

## Quasi-Static Transient Thermal Stresses in a Dirichlet's thin Solid Cylinder with Internal Moving Heat Source

<sup>1</sup>D. T. Solanke, <sup>2</sup>M. H. Durge

<sup>1</sup>Sudhakar Naik and Umashankar Khetan college Akola, Maharashtra State, India.

<sup>2</sup>Anand Niketan Collge, Warora, Maharashtra State, India.

**Abstract:** This paper concern with transient non-homogeneous thermoelastic problem with Dirichlet's boundary condition in thin solid cylinder of isotropic material of radius  $a$ , and height  $h$ , occupying the space  $0 \leq r \leq a$ ,  $-\frac{h}{2} \leq z \leq \frac{h}{2}$ ,  $0 \leq \phi \leq 2\pi$  having initial temperature  $f(r, \phi, z)$  placed in an ambient temperature zero. The cylinder is subjected to the activity of moving heat source along circular trajectory of radius  $r_0$ , where  $0 < r_0 < a$ , around the centre of the cylinder with constant angular velocity  $\omega$ . Heat conduction equation containing heat generation term is solved by applying integral transform technique and Green's theorem is adopted in deducing the solution of heat conduction equation. The solution is obtained in a series form of Bessel function and trigonometric function and derived thermal stresses.

**Keywords:** Dirichlet's solid thin cylinder, moving heat source, thermal stresses, Green's theorem.

### I. Introduction:

During the second half of 20<sup>th</sup> century, non-isothermal problems of the theory of elasticity became increasingly important. This is due to their wide application in diverse fields. The high velocities of modern aircraft give rise to aerodynamic heating, which produces intense thermal stresses that reduce the strength of aircraft structure.

In this present paper we determine temperature, thermal stresses, in a Dirichlet's thin solid cylinder, determined by  $0 \leq r \leq a$ ,  $-\frac{h}{2} \leq z \leq \frac{h}{2}$ ,  $0 \leq \phi \leq 2\pi$  with internal moving heat source. Heat conduction equation with heat generation term is solved by applying integral transform technique and Green's theorem. Solution is obtained in series form of Bessel function and trigonometric function. The direct problems is very important in view of its relevance to various industrial mechanics subjected to heating such as main shaft of lathe, turbines and the role of rolling mill for base of furnace boiler of thermal power plant, gas power plant and measurement of aerodynamic heating.

### II. Formulation of the Heat conduction problem:

Consider a thin solid cylinder of isotropic material of radius  $a$  and thickness  $h$  occupying the space  $0 \leq r \leq a$ ,  $-\frac{h}{2} \leq z \leq \frac{h}{2}$ ,  $0 \leq \phi \leq 2\pi$  having initial temperature  $f(r, \phi, z)$  placed in an ambient temperature zero. The cylinder is subjected to the activity of moving heat source which changes its place along circular trajectory of radius  $r_0$ , where  $0 < r_0 < a$ , around the centre of the cylinder with constant angular velocity  $\omega$ . The activity of moving heat source and initial temperature of the cylinder may cause the generation of heat due to nuclear interaction that may be a function of position and time in the form  $g(r, \phi, z, t)$  w/s<sup>3</sup>. The temperature distribution of the solid cylinder is described by the differential equation of heat conduction with heat generation term as in [2] page no. 232 is given by

$$\nabla^2 T + \frac{1}{k} g = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

Where  $T = T(r, \phi, z)$  is temperature distribution,  $k$  is thermal conductivity of the material of the cylinder,

$\alpha = \frac{k}{\rho C_p}$  is thermal diffusivity,  $\rho$  is density,  $C_p$  is specific heat of the material and  $g$  is the volumetric

energy generation term in the cylinder. Where  $\nabla^2$  is Laplacian operator in cylindrical coordinates and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

Now consider an instantaneously moving heat source  $g_s^i$  located at a point  $(r_0, \phi', \xi)$  and releasing its energy spontaneously at time  $\tau$ . Such volumetric heat source in cylindrical coordinates is given by

$$g(r, \phi, z, t) = g_s^i \frac{1}{2\pi r} \delta(r - r_0) \delta(\phi - \phi') \delta(z - \xi) \delta(t - \tau)$$

