Paranormed Sequence spaces \( w(u, v; p) \), \( w_0(u, v; p) \) and 
\( w_\omega(u, v; p) \) generated by weighted mean

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**Abstract:** The sequence spaces \( w(p) \), \( w_0(p) \) and \( w_\omega(p) \) were introduced and studied by I.J. Maddox [1,2,3,4]. In [8,9,10], the authors have introduced sequence spaces \( c,u(v, p), l_\infty(u, v, p) \) and \( l(u, v, p) \) and established some properties. In this paper we introduce the sequence spaces \( w(u, v; p) \), \( w_0(u, v; p) \) and \( w_\omega(u, v; p) \); study some properties, find \( \beta \)- dual of \( w(u, v; p) \). We also characterize the matrix classes \( \left( w(u, v; p), l_\infty \right), \left( w(u, v; p), c \right) \) and \( \left( w(u, v; p), c_0 \right) \).

**Key Words:** Paranormed sequence spaces, \( \beta \)- dual, matrix transformation, generalized weighted mean.

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**I. Introduction**

By \( \omega \) we mean the space of all real valued sequences. A vector subspace of \( \omega \) is called a sequence space. We shall write \( , \) with usual notation, \( l_\infty, c \) and \( c_0 \) for the spaces of all bounded, convergent and null sequence respectively. A linear topological space \( X \) over the field \( \mathbb{R} \) is said to be a paranormed space if there is a sub-additive function \( g : X \rightarrow \mathbb{R} \) such that \( g(\theta) = 0 \), \( g(x) = g(-x) \) and scalar multiplication is continuous i.e. \( |a_n - \alpha| \to 0 \) and \( g(x_n - x) \to 0 \) imply \( g(a_n x_n - \alpha x) \to 0 \), where \( \theta \) is the zero vector in the linear space \( X \).

If \( p = \{ p_k \} \) be a bounded sequence of strictly positive real numbers, I.J. Maddox defined the sequence spaces \( w(p), w_0(p) \) and \( w_\omega(p) \) as:

\[
w(p) = \left\{ x = (x_k) \in \omega : \frac{1}{n} \sum_{k=1}^{n} |x_k|^{p_1} \to 0 \; \text{for some} \; p_1 \in \mathbb{R}, n \to \infty \right\}
\]

\[
w_0(p) = \left\{ x = (x_k) \in \omega : \frac{1}{n} \sum_{k=1}^{n} |x_k|^{p_1} \to 0 \; , \; n \to \infty \right\}
\]

\[
w_\omega(p) = \left\{ x = (x_k) \in \omega : \frac{1}{n} \sum_{k=1}^{n} |x_k|^{p_1} < \infty \; , \; n \to \infty \right\}.
\]

The spaces \( w(p) \) and \( w_0(p) \) are paranormed spaces paranormed by

\[
g(x) = \sup \left( \frac{1}{n} \sum_{k=1}^{n} |x_k|^{p_1} \right)^{\frac{1}{M}}
\]

or equivalently

\[
g(x) = \sup \left( 2^{r} \sum_{r} |x_k|^{p_k} \right)^{\frac{1}{M}}
\]

(1.1)

where \( \sum_r \) is the sum over the range \( 2' \leq r < 2^{\mathbb{R}} \) and \( M = (1, \sup p_k) \). Further \( w_\omega(p) \) is the paranorm space paranormed by (1.1) if and only if \( 0 < \inf p_k \leq \sup p_k < \infty \) [1].

Let \( X \) and \( Y \) be any two sequence spaces and \( A = (a_{nk}) ; n, k \in \mathbb{R} \) be infinite matrix of complex numbers \( a_{nk} \). Then we say that \( A \) defines a matrix mapping \( X \to Y \); and it is denoted by writing \( A : X \to Y \) if for every sequence \( x = (x_k) \in X \), the sequence \( (Ax)_n \) is in \( Y \), where
(Ax)ₙ = Σₖ₌₁ⁿ aₙₖ xₖ ; (n ∈ ℤ)  

(1.2)

By (X, Y) we denote the class of all matrices A such that A:X → Y. Thus, A ∈ (X, Y) if and only if the series on right side of (1.2) converges for each n ∈ ℤ and every x ∈ X; and we write,

Ax = \{ (Ax)ₙ \}_{n∈ℤ} ∈ Y for all x ∈ X.

