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On Semi-Invariant Submanifoldsof A Nearly Hyperbolic Kenmotsu Manifold With Semi-Symmetric Semi-Metric Connection

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Abstract.*Weconsiders a nearly hyperbolic Kenmotsu manifold admitting a semi-symmetric semi-metric connection and study semi-invariant submanifolds of a nearly hyperbolic Kenmotsu manifold with semi-symmetric semi-metric connection. We also find the integrability conditions of some distributions on nearly hyperbolic Kenmotsu manifold and study parallel distributions (horizontal & vertical distributions) on nearly hyperbolic Kenmotsu manifold.*
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I. Introduction

The study of CR-submanifolds of Kaehler manifold as generalization of invariant and anti-invariant submanifolds was initiated by A. Bejancu in [8]. A semi-invariant submanifold is the extension of a CR-submanifold of a Kaehler manifold to submanifolds of almost contact manifolds. The study of semi- invariant submanifolds of Sasakian manifolds was initiated by Bejancu-Papaghuic in [10]. The same concept was studied under the name contact CR-submanifold by Yano-Kon in [19] and K. Matsumoto in [16]. The study of semi-invariant submanifolds in almost contact manifold was enriched by several geometers (see, [2], [4], [5], [6], [7], [8], [12], [13], [17]). On the otherhand, almost hyperbolic(f,ξ,η,g), structure was defined and studied by Upadhyay and Dube in [18].Joshi and Dube studied semi-invariant submanifolds of an almost r-contact hyperbolic metric manifold in [15].Motivated by the studies in ([2], [3], [4], [5], [6], [7], [8], [9]),in this paper, we study semi-invariant submanifolds of a nearly hyperbolic Kenmotsu manifold admitting a semi-symmetric semi-metric connection.

Let ∇be a linear connection in an n-dimensional differentiable manifold$\bar{M}$. The torsion tensor $T$and curvature tensor$R$ of ∇ given respectively by

$$T(X,Y)=\nabla_X Y-\nabla_Y X-[X,Y],$$
$$R(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z.$$

The connection ∇ is symmetric if its torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is metric connection if there is a Riemannian metric g in $\bar{M}$ such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection. A. Friedmann and J. A. Schouten introduced the idea of a semi-symmetric connection. A linear connection is said to be a semi-symmetric connection if its torsion tensor T is of the form:

$$T(X,Y)=\eta(X)Y-\eta(Y)X.$$

Many geometers (see, [2], [20], [21]) have studied certain properties of semi-symmetric semi-metric connection. This paper is organized as follows. In section 2, we give a brief description of nearly hyperbolic Kenmotsu manifold admitting a semi-symmetric semi-metric connection. In section 3, we study some properties of semi-invariant submanifolds of a nearly hyperbolic Kenmotsu manifold with a semi-symmetric semi-metric connection. In section 4, we discuss the integrability conditions of some distributions on nearly hyperbolic Kenmotsu manifold with a semi-symmetric semi-metric connection. In section 5, we obtain parallel horizontal distribution on nearly hyperbolic Kenmotsu manifold with a semi-symmetric semi-metric connection.

II. Preliminaries

Let $\bar{M}$ be an n-dimensional almost hyperbolic contact metric manifold with the almost hyperbolic contact metric structure$(\emptyset,\xi,\eta,g)$, where a tensor $\emptyset$ of type $(1,1)$ a vector field ξ, called structure vector field and η, the dual 1-form of $\xi$and the associated Riemannian metric $g$satisfying the following

$$\emptyset^2X=X+\eta(X)\xi,\qquad g(X,\xi)=\eta(X),\tag{2.1}$$
$$\eta(\xi)=-1,\quad \emptyset(\xi)=0,\quad \eta o\emptyset=0,\tag{2.2}$$
$$g(\emptyset X,\emptyset Y)=-g(X,Y)-\eta(X)\eta(Y)\tag{2.3}$$

for any X, Y tangent to $\bar{M}$ [19]. In this case

$$g(\emptyset X, Y) = -g(\emptyset Y, X). \quad (2.4)$$

An almost hyperbolic contact metric structure $(\emptyset, \xi, \eta, g)$ on $\bar{M}$ is called hyperbolic Kenmotsu manifold [7] if and only if

$$(\nabla_X \emptyset)Y = g(\emptyset X, Y)\xi - \eta(Y)\emptyset X \quad (2.5)$$

for all X, Y tangent to $\bar{M}$.

