonexpansive mapping for real functions. Later Diaz and Metcalf [2] and Dotson [3] studied quasi-nonexpansive mappings in Banach spaces. Recently, this concept was given by Kirk [6] in metric spaces which we adapt to a normed space as the following: $T$ is called a quasi-nonexpansive mapping provided for all $x \in K$ and $f \in F(T)$.

$$ \| Tx - f \| \leq \| x - f \| $$

Recall that a Banach space $E$ is said to satisfy Opial’s condition [10] if, for each sequence $\{x_n\}$ in $E$, the condition $x_n \rightharpoonup x$ implies that
\[
\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\| \tag{1.6}
\]

for all \( y \in E \) with \( y \neq x \). It is well known from [10] that all \( L_p \) spaces for \( 1 < p < \infty \) have this property. However, the \( L_p \) spaces do not, unless \( p = 2 \). The following definitions and statements are needed for the proof of our theorem.

Let \( K \) be a closed convex bounded subset of uniformly convex Banach spaces \( E \) and \( T \) a self-mapping of \( E \). Then \( T \) is called nonexpansive on \( K \) if
\[
\|Tx - Ty\| \leq \|x - y\| \tag{1.7}
\]
for all \( x, y \in K \). Let \( F(T) := \{ x \in K : Tx = x \} \) be denote the set of fixed points of a mapping \( T \).

Let \( K \) be a subset of a normed space \( E \) and \( T \) and \( I \) self-mappings of \( K \). Then \( T \) is called \( I \)-nonexpansive on \( K \) if
\[
\|Tx - Ty\| \leq \|Ix - Iy\| \tag{1.8}
\]
for all \( x, y \in K \) [14].

A mapping \( T \) is called \( I \)-quasi-nonexpansive on
\[
\|Tx - f\| \leq \|Ix - f\| \tag{1.9}
\]
for all \( x, y \in K \) and \( f \in F(T) \cap F(I) \).

A subset \( K \) of \( E \) is said to be a retract of \( E \) if there exists a continuous map \( P : E \to K \) such that \( P x = x \) for all \( x \in K \). A map \( P : E \to E \) is said to be a retraction if \( P^2 = P \). It follows that if a map \( P \) is a retraction, then \( Py = y \) for all \( y \) in the range of \( P \). A set \( K \) is optimal if each point outside \( K \) can be moved to be closer to all points of \( K \). Note that every nonexpansive retract is optimal. In strictly convex Banach spaces, optimal sets are closed and convex. However, every closed convex subset of a Hilbert space is optimal and also a nonexpansive retract.

**Remark 1.1.** From the above definitions it is easy to see that if \( F(T) \) is nonempty, a nonexpansive mapping must be quasi-nonexpansive, and linear quasi- nonexpansive mappings are nonexpansive. But it is easily seen that there exist nonlinear continuous quasi-nonexpansive mappings which are not nonexpansive. There are many results on fixed points on nonexpansive and quasi-nonexpansive mappings in Banach spaces and metric spaces. For example, Petrichyn and Williamson [12] studied the strong and weak convergence of the sequence of certain iterates to a fixed point of quasi-nonexpansive mapping. Their analysis was related to the convergence of Mann iterates studied by Dotson [3]. Subsequently, Ghosh and Debnav [4] considered the convergence of Ishikawa iterates of quasi- nonexpansive mappings in Banach spaces. Later Temir and Gul [15] proved the weakly convergence theorem for \( I \)-asymptotically quasi-nonexpansive mapping defined in Hilbert space. In [16], the convergence theorems of iterative schemes for nonexpansive mappings have been presented and generalized.

In [13], Rhoades and Temir considered \( T \) and \( I \) self-mappings of \( K \), where \( T \) is \( I \)-nonexpansive mapping. They established the weak convergence of the sequence of Mann iterates to a common fixed point of \( T \) and \( I \). More precisely, they proved the following theorems.

**Theorem (Rhoades and Temir [13]):** Let \( K \) be a closed convex bounded subset of uniformly convex Banach space \( E \), which satisfies Opial’s condition, and let \( T, I \) self-mappings of \( K \) with \( T \) be an \( I \)-nonexpansive mapping, \( I \) a nonexp-ansive on \( K \). Then, for \( x_0 \in K \), the sequence \( \{ x_n \} \) of modified Noor iterates converges weakly to a common fixed point of \( F(T) \cap F(I) \).

