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Paranormed Sequence spaces $w(u,v;p)$, $w_0(u,v;p)$ and $w_\infty(u,v;p)$ generated by weighted mean

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Abstract: *The sequence spaces* $w(p), w_0(p)$ *and* $w_\infty(p)$ *were introduced and studied by I.J. Maddox [1,2,3,4]. In [8,9,10] , the authors have introduced sequence spaces* $c_0(u,v,p), c(u,v,p), l_\infty(u,v,p)$ *and* $l(u,v,p)$ *and established some properties. In this paper we introduce the sequence spaces* $w(u,v;p)$ *,* $w_0(u,v;p)$ *and* $w_\infty(u,v;p)$*; study some properties, find* β *- dual of* $w(u,v;p)$*. We also characterize the matrix classes* $\left(w(u,v;p), l_\infty\right)$*,* $\left(w(u,v;p), c\right)$ *and* $\left(w(u,v;p), c_0\right)$*.*

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I. Introduction

By ω we mean the space of all real valued sequences. A vector subspace of ω is called a sequence space. We shall write , with usual notation , l_∞, c and c_0 for the spaces of all bounded, convergent and null sequence respectively. A linear topological space X over the field ℝ is said to be a paramormed space if there is a sub-additive function $g: X \to \mathbb{R}$ such that $g(\theta)=0$, $g(x)=g(-x)$ and scalar multiplication is continuous i.e. $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$, where θ is the zero vector in the linear space X .

If $p = \{p_k\}$ be a bounded sequence of strictly positive real numbers, I.J. Maddox defined the sequence spaces $w(p), w_0(p)$ and $w_\infty(p)$ as:

$$w(p)=\left\{x=(x_k)\in\omega : \frac{1}{n}\sum_{k=1}^{n}|x_k - l|^{p_k} \to 0;\ for\ some\ l \in \mathbb{R}, n\to\infty\right\}$$

$$w_0(p)=\left\{x=(x_k)\in\omega : \frac{1}{n}\sum_{k=1}^{n}|x_k|^{p_k} \to 0,\quad n\to\infty\right\} \text{ and}$$

$$w_\infty(p)=\left\{x=(x_k)\in\omega : \frac{1}{n}\sum_{k=1}^{n}|x_k|^{p_k} < \infty,\quad n\to\infty\right\}.$$

The spaces $w(p)$ and $w_0(p)$ are paranormed spaces paranormed by

$$g(x)=\sup\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|^{p_k}\right)^{\frac{1}{M}} \text{ or equivalently } g(x)=\sup_r\left(2^{-r}\sum_r |x_k|^{p_k}\right)^{\frac{1}{M}} \quad (1.1)$$

where $\sum_r$ is the sum over the range $2^r \le r < 2^{r+1}$ and $M=(1,\sup p_k)$. Further $w_\infty(p)$ is the paranorm space paranormed by (1.1) if and only if $0 < \inf p_k \le \sup p_k < \infty$ [1].

Let X and Y be any two sequence spaces and $A=(a_{nk});\ n,k\in\mathbb{N}$ be infinite matrix of complex numbers a_{nk} . Then we say that A defines a matrix mapping X into Y ; and it is denoted by writing $A: X \to Y$ if for every sequence $x=(x_k)\in X$, the sequence $\left((Ax)_n\right)$ is in Y , where

$$(Ax)_n = \sum_{k=1}^{\infty} a_{nk} x_k \; ; \; (n \in \mathbb{N})$$
(1.2)

By (X,Y) we denote the class of all matrices A such that $A: X \to Y$. Thus, $A \in (X,Y)$ if and only if the series on right side of (1.2) converges for each $n \in \mathbb{N}$ and every $x \in X$; and we write,

$$Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in Y \text{ for all } x \in X .$$

We denote by U for the set of all sequences $u = (u_n)$ such that $u_n \neq 0$ for all $n \in \mathbb{N}$. For $u \in$ U, let $\frac{1}{u} = \left(\frac{1}{u_n}\right)$. Let us define the matrix $G(u,v) = (g_{nk})$ as:

$$g_{nk} = \begin{cases} u_n v_k; & 0 \le k \le n \\ 0; & k > n \end{cases}$$
(1.3)

for all $n,k \in \mathbb{N}$,where u_n depends only on n and v_k only on k. The matrix $G(u,v) = (g_{nk})$ is called generalized weighted mean or factorable matrix.