On a hyperbolic Kenmotsumanifold$\bar{M}$, we have

$$\nabla_X \xi = X + \eta(X)\xi \quad (2.6)$$

for a Riemannian metric g and Riemannian connection∇.

Further, an almost hyperbolic contact metric manifold $\bar{M}$ on $(\emptyset, \xi, \eta, g)$ is called a nearly hyperbolic Kenmotsu manifold [7], if

$$(\nabla_X \emptyset)Y + (\nabla_Y \emptyset)X = -\eta(X)\emptyset Y - \eta(Y)\emptyset X, \quad (2.7)$$

where∇ is Riemannian connection $\bar{M}$.

Now, we define a semi-symmetric semi-metric connection

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(X)Y + g(X, Y)\xi \quad (2.8)$$

such that $(\bar{\nabla}_X g)(Y, Z) = 2\eta(X)g(Y, Z) - \eta(Y)g(X, Z) - \eta(Z)g(X, Y)$.

From (2.7)& (2.8), we obtain

$$(\bar{\nabla}_X \emptyset)Y + (\bar{\nabla}_Y \emptyset)X = -\eta(X)\emptyset Y - \eta(Y)\emptyset X \quad (2.9)$$

$$\bar{\nabla}_X \xi = X + \eta(X)\xi. \quad (2.10)$$

An almost hyperbolic contact metric manifold with almost hyperbolic contact structure $(\emptyset, \xi, \eta, g)$ is called nearly hyperbolic Kenmotsu manifold with semi-symmetric semi-metric connection if it satisfy (2.9) and (2.10).

For semi-symmetric semi-metric connection, the Nijenhuis tensor is expressed as

$$N(X, Y) = (\bar{\nabla}_{\emptyset X} \emptyset)Y - (\bar{\nabla}_{\emptyset Y} \emptyset)X - \emptyset(\bar{\nabla}_X \emptyset)Y + \emptyset(\bar{\nabla}_Y \emptyset)X. \quad (2.11)$$

Now from (2.9), we get

$$(\bar{\nabla}_{\emptyset X} \emptyset)Y = -\eta(Y)X - \eta(X)\eta(Y)\xi - (\bar{\nabla}_Y \emptyset)\emptyset X. \quad (2.12)$$

Differentiating (2.1) conveniently along the vector$Y$and using (2.10), we have

$$(\bar{\nabla}_Y \emptyset)\emptyset X = (\bar{\nabla}_Y \eta)(X)\xi + \eta(X)Y + \eta(X)\eta(Y)\xi - \emptyset(\bar{\nabla}_Y \emptyset)X. \quad (2.13)$$

From (2.12) and (2.13), we have

$$(\bar{\nabla}_{\emptyset X} \emptyset)Y = -\eta(X)Y - \eta(Y)X - 2\eta(X)\eta(Y)\xi - (\bar{\nabla}_Y \eta)(X)\xi + \emptyset(\bar{\nabla}_Y \emptyset)X. \quad (2.14)$$

Interchanging $X$and,Y we have

$$(\bar{\nabla}_{\emptyset Y} \emptyset)X = -\eta(Y)X - \eta(X)Y - 2\eta(X)\eta(Y)\xi - (\bar{\nabla}_X \eta)(Y)\xi + \emptyset(\bar{\nabla}_X \emptyset)Y. \quad (2.15)$$

Using equation (2.14), (2.15) in (2.11), we obtain

$$N(X, Y) = 2d\eta(X, Y)\xi + 2\emptyset(\bar{\nabla}_Y \emptyset)X - 2\emptyset(\bar{\nabla}_X \emptyset)Y \quad (2.16)$$