Hence above equation reduces to

$$\nabla^2 T + g_s^i \frac{1}{2\pi kr} \delta(r - r_0) \delta(\phi - \phi') \delta(z - \xi) \delta(t - \tau) = \frac{1}{\alpha} \frac{\partial T}{\partial t} \tag{3.1}$$

$$\phi' = \omega t \tag{3.2}$$

With initial and homogeneous boundary conditions,

$$T = 0 \text{ at } r = a \tag{3.3}$$

$$T = 0 \text{ at } z = -\frac{h}{2} \tag{3.4}$$

$$T = 0 \text{ at } z = \frac{h}{2} \tag{3.5}$$

$$T = f(r, \phi, z) \text{ at } t = 0 \tag{3.6}$$

### III. Formulation of thermoelastic Problem:

Let us introduce a thermal stress function  $\chi$  related to component of stress in the cylindrical coordinates system as in [3]

$$\sigma_{rr} = \frac{1}{r} \frac{\partial \chi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \chi}{\partial r^2} \tag{4.1}$$

$$\sigma_{\phi\phi} = \frac{\partial^2 \chi}{\partial r^2} \tag{4.2}$$

$$\sigma_{r\phi} = -\frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \chi}{\partial \phi} \right) \tag{4.3}$$

The boundary condition for a traction free body are

$$\sigma_{rr} = 0, \sigma_{r\phi} = 0 \text{ at } r = a \tag{4.4}$$

$$\text{Where } \chi = \chi_c + \chi_p \tag{4.5}$$

Where  $\chi_c$  is complementary solution and  $\chi_p$  is particular solution and

$$\chi_c \text{ satisfies the equation } \nabla^4 \chi_c = 0 \tag{4.6}$$

$$\chi_p \text{ satisfies the equation } \nabla^4 \chi_p = -\lambda E \nabla^2 \Gamma \tag{4.7}$$

Where  $\Gamma$  is temperature change  $\Gamma = T - T_i$ ,  $T_i$  is initial temperature

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \text{ since cylinder is thin, } z \text{ component is negligible}$$

### IV. Solution of the Heat conduction problem:

We define integral transform of  $T(r, \phi, z)$  by

$$\bar{T} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^{2\pi} \int_0^a T(r, \phi, z) r J_\nu(\beta_m r) \cos \nu(\phi - \phi') \sin \alpha_n(z + \frac{h}{2}) dr d\phi dz \tag{5.1}$$

And its inverse by

$$T = \frac{4}{\pi h a^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{\nu=0}^{\infty} \bar{T} \frac{J_\nu(\beta_m r) \cos \nu(\phi - \phi') \sin \alpha_n(z + \frac{h}{2})}{[J_{\nu+1}(\beta_m a)]^2} \tag{5.2}$$

$$\text{Where } \nu = 0, 1, 2, 3, \dots \tag{5.3}$$

$$\beta_m \text{ Is root of the transcendental equation } J_\nu(\beta_m a) = 0 \tag{5.4}$$

$$\alpha_n = \frac{p\pi}{h}, \quad p = 0, 1, 2, 3, \dots \quad (5.5)$$

By taking integral transform of equation (3.1) and using following Green's theorem

$$\int_R \nabla^2 T \psi_k dv = \int_R T \nabla^2 \psi_k dv + \sum_{i=1}^N \int_{s_i} \left[ \psi_k \frac{\partial T}{\partial n_i} - T \frac{\partial \psi_k}{\partial n_i} \right] ds_i \quad (5.6)$$

Which yield as

$$\frac{d\bar{T}}{dt} + \alpha(\beta_m^2 + \alpha_n^2)\bar{T} = g_s^i \frac{\alpha}{2\pi k} J_\nu(\beta_m r_0) \sin \alpha_n \left(\xi + \frac{h}{2}\right) \delta(t - \tau) \quad (5.7)$$

This is linear differential equation of first order. By solving this we arrive at

$$\bar{T} = \left[ \bar{f}(\beta_m, \nu, \alpha_n) + \frac{\alpha}{2\pi k} g_s^i J_\nu(\beta_m r_0) \sin \alpha_n \left(\xi + \frac{h}{2}\right) e^{\alpha(\beta_m^2 + \alpha_n^2)\tau} \right] e^{-\alpha(\beta_m^2 + \alpha_n^2)t} \quad (5.8)$$

