IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN:2319-765X. Volume 10, Issue 2 Ver. VII (Mar-Apr. 2014), PP 52-58
www.iosrjournals.org

Total Dominating Sets and Total Domination Polynomials of Square Of Paths

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Abstract: *Let G= (V , E) be a simple connected graph. A set S ⊆ V is a total dominating set of G if every vertex is adjacent to an element of S.Let* $D_t(P_n^2,i)$ *be the family of all total dominating sets of the graph* $P_n^2, n \geq 2$ *with cardinality i, and let* $d_t(P_n^2,i) = |D_t(P_n^2,i)|$. *In this paper we computed* $d_t(P_n^2,i)$,*and obtain the polynomial* $D_t(P_n^2,x) = \sum_{i=\gamma_t(P_n^2)}^{n} d_t(P_n^2,i)x^i$ *which we call total domination polynomial of* P_n^2, *n ≥2 and obtain some properties of this polynomial.*

Keywords: *total domination set, total domination polynomial, square of path*

I. Introduction

Let G = (V,E) be a simple connected graph. A set S ⊆ V is a dominating set of G, if every vertex in V-S is adjacent to atleast one vertex in S. A set S ⊆ V is total dominating set if every vertex of the graph is adjacent to an element of S. The total domination number of a graph G is the minimum cardinality of a total dominating set in G, and it is denoted by $\gamma_t(G)$. Obviously $\gamma_t(G) < |V|$. A path is a connected graph in which two vertices have degree 1 and the remaining vertices have degree 2. Let P_n be the path with n vertices. The square of a simple connected graph G is a graph with same set of vertices of G and an edge between two vertices if and only if there is a path of length at most 2 between them. It is denoted by G^2. We use the notation $\lfloor x \rfloor$ for the largest integer less than or equal to x and $\lceil x \rceil$ for the smallest integer greater than or equal to x. Also we denote the set {1,2,…………,n} by [n], throughout this paper.

Let P_n^2, n ≥ 2 be the square of the path P_n,n ≥2 and let $D_t(P_n^2, i)$ be the family of total dominating sets of the graph P_n^2 ,n ≥ 2 with cardinality i and let $d_t(P_n^2,i) = |D_t(P_n^2,i)|$.
Table 1 in page number 12 gives the number of total dominating sets of P_n^2,with cardinality i. The total domination polynomial $D_t(P_n^2, x)$ of P_n^2, n ≥ 2 is defined as
$D_t(P_n^2,x) = \sum_{i=\gamma_t(P_n^2)}^{n} d_t(P_n^2,i)x^i$, where $\gamma_t(P_n^2)$ is the total domination number of P_n^2, *n ≥ 2* .

II- Total Dominating Sets of Square Of Paths

Let $D_t(P_n^2,i)$ be the family of total dominating sets of P_n^2,n ≥2 with cardinality i. We will investigate total dominating sets of P_n^2,n ≥ 2.

Lemma 2.1

$$\gamma_t(P_n^2) = \begin{cases} \lceil \frac{n}{5} \rceil + 2 & \text{if } n \equiv 5 \pmod 5 \\ \lceil \frac{n}{5} \rceil + 1 & \text{if } n \not\equiv 5 \pmod 5 \end{cases}$$

Lemma 2.2

Let P_n^2 ,n ≥ 2 be the square of path P_n , with $|V(P_n^2)| = n$. Then $d_t(P_n^2,i) = 0$ if $i < \lceil \frac{n}{5} \rceil + 1$ or i > n and $d_t(P_n^2,i) > 0$ if $\lceil \frac{n}{5} \rceil + 1 \leq i \leq n$.

Proof:

If n ≡5(mod5), then the total domination number of the square of path P_n^2 is $\gamma_t(P_n^2) = \lceil \frac{n}{5} \rceil + 2$. Therefore $d_t(P_n^2, i) = 0$ if $i < \lceil \frac{n}{5} \rceil + 2$ or i > n and $d_t(P_n^2,i) > 0$ if $\lceil \frac{n}{5} \rceil + 2 \leq i \leq n$.

On the other hand, if $n \not\equiv 5 \pmod 5$,then the total domination number of P_n^2 is $\gamma_t(P_n^2) = \lceil \frac{n}{5} \rceil + 1$.

Therefore $d_t(P_n^2,i) = 0$ if $i < \lceil \frac{n}{5} \rceil + 1$ or i > n and $d_t(P_n^2,i) > 0$ if $\lceil \frac{n}{5} \rceil + 1 \leq i \leq n$.

Hence, on both the cases we have $d_t(P_n^2, i) = 0$ if $i < \lceil \frac{n}{5} \rceil + 1$ or i > n and $d_t(P_n^2, i) > 0$ if $\lceil \frac{n}{5} \rceil + 1 \leq i \leq n$.

Lemma 2.3

Let P_n^2 ,n ≥ 2 be the square of path with $|V(P_n^2)| = n$. Then we have

(i) $D_t(P_n^2, i) = \varphi$ if $i < \Upsilon_t(P_n^2)$ or $i > n$.
(ii) $D_t(P_n^2, x)$ has no constant term and first degree terms.
(iii) $D_t(P_n^2, x)$ is a strictly increasing function on $[0,\infty)$.

Proof of (i)

Since P_n^2 has n vertices, there is only one way to choose all these vertices.Therefore $d_t(P_n^2,n) = 1$.
Out of these n vertices, every combination of n-1 vertices can dominate totally only if $\delta(P_n^2) > 1$.
Therefore $d_t(P_n^2,n-1) = n$ if $\delta(P_n^2) > 1$.
Therefore $D_t(P_n^2, i) = \varphi$ if $i < \gamma_t(P_n^2)$ and $D_t(P_n^2, n+k) = \varphi$, k = 1,2,3,………..
Thus we have $d_t(P_n^2, i) = 0$ for $i < \gamma_t(P_n^2)$ and $d_t(P_n^2, n+i) = 0$,for i = 1,2,3,……….