In the above theorem, \( T \) remains self-mapping of a nonempty closed convex subset \( K \) of a uniformly convex Banach space. If, however, the domain \( K \) of \( T \) is a proper subset of \( E \) and \( T \) maps \( K \) into \( E \), then, the iteration formula (1.1) may fail to be well defined. One method that has been used to overcome this in the case of single operator \( T \) is to introduce a retraction \( P : E \to K \) in the recursion formula (1.1) as follows: \( x_1 \in K, x_{n+1} = (1 - \alpha_n)x_n + \alpha_nPTx_n, n \geq 1 \).

In the above, \( x_{n+1} \) is the \( (n+1) \)-th iterate of the \( (n-1) \)-th iterate of the sequence. The modified Ishikawa iterative scheme to a common fixed point of \( T \) and \( I \).

\[
y_n = P((1 - \beta_n)x_n + \beta_nTn)(x_n)
\]
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\[ x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n Ty_n), \quad n \geq 1. \]  

(1.10)

In this paper, we consider \( T \) and \( I \) are nonself mappings of \( K \), where \( T \) is an \( I \)-nonexpansive mappings. We prove the weak convergence of the sequence of modified Noor iterative scheme to a common fixed point of \( F(T) \cap F(I) \).

II. Main Results

In this section, we prove the weak convergence theorem.

**Theorem 2.1.** Let \( K \) be a closed convex bounded subset of a uniformly convex Banach space \( E \) which satisfies Opial’s condition, and let \( T, I \) nonself mappings of \( K \) with \( T \) be an \( I \)-nonexpansive mapping, \( I \) a nonexpansive on \( K \). Then, for \( x_0 \in K \), the sequence \( \{x_n\} \) of modified Khan iterates defined by \( x_n \in K \),

\[
\begin{align*}
Z_n &= (1 - \gamma_n)x_n + \gamma_n T^n x_n, \\
y_n &= (1 - \beta_n)x_n + \beta_n S^n z_n, \\
x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n P^n y_n,
\end{align*}
\]

(2.1)

converges weakly to common fixed point of \( F(T) \cap F(I) \).

**Proof.** If \( F(T) \cap F(I) \) is nonempty and a singleton, then the proof is complete. We will assume that \( F(T) \cap F(I) \) is nonempty and that \( F(T) \cap F(I) \) is not a singleton.

\[
\|x_{n+1} - f\| = \left\| (1 - \alpha_n)x_n + \alpha_n P^n y_n - f \right\|
\]

\[
= \left\| (1 - \alpha_n)x_n + \alpha_n P^n y_n - (1 - \alpha_n + \alpha_n)f \right\|
\]

\[
\leq (1 - \alpha_n)\|x_n - f\| + \alpha_n\|P^n y_n - f\|
\]

\[
\leq (1 - \alpha_n)\|x_n - f\| + \alpha_n K\|y_n - f\|
\]

(2.2)

and

\[
\|y_n - f\| = \left\| (1 - \beta_n)x_n + \beta_n S^n z_n - f \right\|
\]

\[
= \left\| (1 - \beta_n)x_n + \beta_n (S^n z_n - f) \right\|
\]

\[
\leq (1 - \beta_n)\|x_n - f\| + \beta_n\|S^n z_n - f\|
\]

\[
\leq (1 - \beta_n)\|x_n - f\| + \beta_n K\|z_n - f\|
\]

(2.3)

and also, we get

\[
\|z_n - f\| = \left\| (1 - \gamma_n)x_n + \gamma_n T^n x_n - f \right\|
\]

\[
= \left\| (1 - \gamma_n)x_n + \gamma_n (T^n x_n - f) \right\|
\]

\[
\leq (1 - \gamma_n)\|x_n - f\| + \gamma_n\|T^n x_n - f\|
\]

\[
\leq (1 - \gamma_n)\|x_n - f\| + \gamma_n K\|x_n - f\|
\]

(2.4)

Substituting (2.4) in (2.3), we have

\[
\|y_n - f\| \leq (1 - \beta_n)\|x_n - f\| + K_n\beta_n (1 - \gamma_n + K_n\gamma_n)\|x_n - f\|
\]

(2.5)