Where 
$$\bar{f} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^{2\pi} \int_0^a f(r, \phi, z) r J_\nu(\beta_m r) \cos \nu(\phi - \phi') \sin \alpha_n \left(z + \frac{h}{2}\right) dr d\phi dz \quad (5.9)$$

Taking inverse integral transform we finally arrive at

$$T = \frac{4}{\pi h a^2} \sum_{m,n,\nu=0}^{\infty} \frac{J_\nu(\beta_m r) \cos \nu(\phi - \phi') \sin \alpha_n \left(z + \frac{h}{2}\right)}{[J_{\nu+1}(\beta_m a)]^2} \left[ \bar{f}(\beta_m, \nu, \alpha_n) + \frac{\alpha}{2\pi k} g_s^i J_\nu(\beta_m r_0) \sin \alpha_n \left(\xi + \frac{h}{2}\right) e^{\alpha(\beta_m^2 + \alpha_n^2)\tau} \right] e^{-\alpha(\beta_m^2 + \alpha_n^2)t} \quad (5.10)$$

$$\Gamma = \frac{4}{\pi h a^2} \sum_{m,n,\nu=0}^{\infty} \frac{J_\nu(\beta_m r) \cos \nu(\phi - \phi') \sin \alpha_n \left(z + \frac{h}{2}\right)}{[J_{\nu+1}(\beta_m a)]^2} \left[ \bar{f}(\beta_m, \nu, \alpha_n) + \frac{\alpha}{2\pi k} g_s^i J_\nu(\beta_m r_0) \sin \alpha_n \left(\xi + \frac{h}{2}\right) e^{\alpha(\beta_m^2 + \alpha_n^2)\tau} \right] \left[ e^{-\alpha(\beta_m^2 + \alpha_n^2)t} - 1 \right] \quad (5.11)$$

### V. Solution of Thermoelastic Problem:

Let suitable form of  $\chi_c$  satisfying equation (4.6) be

$$\chi_c = \sum_{\nu=0}^{\infty} \left( A r^{\nu+2} + B r^{-\nu+2} \right) \cos \nu(\phi - \phi') + \left( C r^{\nu+2} + D r^{-\nu+2} \right) \sin \nu(\phi - \phi') \quad (6.1)$$

Let suitable form of  $\chi_p$  satisfying equation (4.7) be

$$\chi_p = \sum_{m,n,\nu=0}^{\infty} \frac{\lambda E}{\beta_m^2} J_\nu(\beta_m r) \cos \nu(\phi - \phi') \sin \alpha_n \left(z + \frac{h}{2}\right) \left[ \bar{f}(\beta_m, \nu, \alpha_n) + \frac{\alpha}{2\pi k} g_s^i J_\nu(\beta_m r_0) \sin \alpha_n \left(\xi + \frac{h}{2}\right) e^{\alpha(\beta_m^2 + \alpha_n^2)\tau} \right] \left[ e^{-\alpha(\beta_m^2 + \alpha_n^2)t} - 1 \right] \quad (6.2)$$

$\chi = \chi_c + \chi_p$  gives

$$\chi = \sum_{\nu=0}^{\infty} \left( A r^{\nu+2} + B r^{-\nu+2} \right) \cos \nu(\phi - \phi') + \left( C r^{\nu+2} + D r^{-\nu+2} \right) \sin \nu(\phi - \phi') + \sum_{m,n,\nu=0}^{\infty} \frac{\lambda E}{\beta_m^2} J_\nu(\beta_m r) \cos \nu(\phi - \phi') \sin \alpha_n \left(z + \frac{h}{2}\right) \left[ \bar{f}(\beta_m, \nu, \alpha_n) + \frac{\alpha}{2\pi k} g_s^i J_\nu(\beta_m r_0) \sin \alpha_n \left(\xi + \frac{h}{2}\right) e^{\alpha(\beta_m^2 + \alpha_n^2)\tau} \right] \left[ e^{-\alpha(\beta_m^2 + \alpha_n^2)t} - 1 \right] \quad (6.3)$$