The main purpose of the present paper is to introduce the sequence spaces $w(u,v;p), w_0(u,v;p)$ and $w_\infty(u,v;p)$; which are the set of all sequences whose $G(u,v)$- transforms are in the spaces $w(p), w_0(p)$ and $w_\infty(p)$ respectively, where $G(u,v)$ denotes the matrix as defined in (1.3). We have discussed some topological properties of $w(u,v;p), w_0(u,v;p)$ and $w_\infty(u,v;p)$; investigated β –dual for the new space $w(u,v;p)$. Moreover we have characterized the matrix classes $\left(w(u,v;p), l_\infty\right)$, $\left(w(u,v;p), c\right)$ and $\left(w(u,v;p), c_0\right)$.

II. The paranormed sequence spaces $w(u,v;p)$, $w_0(u,v;p)$ and $w_\infty(u,v;p)$.

Before introducing these sequence spaces we would like to present some remarks. Malkowsky and Savas [10] have defined the sequence spaces Z(u,v,X) which consists of all sequences whose G(u,v)-transforms are in $X \in \{l_\infty, c, c_0, l(p)\}$ where $u,v \in U$. Chaudhary B. and Mishra S.K. [6] have defined the sequence space $\overline{l(p)}$ which consists of all sequences whose S- transforms are in $l(p)$;where $S = (s_{nk})$ is defined by $s_{nk} = \begin{cases} 1; & 0 \le k \le n \\ 0; & k > n \end{cases}$

Moreover I.J. Maddox [1] introduced the sequence space $w(p), w_0(p)$ and $w_\infty(p)$ which consists of all strongly summable , strongly summable to zero and bounded sequences respectively whose C-transforms are in the spaces $l(p), c_0(p)$ and $l_\infty(p)$ respectively ; where

$$C = (c_{nk}) = \begin{cases} \frac{1}{n}; & 1 \le k \le n \\ 0; & k > n \end{cases}$$

and $C = (c_{nk})$ is called the Ceasaro matrix of order 1 or the matrix of arithmetic mean.

The matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A = \{x = (x_k) \in \omega : Ax \in X\}$$
(2.1)

, which is a sequence space.

With the notation as in (2.1) , we can have the following representations:
$X(u,v,p)=[X]_Z$, for $X\in\{l_\infty,c,c_0,l(p)\}$
$\overline{l(p)}=[l(p)]_S$.
Following the works of the authors [1,6,9,10] , for $p=\{p_k\}$ is a bounded sequence of a strictly positive real numbers , we now define the new sequence spaces $\mu(u,v;p)$ for $\mu\in\{w(p),w_0(p),w_\infty(p)\}$ by

$$\mu(u,v;p)=\left\{x=(x_k)\in\omega:\left(\sum_{k=1}^{n}u_nv_kx_k\right)\in\mu\right\}$$

(2.2)
We may write, using (2.1),
$\mu(u,v;p)=[\mu]_{G(u,v)}$; for $\mu\in\{w(p),w_0(p),w_\infty(p)\}$
If $p_k=1$ for all $k\in\mathbb{N}$, we write $\mu(u,v)$ instead of $\mu(u,v;p)$
We shall first establish following some simple properties.

Proposition 2.1
The sequence spaces $\mu(u,v;p)$ are complete paranorm space paramormed by

$$h(x)=\sup_{n\in\mathbb{N}}\left\{\frac{1}{n}\sum_{k=1}^{n}|u_nv_kx_k|^{p_k}\right\}^{\frac{1}{M}};\ \text{or equivalently}\ g(x)=\sup_r\left(2^{-r}\sum_r|u_nv_kx_k|^{p_k}\right)^{\frac{1}{M}}$$

where $\sum\limits_r$ is the sum over r in the range $2^r\le k<2^{r+1}$. For the space $w_\infty(u,v;p)$, $h(x)$ is a paranorm if and only if $0<\inf p_k\le\sup p_k<\infty$.

Proof: The proof of this proposition follows from the similar arguments as in the theorems 5,6 in [4] and theorem 2.1 in [9]. If $\{x^n\}$ is a Cauchy sequence in $\mu(u,v;p)$; then $\{G(u,v)x^n\}$ is a Cauchy sequence in μ. Now it is a routine work to show $\mu(u,v;p)$ is complete paranorm space under the usual paranorm.