Using equation (2.9), we have

$$N(X, Y) = 2g(\emptyset X, Y)\xi + 4\emptyset(\bar{\nabla}_Y \emptyset)X + 2\eta(X)Y + 2\eta(Y)X + 4\eta(X)\eta(Y)\xi \quad (2.17)$$

As we know that $(\bar{\nabla}_Y \emptyset)X = \bar{\nabla}_Y \emptyset X - \emptyset(\bar{\nabla}_Y X)$, usingGuass formula, we have

$$\emptyset(\bar{\nabla}_Y \emptyset)X = \emptyset(\nabla_Y \emptyset X) + \emptyset h(Y, \emptyset X) - \nabla_Y X - \eta(\nabla_Y X)\xi - h(Y, X)$$

Using this equation in (2.17), we have

$$N(X, Y) = 2\eta(X)Y + 2\eta(Y)X + 4\eta(X)\eta(Y)\xi + 4\emptyset(\nabla_Y \emptyset X) + 4\emptyset h(Y, \emptyset X) \quad (2.18)$$
$$-4(\nabla_Y X) - 4\eta(\nabla_Y X)\xi - 4h(Y, X) + 2g(\emptyset X, Y)\xi$$

III. Semi-invariant Submanifold

Let M be submanifold immersed in $\bar{M}$, we assume that the vector ξ is tangent to M, denoted by $\{\xi\}$ the 1-dimentional distribution spanned by ξ on M, then M is called a semi-invariant submanifold [9] of M if there exist two differentiable distribution $D \& D^\perp$ on M satisfying

(i)$TM = D \oplus D^\perp \oplus \xi$, where D, $D^\perp$and $\xi$are mutually orthogonal to each other,

(ii) the distribution D is invariant under$\emptyset$that is $\emptyset D_X = D_X$for each $X \in M$,

(iii) the distribution $D^\perp$is anti-invariant under$\emptyset$, that is $\emptyset D^\perp{}_X \subset T^\perp M$ for each$X \in M$,where $TM$and$T^\perp M$ be the Lie algebra of vector fields tangential and normal to M respectively.Let $g$be the Riemannian metricand ∇ be induced Levi-Civita connection on M, then the Guass and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (3.1)$$

$$\bar{\nabla}_X N = -A_N X + \nabla_X^\perp N - \eta(X)N \quad (3.2)$$

for any $X, Y \in TM$ and $N \in T^\perp M$, where $\nabla^\perp$ is a connection on the normal bundle $T^\perp M$, $h$is the second fundamental form and A_N is the Weingarten map associated with N as

$$g(A_N X, Y) = g(h(X, Y), N). \quad (3.3)$$

Any vector X tangent to M is given as

$$X = PX + QX + \eta(X)\xi, \quad (3.4)$$

where $PX \in D$ and $QX \in D^{\perp}$.
Similarly, for N normal to M, we have

$$\emptyset N = BN + CN, \tag{3.5}$$

where BN (resp. CN) is the tangential component (resp. normal component) of $\emptyset N$.

Theorem 3.1. The connection induced on a semi-invariant submanifold of a nearly hyperbolic Kenmotsu manifold with semi-symmetric semi-metric connection is also semi-symmetric semi-metric.
Proof: Let ∇ be induced connection with respect to the normal N on semi-invariant submanifold of nearly hyperbolic Kenmotsu manifold with semi-symmetric semi-metric connection $\overline{\nabla}$, then

$$\overline{\nabla}_X Y = \nabla_X Y + m(X,Y), \tag{3.6}$$

where m is the tensor filed of type (0,2) on semi-invariant submanifold M. If ∇^* be the induced connection on semi-invariant submanifold from Riemannian connection $\overline{\overline{\nabla}}$, then

$$\overline{\overline{\nabla}}_X Y = \nabla^*_X Y + h(X,Y), \tag{3.7}$$

where h is second fundamental tensor & we know that semi-symmetric semi-metric connection on nearly hyperbolic Kenmotsu manifold

$$\overline{\nabla}_X Y = \overline{\overline{\nabla}}_X Y - \eta(X)Y + g(X,Y)\xi. \tag{3.8}$$