We denote by U for the set of all sequences u = (uₙ) such that uₙ ≠ 0 for all n ∈ ℤ. For u ∈ U, let 1/u = \left( \frac{1}{uₙ} \right). Let us define the matrix G(u, v) = (gₙₖ) as:

\[ gₙₖ = \begin{cases} uₙ vₖ ; & 0 ≤ k ≤ n \\ 0 ; & k > n \end{cases} \]

(1.3)

for all n,k ∈ ℤ , where uₙ depends only on n and vₖ only on k. The matrix G(u, v) = (gₙₖ) is called generalized weighted mean or factorable matrix.

The main purpose of the present paper is to introduce the sequence spaces \( w(u, v; p), w₀(u, v; p) \) and \( w∞(u, v; p) \); which are the set of all sequences whose G(u,v)-transforms are in the spaces \( w(p), w₀(p) \) and \( w∞(p) \) respectively, where G(u,v) denotes the matrix as defined in (1.3). We have discussed some topological properties of \( w(u, v; p), w₀(u, v; p) \) and \( w∞(u, v; p) \); investigated β – dual for the new space \( w(u, v; p) \). Moreover we have characterized the matrix classes \( \{ w(u, v; p), l∞ \}, \{ w(u, v; p), c \} \) and \( \{ w(u, v; p), c₀ \} \).

II. The paranormed sequence spaces \( w(u, v; p), w₀(u, v; p) \) and \( w∞(u, v; p) \).

Before introducing these sequence spaces we would like to present some remarks. Malkowsky and Savas [10] have defined the sequence spaces Z(u,v,X) which consists of all sequences whose G(u,v) - transforms are in \( \{ l∞, c, c₀, l(p) \} \) where u,v ∈ U. Chaudhary B. and Mishra S.K. [6] have defined the sequence space \( l(p) \) which consists of all sequences whose C-transforms are in \( l(p) \); where \( S = (sₙₖ) \) is defined by \( sₙₖ = \begin{cases} 1 ; & 0 ≤ k ≤ n \\ 0 ; & k > n \end{cases} \)

Moreover I.J. Maddox [11] introduced the sequence space \( w(p), w₀(p) \) and \( w∞(p) \) which consists of all strongly summable , strongly summable to zero and bounded sequences respectively whose C-transforms are in the spaces \( l(p), c₀(p) \) and \( l∞(p) \) respectively ; where

\[ C = (cₙₖ) = \begin{cases} \frac{1}{n} ; & 1 ≤ k ≤ n \\ 0 ; & k > n \end{cases} \]

and \( C = (cₙₖ) \) is called the Cesaro matrix of order 1 or the matrix of arithmetic mean.

The matrix domain \( X_A \) of an infinite matrix A in a sequence space X is defined by

\[ X_A = \{ x = (xₙ) ∈ ω : Ax ∈ X \} \]

(2.1)

which is a sequence space.
With the notation as in (2.1), we can have the following representations:

\[ X(u, v, p) = \left[ X \right] \alpha, \quad \text{for } X \in \{ \ell_{\infty}, \ell, c_0, \ell(p) \} \]

Following the works of the authors [1, 6, 9, 10], for \( p = \{ p_k \} \) is a bounded sequence of a strictly positive real numbers, we now define the new sequence spaces \( \nu(u, v; p) \) for \( \nu \in \{ w(p), \nu_0(p), w_\nu(p) \} \) by

\[ \nu(u, v; p) = \{ x = (x_k) \in \alpha : \left( \sum_{k=1}^{n} u_n v_k x_k \right) \in \nu \} \]

(2.2)

We may write, using (2.1),

\[ \nu(u, v; p) = \nu_{G(u, v)} \; \text{for} \; \nu \in \{ w(p), \nu_0(p), w_\nu(p) \} \]

If \( p_k = 1 \) for all \( k \in \mathbb{N} \), we write \( \nu(u, v) \) instead of \( \nu(u, v; p) \)

We shall first establish following some simple properties.

**Proposition 2.1**

The sequence spaces \( \nu(u, v; p) \) are complete paranorm space paramormed by

\[ h(x) = \sup_{n \in \mathbb{N}} \left\{ \frac{1}{n} \sum_{k=1}^{n} |u_k v_k x_k|^{p_k} \right\}^{\frac{1}{p_k}} \; \text{or equivalently} \; g(x) = \sup_{n \in \mathbb{N}} \left\{ 2^{-r} \sum_{r} |u_n v_k x_k|^{p_k} \right\}^{\frac{1}{M}} \]

where \( \sum_{r} \) is the sum over \( r \) in the range \( 2^r \leq k < 2^{r+1} \). For the space \( w(u, v; p) \), \( h(x) \) is a paranorm if and only if \( 0 < \inf p_k \leq \sup p_k < \infty \).