Proof of (ii)

A single vertex of P_n^2 cannot totally dominate all the vertices of $P_n^2, n \geq 2$.So the set of all vertices of P_n^2 is totally dominated by atleast two of the vertices of P_n^2 Hence the total domination polynomial has no constant term as well as first degree term.

Proof of (iii)

By the definition of total domination, every vertex of P_n^2 is adjacent to an element of total dominating set.

That is $D_t(P_n^2,x) = \sum_{i=\Upsilon_t(P_n^2)}^{n} d_t(P_n^2, i)x^i$

Therefore $D_t(P_n^2, x)$ is a strictly increasing function on $[0, \infty)$. ■

Lemma 2.4

Let P_n^2, $n \geq 2$ be the square of path with $|V(P_n^2)| = n$. Then we have

(i) If $D_t(P_{n-1}^2, i-1) = \varphi, D_t(P_{n-3}^2, i-1) = \varphi$, then $D_t(P_{n-2}^2, i-1) = \varphi$
(ii) If $D_t(P_{n-1}^2, i-1) \neq \varphi, D_t(P_{n-3}^2, i-1) \neq \varphi$, then $D_t(P_{n-2}^2, i-1) \neq \varphi$
(iii) If $D_t(P_{n-1}^2, i-1) = \varphi, D_t(P_{n-2}^2, i-1) = \varphi$, then $D_t(P_{n-3}^2, i-1) = \varphi$
$D_t(P_{n-4}^2, i-1) = \varphi, D_t(P_{n-5}^2, i-1) = \varphi$, then $D_t(P_n^2, i) = \varphi$.

Proof of (i)

Since $D_t(P_{n-1}^2, i-1) = \varphi$ and $D_t(P_{n-3}^2, i-1) = \varphi \Rightarrow d_t(P_{n-1}^2, i-1) = 0$ and $d_t(P_{n-3}^2, i-1) = 0$

Then $i-1 < \lceil\frac{n-1}{5}\rceil+1$ or $i-1 > n-1$ and $i-1 < \lceil\frac{n-3}{5}\rceil+1$ or $i-1 > n-3$.

If $i-1 < \lceil\frac{n-1}{5}\rceil+1$ and $i-1 < \lceil\frac{n-3}{5}\rceil+1$ then $i-1 < \lceil\frac{n-2}{5}\rceil+1$. Therefore $D_t(P_{n-2}^2, i-1) = \varphi$.

If $i-1 > n-1$ and $i-1 > n-3$ then $i-1 > n-2$.Therefore $D_t(P_{n-2}^2, i-1) = \varphi$

Hence in all the cases $D_t(P_{n-2}^2, i-1) = \varphi$.

Proof of (ii)

Since $D_t(P_{n-1}^2, i-1) \neq \varphi$ and $D_t(P_{n-3}^2, i-1) \neq \varphi \Longrightarrow d_t(P_{n-1}^2, i-1) \neq 0$ and $d_t(P_{n-3}^2, i-1) \neq 0$

Then $\lceil\frac{n-1}{5}\rceil+1 \leq i-1 \leq n-1$ and $\lceil\frac{n-3}{5}\rceil+1 \leq i-1 \leq n-3 \Longrightarrow \lceil\frac{n-3}{5}\rceil+1 \leq \lceil\frac{n-2}{5}\rceil+1 \leq i-1 \leq n-3 < n-2$, Since $\lceil\frac{n-1}{5}\rceil+1 \leq i-1$

$\Longrightarrow \lceil\frac{n-2}{5}\rceil+1 \leq i-1 \leq n-2 \Longrightarrow d_t(P_{n-2}^2, i-1) \neq 0 \Longrightarrow D_t(P_{n-2}^2, i-1) \neq \varphi$.

Proof of (iii)

Since,$D_t(P_{n-1}^2, i-1) = \varphi$, $D_t(P_{n-2}^2, i-1)) = \varphi$, $D_t(P_{n-3}^2, i-1) = \varphi$,$D_t(P_{n-4}^2, i-1) = \varphi$ and $D_t(P_{n-5}^2, i-1) = \varphi$

$\Longrightarrow d_t(P_{n-1}^2, i-1) = 0$, $d_t(P_{n-2}^2, i-1)) = 0$ $d_t(P_{n-3}^2, i-1) = 0$, $d_t(P_{n-4}^2, i-1) = 0$ and $d_t(P_{n-5}^2, i-1) = 0$.

Then $i-1 < \lceil\frac{n-1}{5}\rceil+1$ or $i-1 > n-1$; $i-1 < \lceil\frac{n-2}{5}\rceil+1$ or $i-1 > n-2$; $i-1 < \lceil\frac{n-3}{5}\rceil+1$ or $i-1 > n-3$; $i-1 < \lceil\frac{n-4}{5}\rceil+1$ or $i-1 > n-4$ and $i-1 < \lceil\frac{n-5}{5}\rceil+1$ or $i-1 > n-5$.

If $i < \lceil\frac{n-1}{5}\rceil+2$; $i < \lceil\frac{n-2}{5}\rceil+2$; $i < \lceil\frac{n-3}{5}\rceil+2$; $i < \lceil\frac{n-4}{5}\rceil+2$ and $i < \lceil\frac{n-5}{5}\rceil+2 \Rightarrow i < \lceil\frac{n-5}{5}\rceil+2 \leq \lceil\frac{n}{5}\rceil+2 \Longrightarrow i < \lceil\frac{n}{5}\rceil+1$

$\Longrightarrow d_t(P_n^2,i) = 0$.Therefore $D_t(P_n^2, i) = \varphi$.