Using (3.6), (3.7) in (3.8), we have

$$\nabla_X Y + m(X,Y) = \nabla^*_X Y + h(X,Y) - \eta(X)Y + g(X,Y)\xi. \tag{3.9}$$

Comparing tangent and normal parts, we have

$$\nabla_X Y = \nabla^*_X Y - \eta(X)Y + g(X,Y)\xi,$$
$$m(X,Y) = h(X,Y).$$

Thus ∇ is also semi-symmetric semi-metric connection.

⊡

Lemma 3.2. Let M be a semi-invariant submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with semi-symmetric semi-metric connection, then

$$2(\overline{\nabla}_X \emptyset)Y = \nabla_X \emptyset Y - \nabla_Y \emptyset X + h(X,\emptyset Y) - h(Y,\emptyset X) - \emptyset[X,Y]$$

for each $X, Y \in D$.
Proof. From Gauss formula (3.1), we have

$$\overline{\nabla}_X \emptyset Y - \overline{\nabla}_Y \emptyset X = \nabla_X \emptyset Y - \nabla_Y \emptyset X + h(X,\emptyset Y) - h(Y,\emptyset X). \tag{3.10}$$

By covariant differentiation, we have

$$\overline{\nabla}_X \emptyset Y - \overline{\nabla}_Y \emptyset X = (\overline{\nabla}_X \emptyset)Y - (\overline{\nabla}_Y \emptyset)X + \emptyset[X,Y]. \tag{3.11}$$

From (3.10) and (3.11), we have

$$(\overline{\nabla}_X \emptyset)Y - (\overline{\nabla}_Y \emptyset)X = \nabla_X \emptyset Y - \nabla_Y \emptyset X + h(X,\emptyset Y) - h(Y,\emptyset X) - \emptyset[X,Y]. \tag{3.12}$$

Adding (2.9) and (3.12), we obtain

$$2(\overline{\nabla}_X \emptyset)Y = \nabla_X \emptyset Y - \nabla_Y \emptyset X + h(X,\emptyset Y) - h(Y,\emptyset X) - \emptyset[X,Y]$$

for each $X, Y \in D$.

⊡

Lemma 3.3. Let M be a semi-invariant submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with semi-symmetric semi-metric connection, then

$$2(\overline{\nabla}_X \emptyset)Y = -A_{\emptyset Y}X + \nabla^{\perp}_X \emptyset Y - \nabla_Y \emptyset X - h(Y,\emptyset X) - \emptyset[X,Y]$$

for all $X \in D$ and $Y \in D^{\perp}$.
Proof. By Gauss formula (3.1), we have

$$\overline{\nabla}_Y \emptyset X = \nabla_Y \emptyset X + h(Y,\emptyset X).$$

Also, by Weingarten formula (3.2), we have

$$\overline{\nabla}_X \emptyset Y = -A_{\emptyset Y}X + \nabla^{\perp}_X \emptyset Y - \eta(X)\emptyset Y.$$

Subtracting abovetwo equations, we have

$$\overline{\nabla}_X \emptyset Y - \overline{\nabla}_Y \emptyset X = -A_{\emptyset Y}X + \nabla^{\perp}_X \emptyset Y - \eta(X)\emptyset Y - \nabla_Y \emptyset X - h(Y,\emptyset X). \tag{3.13}$$

Comparing equation (3.11) and (3.13), we have

$$(\overline{\nabla}_X \emptyset)Y - (\overline{\nabla}_Y \emptyset)X = -A_{\emptyset Y}X + \nabla^{\perp}_X \emptyset Y - \eta(X)\emptyset Y - \nabla_Y \emptyset X - h(Y,\emptyset X) - \emptyset[X,Y].$$

Adding equation (2.9) in above, we get

$$2(\overline{\nabla}_X \emptyset)Y = -A_{\emptyset Y}X + \nabla^{\perp}_X \emptyset Y - \nabla_Y \emptyset X - h(Y,\emptyset X) - \emptyset[X,Y]. \tag{3.14}$$

for all $X \in D$ and $Y \in D^{\perp}$.