From (4.1) and (6.3) we obtain

$$\sigma_{rr} = \sum_{\nu=0}^{\infty} \left\{ \left[ A(2 + \nu - \nu^2) r^\nu + B(2 - \nu - \nu^2) r^{-\nu} \right] \cos \nu(\phi - \phi') + \left[ C(2 + \nu - \nu^2) r^\nu + D(2 - \nu - \nu^2) r^{-\nu} \right] \sin \nu(\phi - \phi') \right\} +$$

$$\sum_{m,n,v=0}^{\infty} \frac{\lambda E}{\beta_m^2 r^2} \left[ \beta_m r J_{v-1}(\beta_m r) - (v+v^2) J_v(\beta_m r) \right] \cos v(\phi-\phi') \sin \alpha_n \left( z + \frac{h}{2} \right). \\ \left[ \bar{f}(\beta_m, v, \alpha_n) + \frac{\alpha}{2\pi k} g_s^i J_v(\beta_m r_0) \sin \alpha_n \left( \xi + \frac{h}{2} \right) e^{\alpha(\beta_m^2 + \alpha_n^2)\tau} \right] \left[ e^{-\alpha(\beta_m^2 + \alpha_n^2)t} - 1 \right] \quad (6.4)$$

$$\sigma_{\phi\phi} = \left\{ \left[ A(v^2 + 3v + 2)r^v + B(v^2 - 3v + 2)r^{-v} \right] \cos v(\phi-\phi') + \left[ C(v^2 + 3v + 2)r^v + D(v^2 - 3v + 2)r^{-v} \right] \sin v(\phi-\phi') \right\} + \\ \sum_{m,n,v=0}^{\infty} \frac{\lambda E}{\beta_m^2} \left[ \frac{\beta_m}{r} J_{v+1}(\beta_m r) - \beta_m^2 J_v(\beta_m r) \right] \cos v(\phi-\phi') \sin \alpha_n \left( z + \frac{h}{2} \right). \\ \left[ \bar{f}(\beta_m, v, \alpha_n) + \frac{\alpha}{2\pi k} g_s^i J_v(\beta_m r_0) \sin \alpha_n \left( \xi + \frac{h}{2} \right) e^{\alpha(\beta_m^2 + \alpha_n^2)\tau} \right] \left[ e^{-\alpha(\beta_m^2 + \alpha_n^2)t} - 1 \right] \quad (6.5)$$

$$\sigma_{rr} = \sum_{v=0}^{\infty} \left\{ \left[ A(v^2 + v)r^v + B(-v^2 + v)r^{-v} \right] \sin v(\phi-\phi') + \left[ C(-v^2 - v)r^v + D(v^2 - v)r^{-v} \right] \cos v(\phi-\phi') \right\} + \\ \sum_{m,n,v=0}^{\infty} \frac{\lambda E v}{\beta_m^2 r^2} \left[ \beta_m r J_{v-1}(\beta_m r) - (1+v) J_v(\beta_m r) \right] \sin v(\phi-\phi') \sin \alpha_n \left( z + \frac{h}{2} \right). \\ \left[ \bar{f}(\beta_m, v, \alpha_n) + \frac{\alpha}{2\pi k} g_s^i J_v(\beta_m r_0) \sin \alpha_n \left( \xi + \frac{h}{2} \right) e^{\alpha(\beta_m^2 + \alpha_n^2)\tau} \right] \left[ e^{-\alpha(\beta_m^2 + \alpha_n^2)t} - 1 \right] \quad (6.6)$$

Applying condition (4.4) to (6.4) and (6.6) we obtain

$$A = - \sum_{m,n,v=0}^{\infty} \frac{a^{-v} \lambda E}{2v \beta_m a} J_{v-1}(\beta_m a) \sin \alpha_n \left( z + \frac{h}{2} \right) \cdot \left[ \bar{f}(\beta_m, v, \alpha_n) + \frac{\alpha}{2\pi k} g_s^i J_v(\beta_m r_0) \sin \alpha_n \left( \xi + \frac{h}{2} \right) e^{\alpha(\beta_m^2 + \alpha_n^2)\tau} \right] \\ \left[ e^{-\alpha(\beta_m^2 + \alpha_n^2)t} - 1 \right] \quad (6.7)$$