**Proof:** The proof of this proposition follows from the similar arguments as in the theorems 5, 6 in [4] and theorem 2.1 in [9]. If \( \{ x^n \} \) is a Cauchy sequence in \( \nu(u, v; p) \); then \( \{ G(u, v)x^n \} \) is a Cauchy sequence in \( \nu \). Now it is a routine work to show \( \nu(u, v; p) \) is complete paranorm space under the usual paranorm.

**Proposition 2.2**

The sequence spaces \( \nu(u, v; p) \) are linearly isomorphic to \( \nu \in \{ w(p), \nu_0(p), w_\nu(p) \} \).

Proof: We define the transformation

\[ T : \nu(u, v; p) \rightarrow \nu \; \text{by} \]

\[ x \mapsto y = T(x) \]. Linearity of \( T \) is obvious. Further, if \( Tx = \theta \), then \( x = \theta \). Hence \( T \) is injective. Now, let \( y \in \nu \) and define the sequence \( x = (x_k) \) by \( x_k = \frac{1}{v_k} \left( \frac{y_k}{u_k} - \frac{y_{k-1}}{u_{k-1}} \right) \; k \in \mathbb{N} \).

Then, \( h(x) = \sup_{n \in \mathbb{N}} \left\{ \frac{1}{n} \sum_{k=1}^{n} |u_k v_k x_k|^{p_k} \right\}^{\frac{1}{p_k}} \)

\[ = \sup_{n \in \mathbb{N}} \left\{ \frac{1}{n} \sum_{k=1}^{n} |y_k|^{p_k} \right\}^{\frac{1}{M}} \]

\[ = g(y) < \infty \].

Thus, we deduce that \( x \in \nu(u, v; p) \) and as a consequence we conclude that \( T \) is surjective and is a paranorm preserving. Hence \( T \) is a linear bijection and showing that the sequence spaces \( \nu(u, v; p) \) are linearly isomorphic to \( \nu \).
III. Duals

In this section we find \( \beta \)-dual of \( w(u,v; p) \). If \( X \) be a sequence space , we define \( \beta \)-dual of \( X \) as:

\[
X^\beta = \{ a = (a_k) : \sum_{k=1}^{\infty} a_kx_k \text{ is convergent for each } x \in X \}.
\]

**Theorem 3.1**

Let \( 0 < p_k \leq 1 \) for every \( k \in \mathbb{N} \). Then \( w^\beta (u,v; p) = \Gamma \) where

\[
\Gamma = \left\{ a = (a_k) : \sum_{k=1}^{\infty} \frac{1}{v_k} \left[ \frac{(2^r N^{-1})^{\frac{1}{r_k}}}{u_k} - \frac{(2^s N^{-1})^{\frac{1}{r_{k-1}}}}{u_{k-1}} \right] \text{ converges and } \lim_{m \to \infty} \frac{1}{u_m v_m} | a_m | = O(1) \right\}
\]

**Proof:** We first assume that the conditions hold. Let \( a \in \Gamma \) and \( x \in w(u,v; p) \). Then for \( y \in w(p) \), there exists a positive integer \( N > 1 \) such that

\[
\frac{1}{n} \sum_{k=1}^{n} |y_k|^p < \infty
\]

or equivalently \( \frac{1}{2} \sum_{r} |y_k|^p < \infty \), where sum over \( r \) runs from \( 2r \leq k < 2r+1 \). It follows that,

\[
|y_k| \leq (2^r N^{-1})^{\frac{1}{r} - \frac{1}{p}}. \quad \text{Now,}
\]

\[
\sum_{k=1}^{m} a_k x_k \leq \sum_{k=1}^{m} a_k \left[ \frac{1}{v_k} \left( \frac{y_k}{u_k} - \frac{y_{k-1}}{u_{k-1}} \right) \right] + \frac{1}{u_m v_m} | a_m | | y_m |
\]

\[
\leq \sum_{r} a_k \frac{(2^r N^{-1})^{\frac{1}{r_k}}}{u_k} - \frac{(2^s N^{-1})^{\frac{1}{r_{k-1}}}}{u_{k-1}} + \frac{1}{u_m v_m} (2^r N^{-1})^{\frac{1}{r}}
\]

\[
< \infty.
\]

It follows that \( \sum_{k=1}^{\infty} a_k x_k \) converges for each \( x \in w(u,v; p) \).