⊡

Lemma 3.4. Let M be a semi-invariant submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with semi-symmetric semi-metric connection, then

$$2(\overline{\nabla}_X \emptyset)Y = A_{\emptyset X}Y - A_{\emptyset Y}X + \nabla^{\perp}_X \emptyset Y - \nabla^{\perp}_Y \emptyset X - \emptyset[X,Y]$$

for all $X, Y \in D^{\perp}$.
Proof. Using Weingarten formula (3.2), we have

$$\overline{\nabla}_X \emptyset Y = -A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset X - \eta(X)\emptyset Y. \quad (3.15)$$

Interchanging and$Y$, we have

$$\overline{\nabla}_Y \emptyset X = -A_{\emptyset X}Y + \nabla_Y^{\perp}\emptyset X - \eta(Y)\emptyset X. \quad (3.16)$$

Subtracting equation (3.16) from (3.15), we have

$$\overline{\nabla}_X \emptyset Y - \overline{\nabla}_Y \emptyset X = A_{\emptyset X}Y - A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \nabla_Y^{\perp}\emptyset X + \eta(Y)\emptyset X - \eta(X)\emptyset Y. \quad (3.17)$$

Using equation (3.11) in (3.17), we have

$$(\overline{\nabla}_X \emptyset)Y - (\overline{\nabla}_Y \emptyset)X = A_{\emptyset X}Y - A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \nabla_Y^{\perp}\emptyset X$$
$$+\eta(Y)\emptyset X - \eta(X)\emptyset Y - \emptyset[X,Y]$$

Adding (2.9) in above equation, we have

$$2(\overline{\nabla}_X \emptyset)Y = A_{\emptyset X}Y - A_{\emptyset Y}X + \nabla_X^{\perp}\emptyset Y - \nabla_Y^{\perp}\emptyset X - \emptyset[X,Y]$$

for all$X, Y \in D^{\perp}$.

⊡

Lemma 3.5. Let M be a semi-invariant submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with semi-symmetric semi-metric connection, then

$$P(\nabla_X \emptyset PY) + P(\nabla_Y \emptyset PX) - PA_{\emptyset QY}X - PA_{\emptyset QX}Y = -\eta(Y)\emptyset PX \quad (3.18)$$
$$-\eta(X)\emptyset PY + \emptyset P(\nabla_X Y) + \emptyset P(\nabla_Y X),$$
$$Q(\nabla_X \emptyset PY) + Q(\nabla_Y \emptyset PX) - QA_{\emptyset QY}X - QA_{\emptyset QX}Y = 2Bh(X,Y), \quad (3.19)$$
$$h(Y, \emptyset PX) + h(X, \emptyset PY) + \nabla_X^{\perp}\emptyset QY + \nabla_Y^{\perp}\emptyset QX = \emptyset Q(\nabla_X Y) \quad (3.20)$$
$$+\emptyset Q(\nabla_Y X) + 2Ch(X,Y),$$
$$\eta(\nabla_X \emptyset PY) + \eta(\nabla_Y \emptyset PX) - \eta(A_{\emptyset QY}X) - \eta(A_{\emptyset QX}Y) = 0 \quad (3.21)$$

for all$X, Y \in TM$.