If $i-1 > n-1$, $i-1 > n-2$, $i-1 > n-3$, $i-1 > n-4$ and $i-1 > n-5$, then $i-1 > n-1 > n-2 > n-3 > n-4 > n-5$

$\Longrightarrow i > n > n-1 > n-2 > n-3 > n-4 \Longrightarrow i > n \Longrightarrow d_t(P_n^2,i) = 0 \Longrightarrow D_t(P_n^2,i) = \varphi$.

Lemma 2.5

Let$P_n^2, n \geq 2$ be the square of path with $|V(P_n^2)| = n$.Suppose that $D_t(P_n^2, i) \neq \varphi$, then we have
(i)$D_t(P_{n-2}^2,i-1) = \varphi$, $D_t(P_{n-3}^2,i-1) = \varphi$,$D_t(P_{n-4}^2,i-1) = \varphi$, $D_t(P_{n-5}^2 i-1) = \varphi$and$D_t(P_{n-1}^2, i-1) \neq \varphi$ if and only if n = i.
(ii)$D_t(P_{n-1}^2, i-1) \neq \varphi$,$D_t(P_{n-2}^2, i-1) \neq \varphi$,$D_t(P_{n-3}^2,i-1) \neq \varphi$,$D_t(P_{n-4}^2,i-1) \neq \varphi$and $D_t(P_{n-5}^2,i-1) = \varphi$ if only if i = n-3
(iii)$D_t(P_{n-1}^2, i-1) = \varphi$, $D_t(P_{n-2}^2, i-1) = \varphi$, $D_t(P_{n-3}^2, i-1) = \varphi$, $D_t(P_{n-4}^2, i-1) \neq \varphi$, $D_t(P_{n-5}^2, i-1) \neq \varphi$ if and only if n = 5k + 3 and i = k+2 for some positive integer k.

(iv)$D_t(P^2_{n-1}, i-1) \neq \varphi$, $D_t(P^2_{n-2}, i-1) \neq \varphi$,$D_t(P^2_{n-3}, i-1) \neq \varphi$, $D_t(P^2_{n-4}, i-1) \neq \varphi$ and $D_t(P^2_{n-5}, i-1) \neq \varphi$ if and only if $\lceil\frac{n-1}{5}\rceil+2 \leq i \leq n-4$

Proof of (i)

Suppose, $D_t(P^2_{n-2},i-1) = \varphi$, $D_t(P^2_{n-3},i-1) = \varphi$, $D_t(P^2_{n-4},i-1) = \varphi$, $D_t(P^2_{n-5} i-1) = \varphi$

$\Rightarrow d_t(P^2_{n-2},i-1) = 0$, $d_t(P^2_{n-3},i-1) = 0$, $d_t(P^2_{n-4},i-1) = 0$, $d_t(P^2_{n-5} i-1) = 0$

$\Rightarrow$ $i-1 < \lceil\frac{n-2}{5}\rceil+1$ or $i-1 > n-2$; $i-1 < \lceil\frac{n-3}{5}\rceil+1$ or $i-1 > n-3$;$i-1 < \lceil\frac{n-4}{5}\rceil+1$ or $i-1 > n-4$ and $i-1 < \lceil\frac{n-5}{5}\rceil+1$ or $i-1 > n-5$.

If $i-1 < \lceil\frac{n-5}{5}\rceil+1 \leq \lceil\frac{n-4}{5}\rceil+1 \leq \lceil\frac{n-3}{5}\rceil+1 \leq \lceil\frac{n-2}{5}\rceil+1 \leq \lceil\frac{n-1}{5}\rceil+1$, then $i-1 < \lceil\frac{n-1}{5}\rceil+1$

$d_t(P^2_{n-2},i-1) = 0$ which is a contradiction ,since $d_t(P^2_{n-2},i-1) \neq 0$.Therefore, $i-1 > n-2 > n-3 > n-4 > n-5$

$\Rightarrow i-1 > n-2 \Rightarrow i-1 \geq n-1 \Rightarrow i \geq n$ (1)

Also $d_t(P^2_{n-1},i-1) \neq 0 \Rightarrow \lceil\frac{n-1}{5}\rceil+1 \leq i-1 \leq n-1 \Rightarrow i-1 \leq n-1 \Rightarrow i \leq n$ (2)

From (1) and (2), we have, $n = i$

Conversely, if $n = i$ then,

$D_t(P^2_{n-2},i-1) = D_t(P^2_{n-2},n-1) = \varphi$,$D_t(P^2_{n-3},i-1) = D_t(P^2_{n-3},n-1) = \varphi$,$D_t(P^2_{n-4},i-1) = D_t(P^2_{n-4},n-1) = \varphi$

$D_t(P^2_{n-5},i-1) = D_t(P^2_{n-5},n-1) = \varphi$ and $D_t(P^2_{n-1},i-1) = D_t(P^2_{n-1},n-1) \neq \varphi$, since $D_t(P^2_{n},n) \neq \varphi$.