Hence, \( \Gamma \subseteq w^\beta (u,v; p) \).

On the other hand, let \( a \in w^\beta (u,v; p) \). Then \( \sum_{k=1}^{\infty} a_k x_k \) converges for each \( x \in w(u,v; p) \). Since,

\[
x(1) = \left[ \frac{1}{v_k} \left( \frac{(2^r N^{-1})^{\frac{1}{r_k}}}{u_k} - \frac{(2^s N^{-1})^{\frac{1}{r_{k-1}}}}{u_{k-1}} \right) \right] \in w(u,v; p)
\]

it follows that

\[
\sum_{k=1}^{\infty} a_k \left[ \frac{1}{v_k} \left( \frac{(2^r N^{-1})^{\frac{1}{r_k}}}{u_k} - \frac{(2^s N^{-1})^{\frac{1}{r_{k-1}}}}{u_{k-1}} \right) \right] \text{ converges. We need to show that}
\]
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\[
\lim_{m \to \infty} \left( 2^{-N^{-1}} \right)^{\frac{1}{r_m}} \frac{a_m}{u_m v_m} = O(1)
\]

As a contrary let,

\[
\lim_{n \to \infty} \left( 2^{-N^{-1}} \right)^{\frac{1}{r_n}} \frac{a_m}{u_m v_m} \neq O(1)
\]

, which is immediately against the fact that \( \sum_{k=1}^{\infty} a_k x_k \) converges for each \( x \in w(u, v; p) \) and \( \sum_{k=1}^{\infty} a_k \left[ \frac{1}{v_k} \left( \left( 2^{-N^{-1}} \right)^{\frac{1}{r_k}} \frac{u_k}{u_{k-1}} - \left( 2^{-N^{-1}} \right)^{\frac{1}{r_{k-1}}} \right) \right] \) converges.

Hence we must have,

\[
\lim_{m \to \infty} \left( 2^{-N^{-1}} \right)^{\frac{1}{r_m}} \frac{a_m}{u_m v_m} = O(1)
\]

So, we arrive at the result \( w^\beta(u, v; p) \subseteq \Gamma \); thereby proving \( w^\beta(u, v; p) = \Gamma \).

IV. Matrix Transformation

In this section we give characterization for the matrix classes \( (w(u, v; p), l_\infty), (w(u, v; p), c) \) and \( (w(u, v; p), c_0) \).

**Theorem 4.1**

Let \( 0 < p_k \leq 1 \) for every \( k \in \mathbb{N} \). Then \( A \in (w(u, v; p), l_\infty) \) if and only if

i) there exists an integer \( N > 1 \) such that

\[
\sup_N \sum_r \max_r \left[ a_{nk} \left\{ \frac{1}{v_k} \left( \left( 2^{-N^{-1}} \right)^{\frac{1}{r_k}} \frac{u_k}{u_{k-1}} - \left( 2^{-N^{-1}} \right)^{\frac{1}{r_{k-1}}} \right) \right\} \right] < \infty \text{ and}
\]

ii) \( \lim_{m \to \infty} \left( 2^{-N^{-1}} \right)^{\frac{1}{r_m}} \frac{a_{nm}}{u_m v_m} = O(1) \)

**Proof:** Let the conditions be satisfied. Since,

\[
\sum_{k=1}^{m} a_{nk} x_k = \sum_{k=1}^{m-1} a_{nk} \left[ \frac{1}{v_k} \left( \frac{y_k}{u_k} - \frac{y_{k-1}}{u_{k-1}} \right) \right] + \frac{y_m}{u_m v_m} a_{nm}
\]

\[
\leq \left| \sum_{k=1}^{m-1} a_{nk} \left[ \frac{1}{v_k} \left( \frac{y_k}{u_k} - \frac{y_{k-1}}{u_{k-1}} \right) \right] \right| + \left| \frac{a_{nm}}{u_m v_m} \right| \left| y_m \right|
\]