Proof.Differentiating covariantlyequation (3.4) and using equation (3.1) and (3.2), we have

$$(\overline{\nabla}_X \emptyset)Y + \emptyset(\nabla_X Y) + \emptyset h(X,Y) = \nabla_X \emptyset PY + h(X, \emptyset PY)$$
$$-A_{\emptyset QY}X + \nabla_X^{\perp}\emptyset QY - \eta(X)\emptyset QY.$$

Interchanging$X$and Y, we have

$$(\overline{\nabla}_Y \emptyset)X + \emptyset(\nabla_Y X) + \emptyset h(Y,X) = \nabla_Y \emptyset PX + h(Y, \emptyset PX)$$
$$-A_{\emptyset QX}Y + \nabla_Y^{\perp}\emptyset QX - \eta(Y)\emptyset QX.$$

Adding above two equations, we get

$$(\overline{\nabla}_X \emptyset)Y + (\overline{\nabla}_Y \emptyset)X + \emptyset(\nabla_X Y) + \emptyset(\nabla_Y X) + 2\emptyset h(X,Y) = \nabla_X \emptyset PY + \nabla_Y \emptyset PX$$
$$+h(X, \emptyset PY) + h(Y, \emptyset PX) - A_{\emptyset QY}X - A_{\emptyset QX}Y + \nabla_X^{\perp}\emptyset QY$$
$$+\nabla_Y^{\perp}\emptyset QX - \eta(X)\emptyset QY - \eta(Y)\emptyset QX.$$

Using equation (2.9) in above, we have

$$-\eta(X)\emptyset Y - \eta(Y)\emptyset X + \emptyset(\nabla_X Y) + \emptyset(\nabla_Y X) + 2\emptyset h(X,Y) = \nabla_X \emptyset PY + \nabla_Y \emptyset PX$$
$$+h(X, \emptyset PY) + h(Y, \emptyset PX) - A_{\emptyset QY}X - A_{\emptyset QX}Y + \nabla_X^{\perp}\emptyset QY$$
$$+\nabla_Y^{\perp}\emptyset QX - \eta(X)\emptyset QY - \eta(Y)\emptyset QX.$$

Using equations (3.4),(3.5)and (2.2) in above equation, we have

$$-\eta(X)\emptyset PY - \eta(Y)\emptyset PX + \emptyset P(\nabla_X Y) + \emptyset Q(\nabla_X Y) + \emptyset P(\nabla_Y X) + \emptyset Q(\nabla_Y X)$$
$$+2Bh(X,Y) + 2Ch(X,Y) = P(\nabla_X \emptyset PY) + Q(\nabla_X \emptyset PY) + \eta(\nabla_X \emptyset PY)\xi$$
$$+P(\nabla_Y \emptyset PX) + Q(\nabla_Y \emptyset PX) + \eta(\nabla_Y \emptyset PX)\xi + h(Y, \emptyset PX) + h(X, \emptyset PY)$$
$$-PA_{\emptyset QY}X - QA_{\emptyset QY}X - \eta(A_{\emptyset QY}X)\xi - PA_{\emptyset QX}Y - QA_{\emptyset QX}Y$$
$$-\eta(A_{\emptyset QX}Y)\xi + \nabla_X^{\perp}\emptyset QY + \nabla_Y^{\perp}\emptyset QX.$$

Comparing horizontal, vertical and normal components we get desired result.

⊡

IV. Integrability of Distributions

Theorem4.1. Let M be a semi-invariant submanifold of a nearly hyperbolic Kenmotsu manifold $\overline{M}$ with semi-symmetric semi-metric connection. Then the distribution $D \oplus \langle \xi \rangle$ is integrable if the following conditions are satisfied:

$$S(X,Y) \in (D \oplus \langle \xi \rangle), \quad (4.1)$$
$$h(X, \emptyset Y) = h(\emptyset X, Y) \quad (4.2)$$

for each$X, Y \in (D \oplus \langle \xi \rangle)$.