Proposition 2.2
The sequence spaces $\mu(u,v;p)$ are linearly isomorphic to $\mu\in\{w(p),w_0(p),w_\infty(p)\}$.
Proof: We define the transformation
$T:\mu(u,v;p)\to\mu$ by,
$x\mapsto y=T(x)$. Linearity of T is obvious. Further, if $Tx=\theta$, then $x=\theta$. Hence T is injective. Now, let $y\in\mu$ and define the sequence $x=(x_k)$ by $x_k=\frac{1}{v_k}\left\{\frac{y_k}{u_k}-\frac{y_{k-1}}{u_{k-1}}\right\};\quad k\in\mathbb{N}$.

$$\text{Then, } h(x)=\sup_{n\in\mathbb{N}}\left\{\frac{1}{n}\sum_{k=1}^{n}|u_nv_kx_k|^{p_k}\right\}^{\frac{1}{M}}$$
$$=\sup_{n\in\mathbb{N}}\left\{\frac{1}{n}\sum_{k=1}^{n}|y_k|^{p_k}\right\}^{\frac{1}{M}}$$
$$=g(y)$$
$$<\infty.$$

Thus , we deduce that $x\in\mu(u,v;p)$ and as a consequence we conclude that T is surjective and is a paranorm preserving. Hence T is a linear bijection and showing that the sequence spaces $\mu(u,v;p)$ are linearly isomorphic to μ.

III. Duals

In this section we find β - dual of $w(u.v;p)$. If X be a sequence space , we define $\beta-$dual of X as:

$$X^{\beta}=\left\{a=(a_k):\sum_{k=1}^{\infty}a_kx_k \ \ is\,convergent\ \ for\,each\,x\in X\right\}.$$

Theorem 3.1

Let $0<p_k\le 1$for every $k\in\mathbb{N}$. Then $w^{\beta}(u.v;p)=\Gamma$ where

$$\Gamma=\left\{a=(a_k):\sum_r a_k\left[\frac{1}{v_k}\left(\frac{\left(2^rN^{-1}\right)^{\frac{1}{p_k}}}{u_k}-\frac{\left(2^rN^{-1}\right)^{\frac{1}{p_{k-1}}}}{u_{k-1}}\right)\right]convrges\,and\,\lim_{m\to\infty}\left(2^rN^{-1}\right)^{\frac{1}{p_m}}\frac{a_m}{u_mv_m}=O(1)\right\}$$

.

Proof : We first assume that the conditions hold. Let $a\in\Gamma$ and $x\in w(u,v;p)$.Then for $y\in w(p)$, there exists a positive integer $N>1$Such that

$$\frac{1}{n}\sum_{k=1}^{n}\left|y_k\right|^{p_k}<\infty$$

or equivalently $\frac{1}{2^r}\sum_r\left|y_k\right|^{p_k}<\infty$, where sum over r runs from $2^r\le k<2^{r+1}$. It follows that ,

$\left|y_k\right|\le\left(2^rN^{-1}\right)^{\frac{1}{p_k}}$. Now,

$$\left|\sum_{k=1}^{m}a_kx_k\right|=\left|\sum_{k=1}^{m-1}a_k\left[\frac{1}{v_k}\left(\frac{y_k}{u_k}-\frac{y_{k-1}}{u_{k-1}}\right)\right]+\frac{a_my_m}{u_mv_m}\right|$$

$$\le\left|\sum_{k=1}^{m-1}\frac{a_k}{v_k}\left(\frac{y_k}{u_k}-\frac{y_{k-1}}{u_{k-1}}\right)\right|+\left|\frac{a_m}{u_mv_m}\right|\left|y_m\right|$$

$$\le\sum_r\frac{a_k}{v_k}\left|\frac{\left(2^rN^{-1}\right)^{\frac{1}{p_k}}}{u_k}-\frac{\left(2^rN^{-1}\right)^{\frac{1}{p_{k-1}}}}{u_{k-1}}\right|+\left|\frac{a_m}{u_mv_m}\right|\left(2^rN^{-1}\right)^{\frac{1}{p_m}}$$

$$<\infty.$$

It follows that $\sum_{k=1}^{\infty}a_kx_k$ converges for each $x\in w(u,v;p)$.

Hence, $\Gamma\subseteq w^{\beta}(u.v;p)$.