Proof of (ii)

Suppose, $D_t(P^2_{n-1}, i-1) \neq \varphi$,$D_t(P^2_{n-2}, i-1) \neq \varphi$,$D_t(P^2_{n-3},i-1) \neq \varphi$ and $D_t(P^2_{n-4},i-1) \neq \varphi$

Then, $d_t(P^2_{n-1}, i-1) \neq 0$, $d_t(P^2_{n-2}, i-1) \neq 0$, $d_t(P^2_{n-3},i-1) \neq 0$ and $d_t(P^2_{n-4},i-1) \neq 0$

$\Rightarrow \lceil\frac{n-1}{5}\rceil +1 \leq i-1 \leq n-1$; $\lceil\frac{n-2}{5}\rceil +1\leq i-1 \leq n-2$; $\lceil\frac{n-3}{5}\rceil +1 \leq i-1 \leq n-3$ and $\lceil\frac{n-4}{5}\rceil +1\leq i-1 \leq n-4$

Also, $D_t(P^2_{n-5},i-1) = \varphi$,$d_t(P^2_{n-5},i-1) = 0 \Rightarrow i-1 < \lceil\frac{n-5}{5}\rceil +1$ or $i-1 > n-5$

If $i-1 < \lceil\frac{n-5}{5}\rceil+1$, then $i-1 < \lceil\frac{n-5}{5}\rceil+1 \leq \lceil\frac{n-4}{5}\rceil+1 \Rightarrow i-1< \lceil\frac{n-4}{5}\rceil+1$

$\Rightarrow d_t(P^2_{n-4},i-1) = 0$ which is a contradiction, since $d_t(P^2_{n-4},i-1) \neq 0 \Rightarrow i \leq n-3$ (2)

From (1) and (2), we have, $i = n-3$ Therefore, $i-1 < \lceil\frac{n-5}{5}\rceil +1$ is not possible, so $i-1 > n-5$

$\Rightarrow i > n-4 \Rightarrow i \geq n-3$ (1)

Since, $d_t(P^2_{n-4},i-1) \neq 0 \Rightarrow \lceil\frac{n-4}{5}\rceil+1\leq i-1\leq n-4 \Rightarrow i-1 \leq n-4$

Conversely, if $i= n-3$, then

$D_t(P^2_{n-1}, i-1) = D_t(P^2_{n-1}, n-4) \neq \varphi$,$D_t(P^2_{n-2}, i-1) = D_t(P^2_{n-2}, n-4) \neq \varphi$,$D_t(P^2_{n-3}, i-1) = D_t(P^2_{n-3}, n-4) \neq \varphi$

$D_t(P^2_{n-4}, i-1) = D_t(P^2_{n-4}, n-4) \neq \varphi$ and$D_t(P^2_{n-5}, i-1) = D_t(P^2_{n-5}, n-4) = \varphi$.

Proof of (iii)

Suppose, $D_t (P^2_{n-1}, i-1) = \varphi$, $D_t(P^2_{n-2}, i-1) = \varphi$ and $D_t(P^2_{n-3}, i-1) = \varphi$

Then $d_t(P^2_{n-1}, i-1) = 0$, $d_t(P^2_{n-2}, i-1) = 0$ and $D_t(P^2_{n-3}, i-1) = 0$

$\Rightarrow i-1 < \lceil\frac{n-1}{5}\rceil +1$ or $i-1 > n-1$; $i-1 < \lceil\frac{n-2}{5}\rceil +1$ or $i-1 > n-2$ and$i-1 < \lceil\frac{n-3}{5}\rceil +1$ or $i-1 > n-3$

If $i-1 > n-1>n-2 > n-3 \Rightarrow i-1 > n-1 \Rightarrow i > n$,$D_t(P^2_{n},i) = \varphi$, which is a contradiction ,since $D_t (P^2_{n},i) \neq \varphi$

Therefore $i-1 > n-1$ is not possible So, $i-1 < \lceil\frac{n-3}{5}\rceil+1 \leq \lceil\frac{n-2}{5}\rceil+1 \leq \lceil\frac{n-1}{5}\rceil+1$

$i < \lceil\frac{n-3}{5}\rceil+2 \leq \lceil\frac{n-2}{5}\rceil+2 \leq \lceil\frac{n-1}{5}\rceil+2 \Rightarrow i < \lceil\frac{n-1}{5}\rceil+2$

Also $D_t(P^2_{n-4},i-1) \neq \varphi$ and $D_t(P^2_{n-5},i-1) \neq \varphi$. Then $d_t(P^2_{n-4},i-1) \neq 0$ and $d_t(P^2_{n-5},i-1) \neq 0$

$\Rightarrow \lceil\frac{n-4}{5}\rceil +1\leq i-1 \leq n-4$ and $\lceil\frac{n-5}{5}\rceil +1\leq i-1 \leq n-5 \Rightarrow \lceil\frac{n-4}{5}\rceil+2 \leq i$ and $\lceil\frac{n-5}{5}\rceil+2 \leq i$

$\lceil\frac{n-4}{5}\rceil+2 \leq i$ and $\lceil\frac{n-5}{5}\rceil+2 \leq i$ which holds only for $n = 5k + 3$ and $i = k+2$ for Some $K \in N$

Conversely, if $n = 5k + 3$ and $i = k+2$ for some positive integer k

Then,

$D_t(P^2_{n-1}, i-1) =D_t(P^2_{5k-3}, k) = \varphi$,$D_t(P^2_{n-2}, i-1) = D_t(P^2_{5k-4}, k) = \varphi$,$D_t(P^2_{n-3}, i-1) = D_t(P^2_{5k-5}, k) = \varphi$

$D_t(P^2_{n-4}, i-1) = D_t(P^2_{5k-6}, k) \neq \varphi$ and $D_t(P^2_{n-5}, i-1) = D_t(P^2_{5k-7}, k) \neq \varphi$

Proof of (v)