Proof.The torsion tensor $S(X,Y)$ of an almost hyperbolic contact manifold is given by

$$S(X,Y) = N(X,Y) + 2d\eta(X,Y)\xi,$$

where$N(X,Y)$ is Neijenhuis tensor. Then

$$S(X,Y) = [\emptyset X, \emptyset Y] - \emptyset[\emptyset X, Y] - \emptyset[X, \emptyset Y] + 2d\eta(X,Y)\xi. \quad (4.3)$$

Suppose that $(D \oplus \langle \xi \rangle)$ is integrable, then$N(X,Y) = 0$ for any$X, Y \in (D \oplus \langle \xi \rangle)$.Therefore,

$$S(X,Y) = 2d\eta(X,Y)\xi \in (D \oplus \langle \xi \rangle).$$

From (4.3), (2.18) and comparing normal part, we have

$$\emptyset Q(\nabla_Y \emptyset X) + Ch(Y, \emptyset X) - h(X, Y) = 0, for X, Y \in (D \oplus \langle \xi \rangle).$$

Replacing Y by $\emptyset Z$, where $Z \in D$, we have

$$\emptyset Q(\nabla_{\emptyset Z} \emptyset X) + Ch(\emptyset Z, \emptyset X) - h(X, \emptyset Z) = 0. \tag{4.4}$$

Interchanging X and Z, we have

$$\emptyset Q(\nabla_{\emptyset X} \emptyset Z) + Ch(\emptyset X, \emptyset Z) - h(Z, \emptyset X) = 0. \tag{4.5}$$

Subtracting (4.4) from (4.5), we obtain

$$\emptyset Q[\emptyset X, \emptyset Z] + h(X, \emptyset Z) - h(Z, \emptyset X) = 0.$$

Since $D \oplus \langle \xi \rangle$ is integrable so that $[\emptyset X, \emptyset Z] \in (D \oplus \langle \xi \rangle)$ for $X, Y \in D$. Consequently above equation gives

$$h(X, \emptyset Z) = h(\emptyset X, Z).$$

⊡

Proposition 4.2. Let M be a semi-invariant submanifold of a nearly hyperbolicKenmotsu manifold $\bar{M}$ with semi-symmetric semi-metric connection. Then

$$A_{\emptyset Y} Z - A_{\emptyset Z} Y = \frac{1}{3} \emptyset P[Y, Z]$$

for each $Y, Z \in D^{\perp}$.

Proof. Let $Y, Z \in D^{\perp}$ and $X \in TM$. From (3.3), we have

$$2g(A_{\emptyset Z} Y, X) = g(h(Y, X), \emptyset Z) + g(h(X, Y), \emptyset Z). \tag{4.6}$$

Using (3.1) and (2.9) in (4.6), we have

$$2g(A_{\emptyset Z} Y, X) = -g(\nabla_Y \emptyset X, Z) - g(\nabla_X \emptyset Y, Z) \tag{4.7}$$
$$-\eta(Y) g(\emptyset X, Z) - \eta(X) g(\emptyset Y, Z).$$

From (3.2), we have

$$\bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N - \eta(X) N.$$

Replacing N by $\emptyset Y$

$$\bar{\nabla}_X \emptyset Y = -A_{\emptyset Y} X + \nabla_X^{\perp} \emptyset Y - \eta(X) \emptyset Y.$$

Using (2.5) andabove equation in (4.7), we have

$$2\, g(A_{\emptyset Z} Y, X) = -g(\emptyset \nabla_Y Z, X) + g(A_{\emptyset Y} Z, X).$$

Transvecting X from both sides, we obtain

$$2A_{\emptyset Z} Y = -\emptyset \nabla_Y Z + A_{\emptyset Y} Z. \tag{4.8}$$

Interchanging Y and Z, we have

$$2A_{\emptyset Y} Z = -\emptyset \nabla_Z Y + A_{\emptyset Z} Y. \tag{4.9}$$

Subtracting(4.8) from (4.9), we have

$$(A_{\emptyset Y} Z - A_{\emptyset Z} Y) = \frac{1}{3} \emptyset P[Y, Z]. \tag{4.10}$$

⊡

Theorem4.3. Let M be a semi-invariant submanifold of a nearly hyperbolic Kenmotsu manifold $\bar{M}$ with semi-symmetric semi-metric connection. Then the distribution $D^{\perp}$ is integrable if and only if

$$A_{\emptyset Y} Z - A_{\emptyset Z} Y = 0 \tag{4.11}$$

for all $Y, Z \in D^{\perp}$.