On the other hand , let $a\in w^{\beta}(u,v;p)$. Then , $\sum_{k=1}^{\infty}a_kx_k$ converges for each $x\in w(u,v;p)$. Since ,

$$x=\left\{\frac{1}{v_k}\left(\frac{\left(2^rN^{-1}\right)^{\frac{1}{p_k}}}{u_k}-\frac{\left(2^rN^{-1}\right)^{\frac{1}{p_{k-1}}}}{u_{k-1}}\right)\right\}\in w(u,v;p)$$

;it follows that

$$\sum_{k=1}^{\infty}a_k\left[\frac{1}{v_k}\left(\frac{\left(2^rN^{-1}\right)^{\frac{1}{p_k}}}{u_k}-\frac{\left(2^rN^{-1}\right)^{\frac{1}{p_{k-1}}}}{u_{k-1}}\right)\right]$$

converges. We need to show that

$$\lim_{m\to\infty}\left(2^r N^{-1}\right)^{\frac{1}{p_m}} \frac{a_m}{u_m v_m} = O(1)$$

As a contrary let,

$\lim_{n\to\infty}\left(2^r N^{-1}\right)^{\frac{1}{p_m}} \frac{a_m}{u_m v_m} \neq O(1)$, which is immediately against the fact that $\sum_{k=1}^{\infty} a_k x_k$ converges for each $x \in w(u,v;p)$ and $\sum_{k=1}^{\infty} a_k \left[\frac{1}{v_k}\left(\frac{\left(2^r N^{-1}\right)^{\frac{1}{p_k}}}{u_k} - \frac{\left(2^r N^{-1}\right)^{\frac{1}{p_{k-1}}}}{u_{k-1}}\right)\right]$ converges.

Hence we must have, $\lim_{m\to\infty}\left(2^r N^{-1}\right)^{\frac{1}{p_m}} \frac{a_m}{u_m v_m} = O(1)$

So, we arrive at the result $w^\beta(u.v;p) \subseteq \Gamma$; thereby proving $w^\beta(u.v;p) = \Gamma$.

IV. Matrix Transformation

In this section we give characterization for the matrix classes $(w(u,v;p), l_\infty)$, $(w(u,v;p), c)$ and $(w(u,v;p), c_0)$.

Theorem 4.1

Let $0 < p_k \le 1$ for every $k \in \mathbb{N}$. Then $A \in (w(u,v;p), l_\infty)$ if and only if

i) there exists an integer $N > 1$ such that

$$\sup_n \sum_r \max_r \left[a_{nk} \left\{ \frac{1}{v_k}\left(\frac{\left(2^r N^{-1}\right)^{\frac{1}{p_k}}}{u_k} - \frac{\left(2^r N^{-1}\right)^{\frac{1}{p_{k-1}}}}{u_{k=1}}\right)\right\}\right] < \infty \text{ and}$$

ii) $$\lim_{m\to\infty}\left\{\left(2^r N^{-1}\right)^{\frac{1}{p_m}} \frac{a_{nm}}{u_m v_m}\right\}_{n\in\mathbb{N}} = O(1)$$

Proof: Let the conditions be satisfied. Since,

$$\left|\sum_{k=1}^{m} a_{nk} x_k\right| = \left|\sum_{k=1}^{m-1} a_{nk}\left[\frac{1}{v_k}\left(\frac{y_k}{u_k} - \frac{y_{k-1}}{u_{k-1}}\right)\right] + \frac{y_m}{u_m v_m} a_{nm}\right|$$

$$\le \left|\sum_{k=1}^{m-1} a_{nk}\left\{\frac{1}{v_k}\left(\frac{y_k}{u_k} - \frac{y_{k-1}}{u_{k-1}}\right)\right\}\right| + \left|\frac{a_m}{u_m v_m}\right| |y_m|$$

$$\therefore \sum_{k=1}^{\infty} |a_{nk} x_k| \le \sum_r \max_r \left| a_{nk}\left\{\frac{1}{v_k}\left(\frac{\left(2^r N^{-1}\right)^{\frac{1}{p_k}}}{u_k} - \frac{\left(2^r N^{-1}\right)^{\frac{1}{p_{k-1}}}}{u_{k-1}}\right)\right\}\right| + \left|\frac{a_{nm}}{u_m v_m}\right| \left(2^r N^{-1}\right)^{\frac{1}{p_m}}$$

$$\le \sup_n \sum_r \max_r \left| a_{nk} \left\{ \frac{1}{v_k} \left(\frac{\left(2^r N^{-1}\right)^{\frac{1}{p_k}}}{u_k} - \frac{\left(2^r N^{-1}\right)^{\frac{1}{p_{k-1}}}}{u_{k-1}} \right) \right\} \right| + \left(2^r N^{-1}\right)^{\frac{1}{p_m}} \left| \frac{a_{nm}}{u_m v_m} \right|$$

$< \infty$, by using conditions (i) and (ii).