Suppose,$D_t(P^2_{n-1}, i-1)\neq \varphi$, $D_t(P^2_{n-2}, i-1) \neq \varphi$,$D_t(P^2_{n-3}, i-1) \neq \varphi$, $D_t(P^2_{n-4}, i-1) \neq \varphi$ and $D_t(P^2_{n-5}, i-1) \neq \varphi$

$d_t(P^2_{n-1}, i-1) \neq 0$,$d_t(P^2_{n-2}, i-1) \neq 0$,$d_t(P^2_{n-3}, i-1) \neq 0$, $d_t(P^2_{n-4}, i-1) \neq 0$ and $d_t(P^2_{n-5}, i-1) \neq 0$

$\Rightarrow \lceil \frac{n-1}{5} \rceil +1 \leq i-1 \leq n-1$;$\lceil \frac{n-2}{5} \rceil +1 \leq i-1 \leq n-2$;$\lceil \frac{n-3}{5} \rceil +1 \leq i-1 \leq n-3$;

$\lceil \frac{n-4}{5} \rceil +1 \leq i-1 \leq n-4$ and $\lceil \frac{n-5}{5} \rceil +1 \leq i-1 \leq n-5 \Rightarrow \lceil \frac{n-1}{5} \rceil +1 \leq i-1 \leq n-5 \Rightarrow \lceil \frac{n-1}{5} \rceil +2 \leq i-1 \leq n-4$;

Conversely, if $\lceil \frac{n-1}{5} \rceil +2 \leq i-1 \leq n-4$; $\lceil \frac{n-1}{5} \rceil +1 \leq i-1 \leq n-5$;

$\lceil \frac{n-5}{5} \rceil +1 \leq \lceil \frac{n-4}{5} \rceil +1 \leq \lceil \frac{n-3}{5} \rceil +1 \leq \lceil \frac{n-2}{5} \rceil +1 \leq \lceil \frac{n-1}{5} \rceil +1 \leq i-1 \leq n-5 < n-4 < n-3 < n-2 < n-1$

$\Rightarrow \lceil \frac{n-5}{5} \rceil +1 \leq i-1 \leq n-5$;$\lceil \frac{n-4}{5} \rceil +1 \leq i-1 \leq n-4$;$\lceil \frac{n-3}{5} \rceil +1 \leq i-1 \leq n-3$;

$\lceil \frac{n-2}{5} \rceil +1 \leq i-1 \leq n-2$; $\lceil \frac{n-1}{5} \rceil +1 \leq i-1 \leq n-1$;

$D_t(P^2_{n-1}, i-1) \neq \varphi$,$D_t(P^2_{n-2}, i-1) \neq \varphi$,$D_t(P^2_{n-3}, i-1) \neq \varphi$,$D_t(P^2_{n-4}, i-1) \neq \varphi$ and $D_t(P^2_{n-5}, i-1) \neq \varphi$

Theorem 2.6

For every $n \geq 6$ and $i > \lceil \frac{n}{5} \rceil +1$, then we have

(i)$D_t(P^2_{7n},2n) = \{\{3,5,\ldots\ldots.7n-4,7n-5\}\}$

(ii)If $D_t(P^2_{n-2},i-1)=\varphi$,$D_t(P^2_{n-3},i-1)= \varphi$,$D_t(P^2_{n-4},i-1) = \varphi$,$D_t(P^2_{n-5}i-1) = \varphi$ and $D_t(P^2_{n-1}, i-1) \neq\varphi$ then $D_t(P^2_n,i) = \{[n]\}$

(iii)If $D_t(P^2_{n-2},i-1) \neq \varphi$, $D_t(P^2_{n-3},i-1) \neq \varphi$, $D_t(P^2_{n-4},i-1) \neq \varphi$, $D_t(P^2_{n-5}i-1) \neq \varphi$ and $D_t(P^2_{n-1}, i-1) = \varphi$ then

$D_t(P^2_n,i) = \{[n]-\{x\}/x \in[n]\}\}$

Proof of (i)

From Table 4.1, $D_t(P^2_{7n}, 2n) = | D_t(P^2_{7n}, 2n) | = 1$

For any $n \geq 6$, $D_t(P^2_{7n},2n)$ has the only total dominating set as $D_t(P^2_{7n}, 2n)= \{\{3,5, ,,\ldots\ldots7n-4,7n-5\}\}$.

Proof of (ii)

Since, $D_t(P^2_{n-2},i-1) = \varphi$,$D_t(P^2_{n-3},i-1) =\varphi$,$D_t(P^2_{n-4},i-1) = \varphi$,$D_t(P^2_{n-5}i-1) = \varphi$ and $D_t(P^2_{n-1}, i-1) \neq\varphi$

By lemma 4.6 (i), we have, i = n.Therefore $D_t(P^2_n,i) = D_t(P^2_n,n) =\{\{1,2,3\ldots\ldots\ldots n\}\} = \{[n]\}$

Proof of (iii)

Since, $D_t(P^2_{n-1}, i-1) \neq\varphi$,$D_t(P^2_{n-2}, i-1) \neq\varphi$,$D_t(P^2_{n-3},i-1) \neq \varphi$,$D_t(P^2_{n-4},i-1) \neq\varphi$ and $D_t(P^2_{n-5},i-1) = \varphi$

By Lemma 4.6(ii), we have, i = n-3.Therefore $D_t(P^2_n,i) = D_t(P^2_n,n-3) = \{[n]-\{x\}/x \in[n]\}$.