\[
\therefore \sum_{k=1}^{m} a_{nk} x_k \leq \sum_r \max_r \left[ a_{nk} \left\{ \frac{1}{v_k} \left( \left( 2^{-N^{-1}} \right)^{\frac{1}{r_k}} \frac{u_k}{u_{k-1}} - \left( 2^{-N^{-1}} \right)^{\frac{1}{r_{k-1}}} \right) \right\} \right] + \left| \frac{a_{nm}}{u_m v_m} \right| \left( 2^{-N^{-1}} \right)^{\frac{1}{r_m}}
\]
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\[
\leq \sup_n \sum_{r} \max_{k} \left| a_{nk} \left( \frac{1}{v_k} \left( \frac{1}{u_k} \frac{1}{v_{k-1}} \right) - \frac{1}{u_{k-1}} \right) \right| + \left( 2^{-1} \right) \frac{a_{nn}}{u_n v_n} < \infty ,
\]

by using conditions (i) and (ii).

It follows that \( A_n \in \Gamma \) and hence \( \sum_{k=1}^n a_{nk} x_k = A_n(x) \) converges for each \( x \in w(u, v; p) \) and \( n \in \mathbb{N} \).

Thus \( Ax \in L_\infty \).

On the other hand, let \( A \in (w(u, v; p), L_\infty) \). Since

\[
\left\{ \frac{1}{v_k} \left( \frac{1}{u_k} \frac{1}{v_{k-1}} \right) - \frac{1}{u_{k-1}} \right\} \in w(u, v; p),
\]

the condition (i) holds. In order to see that condition (ii) is necessary, we assume that for \( N > 1 \),

\[
\lim_{m \to \infty} \left( 2^{-1} \right) \frac{a_{nn}}{u_n v_n} \neq O(1),
\]

that is, \( \left\{ \frac{1}{v_k} \left( \frac{1}{u_k} \frac{1}{v_{k-1}} \right) \right\} \notin L_\infty \).

Now, therefore, there exists a sequence \( \{N_j\} \to \infty \) such that

\[
\sup_n \sum_{r} \max_{k} \left| a_{nk} \left( \frac{1}{v_k} \left( \frac{1}{u_k} \frac{1}{v_{k-1}} \right) - \frac{1}{u_{k-1}} \right) \right| = o(1) \text{ and}
\]

\[
\lim_{m \to \infty} \left( 2^{-1} \right) \frac{a_{nn}}{u_n v_n} = o(1).
\]

Hence, \( x_k \mapsto o(w(u,v;p)) \) but \( x_k \mapsto l(w(u,v;p)) \). So, we arrive at the contradiction to our assumption \( A \in (w(u,v;p), L_\infty) \). Thus, condition (ii) is necessary; thereby completing the proof for the theorem.

By using the arguments as in theorem (4.1) it is straightforward matter to prove the following theorems:

**Theorem 4.2**

Let \( 0 < p_k \leq 1 \) for every \( k \in \mathbb{N} \). Then \( A \in (w(u,v;p), c) \) if and only if

i) there exists an integer \( N > 1 \) such that

\[
\sup_n \sum_{r} \max_{k} \left| a_{nk} \left( \frac{1}{v_k} \left( \frac{1}{u_k} \frac{1}{v_{k-1}} \right) - \frac{1}{u_{k-1}} \right) \right| < \infty \text{ and}
\]

\[
\left| \frac{a_{nn}}{u_n v_n} \right| < \infty.
\]
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ii) $\lim_{m \to \infty} \left\{ \left( 2^{N-1} \right)^{\frac{1}{r_n}} \frac{a_{nm}}{u_n v_m} \right\} = o(1)$

iii) $\lim_{n \to \infty} a_{nk} = \alpha_k$ exists for every fixed $k$.

**Theorem 4.3**

Let $0 < p_k \leq 1$ for every $k \in \mathbb{Z}$. Then $A \in (w(u, v; p), c_0)$ if and only if

i) there exists an integer $N > 1$ such that

$$\sup_k \sum_r \max_{m \leq N} \left[ a_{nk} \left\{ \frac{1}{v_k} \left( 2^{N-1} \right)^{\frac{1}{r_k}} - \left( 2^{N-1} \right)^{\frac{1}{r_k+1}} \right\} \right] < \infty$$

ii) $\lim_{m \to \infty} \left\{ \left( 2^{N-1} \right)^{\frac{1}{r_n}} \frac{a_{nm}}{u_n v_m} \right\} = o(1)$ and

iii) $\lim_{n \to \infty} a_{nk} = \alpha_k$ with $\alpha_k = 0$ for every fixed $k$.

**References**