It follows that $A_n \in \Gamma$ and hence $\sum_{k=1}^{\infty} a_{nk} x_k = A_n(x)$ converges for each $x \in w(u,v;p)$ and $n \in \mathbb{N}$. Thus $Ax \in l_\infty$.

On the other hand , let $A \in (w(u,v;p), l_\infty)$. Since ,

$\left\{ \frac{1}{v_k} \left(\frac{\left(2^r N^{-1}\right)^{\frac{1}{p_k}}}{u_k} - \frac{\left(2^r N^{-1}\right)^{\frac{1}{p_{k-1}}}}{u_{k-1}} \right) \right\} \in w(u,v;p)$, the condition (i) holds. In order to see that condition (ii) is necessary, we assume that for $N > 1$,

$$\lim_{m\to\infty} \left\{ \left(2^r N^{-1}\right)^{\frac{1}{p_m}} \frac{a_{nm}}{u_m v_m} \right\}_{n\in\mathbb{N}} \ne O(1),$$

that is, $\left\{ \left(2^r N^{-1}\right)^{\frac{1}{p_m}} \frac{a_{nm}}{u_m v_m} \right\}_{n\in\mathbb{N}} \notin l_\infty$.

Now, therefore, there exists a sequence $\{N_r\} \to \infty$ such that

$$\sup_n \sum_r \max_r \left[a_{nk} \left\{ \frac{1}{v_k} \left(\frac{\left(2^r N_r^{-1}\right)^{\frac{1}{p_k}}}{u_k} - \frac{\left(2^r N_r^{-1}\right)^{\frac{1}{p_{k-1}}}}{u_{k-1}} \right) \right\} \right] = o(1) \text{ and}$$

$$\lim_{m\to\infty} \left\{ \left(2^r N_r^{-1}\right)^{\frac{1}{p_m}} \frac{a_{nm}}{u_m v_m} \right\}_{n\in\mathbb{N}} = o(1).$$

Hence , $x_k \mapsto o(w(u,v;p))$ but $x_k \mapsto l(w(u,v;p))$. So, we arrive at the contradiction to our assumption $A \in (w(u,v;p), l_\infty)$. Thus , condition (ii) is necessary ; thereby completing the proof for the theorem.

By using the arguments as in theorem (4.1) it is straight forward matter to prove the following theorems:

Theorem 4.2

Let $0 < p_k \le 1$ for every $k \in \mathbb{N}$. Then $A \in (w(u,v;p), c)$ if and only if

i) there exists an integer $N > 1$ such that

$$\sup_n \sum_r \max_r \left[a_{nk} \left\{ \frac{1}{v_k} \left(\frac{\left(2^r N^{-1}\right)^{\frac{1}{p_k}}}{u_k} - \frac{\left(2^r N^{-1}\right)^{\frac{1}{p_{k-1}}}}{u_k} \right) \right\} \right] < \infty \text{ and}$$

ii) $\lim_{m\to\infty}\left\{\left(2^r N^{-1}\right)^{\frac{1}{p_m}} \frac{a_{nm}}{u_m v_m}\right\}_{n\in\mathbb{N}} = o(1)$

iii) $\lim_{n\to\infty} a_{nk} = \alpha_k$ exists for every fixed k.

Theorem 4.3

Let $0 < p_k \le 1$ for every $k \in \mathbb{N}$. Then $A \in (w(u,v;p), c_0)$ if and only if

i) there exists an integer $N > 1$ such that

$$\sup_n \sum_r \max_r \left[a_{nk} \left\{ \frac{1}{v_k} \left(\frac{\left(2^r N^{-1}\right)^{\frac{1}{p_k}}}{u_k} - \frac{\left(2^r N^{-1}\right)^{\frac{1}{p_{k-1}}}}{u_k} \right) \right\} \right] < \infty$$

ii) $\lim_{m\to\infty}\left\{\left(2^r N^{-1}\right)^{\frac{1}{p_m}} \frac{a_{nm}}{u_m v_m}\right\}_{n\in\mathbb{N}} = o(1)$ and

iii) $\lim_{n\to\infty} a_{nk} = \alpha_k$ with $\alpha_k = 0$ for every fixed k.

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