$$B = \sum_{m,n,v=0}^{\infty} \frac{a^v \lambda E}{2v \beta_m a} J_{v-1}(\beta_m a) \sin \alpha_n \left( z + \frac{h}{2} \right) \cdot \left[ \bar{f}(\beta_m, v, \alpha_n) + \frac{\alpha}{2\pi k} g_s^i J_v(\beta_m r_0) \sin \alpha_n \left( \xi + \frac{h}{2} \right) e^{\alpha(\beta_m^2 + \alpha_n^2)\tau} \right] \\ \left[ e^{-\alpha(\beta_m^2 + \alpha_n^2)t} - 1 \right] \quad (6.8)$$

$$C = 0, D = 0 \quad (6.9)$$

Substituting (6.7), (6.8) and (6.9) in above equations which finally reduces to

$$\sigma_{rr} = \sum_{m,n,v=0}^{\infty} \left\{ \left[ a^v(2-v-v^2)r^{-v} - a^{-v}(2+v-v^2)r^v \right] \frac{J_{v-1}(\beta_m a)}{2v \beta_m a} + \left[ \frac{\beta_m r J_{v-1}(\beta_m r) - (v+v^2) J_v(\beta_m r)}{\beta_m^2 r^2} \right] \right\} \cdot \\ \lambda E \cos v(\phi-\phi') \sin \alpha_n \left( z + \frac{h}{2} \right) \cdot \left[ \bar{f}(\beta_m, v, \alpha_n) + \frac{\alpha}{2\pi k} g_s^i J_v(\beta_m r_0) \sin \alpha_n \left( \xi + \frac{h}{2} \right) e^{\alpha(\beta_m^2 + \alpha_n^2)\tau} \right] \\ \left[ e^{-\alpha(\beta_m^2 + \alpha_n^2)t} - 1 \right] \quad (6.10)$$

$$\sigma_{\phi\phi} = \sum_{m,n,v=0}^{\infty} \left\{ \left[ a^v(v^2 - 3v + 2)r^{-v} - a^{-v}(v^2 + 3v + 2)r^v \right] \frac{J_{v-1}(\beta_m a)}{2v \beta_m a} + \left[ \frac{\beta_m J_{v+1}(\beta_m r) - r \beta_m^2 J_v(\beta_m r)}{r \beta_m^2} \right] \right\} \cdot \\ \lambda E \cos v(\phi-\phi') \sin \alpha_n \left( z + \frac{h}{2} \right) \cdot \left[ \bar{f}(\beta_m, v, \alpha_n) + \frac{\alpha}{2\pi k} g_s^i J_v(\beta_m r_0) \sin \alpha_n \left( \xi + \frac{h}{2} \right) e^{\alpha(\beta_m^2 + \alpha_n^2)\tau} \right]$$

$$\left[ e^{-\alpha(\beta_m^2 + \alpha_n^2)t} - 1 \right] \quad (6.11)$$

$$\sigma_{r\phi} = \sum_{m,n,v=0}^{\infty} \left\{ \left[ a^v(1-v^2)r^{-v} - a^{-v}(v^2+v)r^v \right] \frac{J_{v-1}(\beta_m a)}{2v\beta_m a} + \frac{v}{\beta_m^2 r^2} \left[ \beta_m r J_{v-1}(\beta_m r) - (1+v)J_v(\beta_m r) \right] \right\} \cdot \lambda E \sin v(\phi - \phi') \sin \alpha_n \left( z + \frac{h}{2} \right) \cdot \left[ \bar{f}(\beta_m, v, \alpha_n) + \frac{\alpha}{2\pi k} g_s^i J_v(\beta_m r_0) \sin \alpha_n \left( \xi + \frac{h}{2} \right) e^{\alpha(\beta_m^2 + \alpha_n^2)\tau} \right] \left[ e^{-\alpha(\beta_m^2 + \alpha_n^2)t} - 1 \right] \quad (6.12)$$

### VI. Conclusion :

In this paper we determined the temperature distribution and thermal stresses in a Dirichlet's thin solid cylinder with moving heat source with analytical approach on the surface is established. By giving particular values to the parameter one can obtain their desired results by putting values of the parameters in the equations (5.10), (6.10), (6.11), (6.12)

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