Theorem 1.7

For every $n \geq 6$ and $i > \lceil \frac{n}{5} \rceil +1$,$D_t(P^2_{n-1},i-1) \neq\varphi$,$D_t(P^2_{n-2}, i-1) \neq\varphi$,$D_t(P^2_{n-3},i-1) \neq\varphi$,

$D_t(P^2_{n-4},i-1) \neq\varphi$, then $D_t(P^2_n, i) = \{\{X_1 \cup \{n\}$, if n-1 or n-2 $\in X_1$ where $X_1 \in D_t(P^2_{n-1},i-1)\} \cup$

$\{X_2 \cup \{n-1\}$ if n-2 or n-3$\in X_2$ where$X_2 \in D_t(P^2_{n-2},i-1)\}\cup\{X_3 \cup\{n-2\}$ if n-3 or n-4 $\in X_3$ where $X_3 \in D_t(P^2_{n-3},i-1)\} \cup$

$\{(X_3 - X_4) - \{n-3\} \cup \{n,n-1\}$ if n-5, n-4, n-3 or n-6, n-4, n-3 $\in X_3 - X_4$ where $X_4 \in D_t (P^2_{n-4},i-1)\} \cup \{X_4-\{n-4\}\cup$

$\{n, n-1\}$if n-4 $\in X_4\} \cup \{X_4 - \{n-4\} \cup \{n,n-2\}$ if n-4 $\in X_4\} \cup$

$\{X_4-\{n-4\}\cup\{n-1,n-2\}$ifn-4 $\in X_4\}\cup\{X_4 - \{n-5\}\cup\{n-1,n-3\}$if n-5 $\in X_4\} \cup\{X_4 - \{n-5\} \cup \{n-2, n-3\}$ if n-5 $\in X_4$ }}

Proof:

The construction of $D_t(P^2_n,i)$ from $D_t(P^2_{n-1}, i-1)$,$D_t(P^2_{n-2}, i-1)$,$D_t(P^2_{n-3}, i-1)$ and $D_t(P^2_{n-4}, i-1)$ is as follows :

Let X_1 be the total dominating set of P^2_{n-1} with cardinality i-1. All the elements of $D_t(P^2_{n-1}, i-1)$ end with n-1 or n-2 or n-3.When n-1 or n-2 $\in X_1$,then adjoin n with X_1.Hence every X_1 of $D_t(P^2_{n-1}, i-1)$ belong to $D_t(P^2_n, i)$ by addition of {n}.

Let X_2 be the total dominating set of P^2_{n-2} with cardinality i-1. All the elements of $D_t(P^2_{n-2}, i-1)$ end with n-2 or n-3 or n-4.When n-2 or n-3 $\in X_2$,then adjoin n -1 with X_2.Hence every X_2 of $D_t(P^2_{n-2}, i-1)$ belong to $D_t(P^2_n, i)$ by addition of {n-1}.

Let X_3 be the total dominating set of P^2_{n-3} with cardinality i-1. All the elements of $D_t(P^2_{n-3}, i-1)$ end with n-3 or n-4 or n-5. When n-3 or n-4 $\in X_3$, then adjoin n -2 with X_3.Hence every X_3 of $D_t(P^2_{n-3}, i-1)$ belong to $D_t(P^2_n, i)$ by addition of {n-2}.

Let X_4 be the total dominating set of P^2_{n-4} with cardinality i-1. When n-5, n-4, n-3 or n-6, n-4, n-3 $\in X_3 - X_4$, remove n-4 from $X_3 - X_4$, and adjoin {n, n-1} with $X_3 - X_4$.

When n-4 $\in X_4$, remove n-4 from X_4 and adjoin {n,n-1},{n,n-2} and {n-1,n-2} with X_4.

When n-5 $\in X_4$, remove n-5 from X_4 and adjoin {n-1, n-3} and {n-2,n-3} with X_4.

Hence, $D_t(P^2_n,i)=\{\{X_1 \cup \{n\}$if n-1 or n-2 $\in X_1$ where $X_1\in D_t(P^2_{n-1}, i-1)\} \cup\{X_2 \cup \{n-1\}$

if n-2 or n-3 ∈ X_2 where X_2∈ $D_t(P^2_{n-2}$, i-1)}∪ {X_3 ∪ {n-2} if n-3 or n-4 ∈ X_3 where
X_3 ∈ $D_t(P^2_{n-3}$, i-1)}∪ {(X_3-X_4)-{n-3} ∪ {n,n-1}if n-5,n-4, n-3 or n-6, n-4, n-3 ∈ X_3-X_4 where
X_4 ∈ $D_t(P^2_{n-4}$, i-1)}∪{X_4−{n-4} ∪ {n,n-1} if n-4 ∈ X_4}∪ { X_4−{n-4} ∪ {n,n-2} if n-4 ∈ X_4}∪
{X_4− {n-4} ∪ {n-1,n-2} if n-4 ∈ X_4}∪ {X_4−{n-5} ∪ {n-1,n-3} if n-5 ∈ X_4}∪ {X_4−{n-5} ∪
{n-2,n-3} if n-5 ∈ X_4 }}.