Proof. Suppose that the distribution $D^{\perp}$ is integrable, that is $[Y, Z] \in D^{\perp}$ for any $Y, Z \in D^{\perp}$, therefore

$$P[Y, Z] = 0.$$

Consequently, from (4.10) we have

$$A_{\emptyset Y} Z - A_{\emptyset Z} Y = 0.$$

Conversely, let (4.11) holds. Then by virtue of (4.10), we have either $P[Y, Z] = 0$ or $P[Y, Z] = k\xi$. But $P[Y, Z] = k\xi$ is not possible as P being a projection operator on D. So, $P[Y, Z] = 0$, this implies that $[Y, Z] \in D^{\perp}$ for all $Y, Z \in D^{\perp}$.

Hence $D^{\perp}$ is integrable.

⊡

V. Parallel Distribution

Definition 5.1. The horizontal (resp., vertical) distribution D(resp., $D^{\perp}$) is said to be parallel [7] with respect to the connection on M if $\nabla_X Y \in D$ (resp., $\nabla_Z W \in D^{\perp}$) for any vector field.

Proposition 5.2. Let M be a semi-invariant submanifold of a nearly hyperbolicKenmotsu manifold $\bar{M}$ with semi-symmetric semi-metric connection. If the horizontal distribution D is parallel then $h(X, \emptyset Y) = h(Y, \emptyset X)$ for all $X, Y \in D$.

Proof. Let $X, Y \in D$, as D is parallel distribution so that $\nabla_X \emptyset Y \in D$ and $\nabla_Y \emptyset X \in D$. From (3.19) and (3.20), we have

$$Q(\nabla_X \emptyset PY) + Q(\nabla_Y \emptyset PX) - QA_{\emptyset QY} X - QA_{\emptyset QX} Y + h(Y, \emptyset PX) + h(X, \emptyset PY)$$
$$+ \nabla_X^{\perp} \emptyset QY + \nabla_Y^{\perp} \emptyset QX = 2Bh(X, Y) + \emptyset Q(\nabla_X Y) + \emptyset Q(\nabla_Y X) + 2Ch(X, Y).$$

AsQ being a projection operator on $D^{\perp}$ then we have

$$h(X, \emptyset Y) + h(Y, \emptyset X) = 2\emptyset h(X, Y). \quad (5.1)$$

Replacing$X by \emptyset X$ in (5.1) and using (2.1), we have

$$h(\emptyset X, \emptyset Y) + h(Y, X) = 2\emptyset h(\emptyset X, Y).(5.2)$$

Replacing $Y$by$\emptyset Y$ in (5.1) and using (2.1),we have

$$h(X, Y) + h(\emptyset Y, \emptyset X) = 2\emptyset h(X, \emptyset Y). \quad (5.3)$$

Comparing (5.2) and (5.3), we get

$$h(X, \emptyset Y) = h(Y, \emptyset X)$$

for all$X, Y \in D$.

⊡

Definition 5.3.A semi-invariant submanifold is said to be mixed totally geodesic if $h(X, Y) = 0$ for all$X \in D$and$Y \in D^{\perp}$.

Proposition 5.4. Let M be a semi-invariant submanifold of a nearly hyperbolicKenmotsu manifold $\bar{M}$ with semi-symmetric semi-metric connection. Then M is a mixed totally geodesic if and only if $A_N X \in D$for all$X \in D$.

Proof.Let $A_N X \in D$for all$X \in D$.Now, $g(h(X, Y), N) = g(A_N X, Y) = 0$ for $Y \in D^{\perp}$, which is equivalent to $h(X, Y) = 0$. Hence M is totally mixed geodesic.

Conversely, Let M is totally mixed geodesic, that is $h(X, Y) = 0$ $forX \in D$and$Y \in D^{\perp}$. Then $g(h(X, Y), N) = g(A_N X, Y)$ gives$g(A_N X, Y) = 0$,which implies that$A_N X \in D$for all$Y \in D^{\perp}$.

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