Substituting (2.5) in (2.2), we have

\[
\|x_{n+1} - f\| \leq (1 - \alpha_n)\|x_n - f\| + \alpha_n K_n (1 - \beta_n + K_n\beta_n - K_n\beta_n\gamma_n + K_n^2\beta_n\gamma_n)\|x_n - f\|
\]

\[
\leq (1 - \alpha_n + K_n\alpha_n - K_n\alpha_n\beta_n + K_n^2\alpha_n\beta_n - K_n^3\alpha_n\beta_n\gamma_n + K_n^3\alpha_n\beta_n\gamma_n)\|x_n - f\|
\]

\[
\leq [1 - \alpha_n(K_n - 1) + \alpha_n\beta_n K_n(K_n - 1) + \alpha_n\beta_n\gamma_n K_n^2(K_n - 1)]\|x_n - f\|
\]

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Thus \( \alpha_n \neq 0, \beta_n \neq 0 \) and \( \gamma \neq 0 \). Since \( \{K_n\} \) is a nonincreasing bounded sequence and hence \( K_n < 1 \) implies that
\[
\sum_{n=1}^{\infty} (K_n - 1) < \infty. \quad \text{Then } \lim_{n \to \infty} \|x_n - f\| \text{ exists.}
\]

Now we show that \( \{x_{n_k}\} \) converges weakly to a common fixed point of \( T \) and \( I \). The sequence \( \{x_{nk}\} \) contains a subsequence which converges weakly to a point in \( K \). Let \( \{x_{nk}\} \) and \( \{x_{nk}\} \) be two subsequences of \( \{x_n\} \) which converge weak to \( f \) and \( q \), respectively. We will show that \( f = q \). Suppose that \( E \) satisfies Opial’s condition and that \( f \neq q \) is in weak limit set of the sequence \( \{x_n\} \). Then \( \{x_{nk}\} \to f \) and \( \{x_{nk}\} \to q \), respectively. Since
\[
\lim_{n \to \infty} \|x_n - f\| \text{ exists for any } f \in F(T) \cap F(I), \text{ by Opial’s condition, we conclude that }
\]
\[
\lim_{n \to \infty} \|x_n - f\| = \lim_{k \to \infty} \|x_{nk} - f\| < \lim_{k \to \infty} \|x_{nk} - q\| = \lim_{j \to \infty} \|x_{nj} - q\| < \lim_{j \to \infty} \|x_{nj} - f\| = \lim_{n \to \infty} \|x_n - f\|
\]
This is a contradiction. Thus \( \{x_n\} \) converges weakly to an element of \( F(T) \cap F(I) \). This completes the proof.

Corollary 2.2. (Kumam et al. [8, Theorem 2.1]) Let \( K \) be a closed convexbounded subset of a uniformly convex Banach space \( X \), which satisfies Opial’s condition, and let \( T, I \) self-mappings of \( K \) with \( T \) be an \( I \)-quasi-nonexpansive mapping, \( I \) a nonexpansive on \( K \). Then, for \( x_0 \in K \), the sequence \( \{x_n\} \) of three-step Noor iterative scheme defined by (1.3) converges weakly to common fixed point of \( F(T) \cap F(I) \).

Corollary 2.3. (Kiziltunc and Ozdemir [7, Theorem 2.1]) Let \( K \) be a closed convex bounded subset of a uniformly convex Banach space \( E \), which satisfies Opial’s condition, and let \( T, I \) nonself mappings of \( K \) with \( T \) be an \( I \)-nonexpansive map- ping, \( I \) a nonexpansive on \( K \). Then, for \( x_1 \in K \), the sequence \( \{x_n\} \) of modified Ishikawa iterates defined by (1.9) converges weakly to common fixed point of \( F(T) \cap F(I) \).

Theorem 2.4. Let \( K \) be a closed convex bounded subset of a uniformly convex Banach space \( E \), which satisfies Opial’s condition, and let \( T, I \) nonself mappings of \( K \) with \( T \) be an \( I \)-nonexpansive mapping, \( I \) a nonexpansive on \( K \). Then, for \( x_1 \in K \), the sequence \( \{x_n\} \) of Mann converges weakly to common fixed point of \( F(T) \cap F(I) \).

Proof. Putting \( \gamma_n = 0 \) and \( \beta_n = 0 \) in Theorem 2.1, we obtain the desired result.

References