III-Total Domination Polynomial of Square Of Paths

Theorem 3.1

If $D_t(P^2_n$,i) is the family of the total dominating sets of P^2_n with cardinality i,Where i > $\lceil \frac{n}{5} \rceil$ + 1, then
$d_t(P^2_n$,i) = $d_t(P^2_{n-1}$, i-1) + $d_t(P^2_{n-2}$, i-1) + $d_t(P^2_{n-3}$, i-1) + $d_t(P^2_{n-4}$, i-1)

Proof:

From theorem (4.7) and (4.8), we consider all the three cases as given below, Where i >$\lceil \frac{n}{5} \rceil$+1,

(i)If $D_t(P^2_{n-2}$,i-1) = φ, $D_t(P^2_{n-3}$,i-1) = φ,$D_t(P^2_{n-4}$,i-1) = φ,$D_t(P^2_{n-5}$i-1)=φand $D_t(P^2_{n-1}$, i-1) ≠φ then $D_t(P^2_n$,i) = {[n]}

(ii)If $D_t(P^2_{n-1}$,i-1) ≠ φ, $D_t(P^2_{n-2}$,i-1) ≠ φ, $D_t(P^2_{n-3}$,i-1) ≠φ ,$D_t(P^2_{n-4}$i-1)≠ φ and $D_t(P^2_{n-5}$, i-1) =φ then
$D_t(P^2_n$,i) = {[n]-{x}/x ∈ [n]}}

(iii)If $D_t(P^2_{n-1}$,i-1) ≠ φ ,$D_t(P^2_{n-2}$,i-1) ≠ φ, $D_t(P^2_{n-3}$,i-1) ≠ φ, $D_t(P^2_{n-4}$i-1) ≠ φ then

$D_t(P^2_n$,i) = {{X_1 ∪{n} if n-1 or n-2 ∈ $X_1$where X_1 ∈ $D_t(P^2_{n-1}$,i-1)}∪{X_2 ∪{n- 1} if n-2 or n-3 ∈ X_2,
where X_2 ∈ $D_t(P^2_{n-2}$,i-1)} ∪{X_3 ∪{n-2} if n-3 or n-4 ∈ X_3,where X_3 ∈ $D_t(P^2_{n-3}$,i-1)}∪
{(X_3 − X_4)-{n-3} ∪{n, n-1} if n-5, n-4, n-3 or n-6, n-4, n-3 ∈ X_3 − X_4 where X_4 ∈ $D_t(P^2_{n-4}$,i-1)}∪
{X_4−{n-4} ∪{n, n-1} if n-4 ∈ X_4}∪{X_4−{n-4} ∪{n, n-2} if n-4 ∈ X_4}}∪
{X_4−{n-4} ∪{n-1, n-2} if n-4 ∈ X_4}∪ {X_4−{n-5} ∪{n-1, n-3} if n-5 ∈ X_4}

By this construction in each case, we obtain that
$D_t(P^2_n$,i) = $D_t(P^2_{n-1}$,i-1)+$D_t(P^2_{n-2}$,i-1)+ $D_t(P^2_{n-3}$,i-1) + $D_t(P^2_{n-4}$,i-1)
Therefore, $d_t(P^2_n$,i) = $d_t(P^2_{n-1}$,i-1)+$d_t(P^2_{n-2}$,i-1)+ $d_t(P^2_{n-3}$,i-1) + $d_t(P^2_{n-4}$,i-1)
Using theorem (1.9), we obtain, $d_t(P^2_n$,i) for 1 ≤ i≤ 15as shown in Table 1.1

Table 1 : $d_t(P^2_n$,i) the number of total dominating sets of P^2_n with cardinality i

i	2	3	4	5	6	7	8	9	10	11	12	13	14	15
n														
2	1													
3	3	1												
4	5	4	1											
5	5	8	5	1										
6	3	10	13	6	1									
7	1	8	23	19	7	1								
8	0	4	31	42	26	8	1							
9	0	1	34	72	68	34	9	1						
10	0	0	31	101	139	102	43	10	1					
11	0	0	23	119	234	240	145	53	11	1				
12	0	0	13	119	334	467	384	198	64	12	1			
13	0	0	5	101	411	775	843	581	262	76	13	1		
14	0	0	1	72	440	1118	1584	1415	842	338	89	14	1	
15	0	0	0	42	411	1419	2600	2956	2247	1179	427	103	15	1

By theorem 3.1 , we obtain d_t (P^2_n,i),n ≥ 2 for 2 ≤ n ≤ 15 as shown in Table 1. There are interesting relationship between numbers in this Table. In the following theorem we obtain some properties of d_t (P^2_n,i).

Theroem 3.2

Let P^2_n, n ≥ 2 be the square of path with $|V(P^2_n)|$ = n. Then the following properties hold for the coefficients of D_t (P^2_n,x):

(i) For n≥ 2, $d_t(P^2_n$,n) =1

(ii) For n≥ 3, $d_t(P^2_n$,n-1) = n

(iii) For n≥ 5, $d_t(P^2_n$,n-2) = nc_2-2

(iv) For $n \geq 7$, $d_t(P_n^2, n-3) = nc_3 - 2(n-1)$

(v) For $n \geq 8$, $d_t(P_n^2, n-4) = nc_4 - (n^2-2n-9)$

(vi) For $k \geq 1$, $d_t(P_{7k}^2, 2k) = 1$

(vii) For $k \geq 1$, $d_t(P_{5k-2}^2, k+1) = k+2$

Proof of (i)

Since for any graph G with n vertices, $d_t(G,n) = 1$, then $d_t(P_n^2, n) = 1$.

Proof of (ii)

To prove $d_t(P_n^2, n-1) = n$, for $n \geq 3$. Since $D_t(P_n^2, n-1) = \{[n]-\{x\}/ x \in [n]$, $|D_t(P_n^2, n-1)| = nc_1 = n$

Therefore $d_t(P_n^2, n-1) = n$.

Proof of (iii)

To prove $d_t(P_n^2, n-2) = nc_2 - 2$, for $n \geq 5$

We apply induction on n. When n = 5

L.H.S = $d_t(P_5^2, 3) = 8$ (from table),R.H.S = $5c_2 - 2 = 8$

Therefore the result is true for n = 5.

Suppose that the result is true for all natural numbers less than n, and we prove it for n.

We have, $d_t(P_n^2, n-2) = d_t(P_{n-1}^2, n-3) + d_t(P_{n-2}^2, n-3) + d_t(P_{n-3}^2, n-3) + d_t(P_{n-4}^2, n-3)$

$= \frac{1}{2}(n-1)(n-2) - 2 + n-1 = \frac{1}{2}(n^2-n-4) = \frac{1}{2}[n(n-1)] - 2 = nc_2 - 2$, for $n \geq 5$

Hence the result is true for all' n'.

Hence by induction hypothesis, we have

$d_t(P_n^2, n-2) = nc_2 - 2$,for $n \geq 5$

Proof of (iv)

To prove $d_t(P_n^2, n-3) = \frac{1}{6}[n-(n-1)(n-2)] - 2(n-1)$, for every $n \geq 7$

We apply induction on n.

When n = 7,L.H.S = $d_t(P_7^2, 4) = 23$ (from table),R.H.S = $\frac{1}{6}[7(7-1)(7-2)] - 2(7-1) = 23$

Therefore the result is true for n = 7.

Suppose that the result is true for all natural numbers less than n, and we prove it for n.

We have, $d_t(P_n^2, n-3) = d_t(P_{n-1}^2, n-4) + d_t(P_{n-2}^2, n-4) + d_t(P_{n-3}^2, n-4) + d_t(P_{n-4}^2, n-4)$

$= \frac{1}{6}[(n-1)(n-2)(n-3)] - 2(n-2) + \frac{1}{2}[(n-2)(n-3)] - 2 + n-3 + 1$

$= \frac{1}{6}[(n-1)(n^2-5n+6)] - 2n + 4 + \frac{1}{2}(n^2-5n+6) - 2 + n-3+1$

$= \frac{1}{6}[n^3-5n^2+6n-n^2+5n-6-6n+3n^2-15n+18] = \frac{1}{6}[n^3-3n^2-10n+12]$

$= \frac{1}{6}[n^3-3n^2+2n--12n+12] = \frac{1}{6}[n^3-3n^2+2n] - \frac{1}{6}(12n-12) = \frac{1}{6}[n(n-1)(n-2)] - 2(n-1)$ for $n \geq 7$

Therefore the result is true for all' n'.Hence by induction hypothesis, we have

$d_t(P_n^2, n-3) = \frac{1}{6}[n(n-1)(n-2)] - 2(n-1)$, for every $n \geq 7$.

Proof of (v)

To prove $d_t(P_n^2, n-4) = nc_4 - (n^2-2n-9)$, for $n \geq 8$.

We apply induction on n.

When n = 8,L.H.S = $d_t(P_8^2, 4) = 31$ (from table),R.H.S = $\frac{1}{24}[8(8-1)(8-2)(8-4)] - (64-16-9) = 31$

Therefore the result is true for n = 8.

Suppose that the result is true for all natural numbers less than n, and we prove it for n.

We have ,$d_t(P_n^2, n-4) = d_t(P_{n-1}^2, n-5) + d_t(P_{n-2}^2, n-5) + d_t(P_{n-3}^2, n-5) + d_t(P_{n-4}^2, n-5)$

$= \frac{1}{24}[(n-1)(n-2)(n-3)(n-4)] + \frac{1}{6}[(n-2)(n-3)(n-4)] + \frac{1}{2}[(n-3)(n-4)] + [-n^2+2n-1+2n-2+9-2n+6+n-6]$

$= \frac{1}{24}[(n^2-3n+2)(n^2-7n+12)] + \frac{1}{6}[(n-2)(n^2-7n+12)] + \frac{1}{2}[(n^2-7n+12)] + [-n^2+3n+6]$

$= \frac{1}{24}[n^4-7n^3+12n^2-3n^3+21n^2-36n+2n^2-14n+24+41n^3-28n^2+48n-8n^2+56n-96+ 12n^2-84n+144- 24n^2+72n+144]$

$= \frac{1}{24}[n^4-6n^3-13n^2+42n+216] = \frac{1}{24}[n^4-6n^3+11n^2-24n^2-6n+48n+216]$

$= \frac{1}{24}[n^4-6n^3+11n^2-6n] + \frac{1}{24}[-24n^2+48n+216] = \frac{1}{24}n(n^3-6n^2+11n-6) + (-n^2+2n+9) = \frac{1}{24}[n(n-1)(n-2)(n-3)]-(n^2-2n-9)$

Therefore the result is true for all' n'.

Hence by induction hypothesis, we have $d_t(P_n^2, n-4) = nc_4-(n^2-2n-9)$, for $n \geq 8$.

Proof of (vi)

To prove $d_t(P_{7k}^2, 2k) = 1$, for $k \geq 1$.

Since, $d_t(P_{7k}^2, 2k) = \{\{3,5, ,,\ldots\ldots 7k-4, 7k-5\}\}$,

Therefore , $d_t(P_{7k}^2, 2k) = D_t(P_{7k}^2, 2k) = 1$.

Proof of (vii)

To prove $d_t(P_{5k-2}^2, k+1) = k+2$, for $k \geq 1$

When K = 1, $D_t(P_3^2, 2) = \{\{1,2\},\{1,3\},\{2,3\}\}$.Therefore, $d_t(P_3^2, 2) = 3$

Assume that the result is true for all natural numbers m less than k.

Therefore, $d_t(P_{5m-2}^2, m+1) = m+2$, for $m < k$,

$d_t(P_{5m+3}^2, m+2) = m+3$

$d_t(P_{5(m+1)-2}^2, m+2) = (m+1)+2, d_t(P_{5k-2}^2, k+1) = k+2$.

II. Conclusion:

We obtain total domination sets and total domination polynomial square of paths. Similarly we can find total domination sets and total domination polynomial of specified graphs.

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