IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 10, Issue 6 Ver. I (Nov - Dec. 2014), PP 22-25
www.iosrjournals.org

Total Dominating Sets and Total Domination Polynomials of Square Of Wheels

T.Premala, Dr.C.Sekar,
Assistant Professor/ Mathematics, C.S.I.Institute of Technology,Thovalai
Associate Professor/ Mathematics, Aditanar college of Arts and Science,Tiruchendur

Abstract: *Let G= (V , E) be a simple connected graph. A set S ⊆ V is a total dominating set of G if every vertex is adjacent to an element of S. Let $D_t(W_n^2,i)$ be the family of all total dominating sets of the graph W_n^2, $n \geq 3$ with cardinality i, and let $d_t(W_n^2,i) = |D_t(W_n^2, i)|$. In this paper we compute $d_t(W_n^2,i)$,and obtain the polynomial $D_t(W_n^2, x) = \sum_{i=\Upsilon_t(W_n^2)}^{n+1} d_t(W_n^2,i)x^i$, which we call total domination polynomial of W_n^2, $n \geq 3$ and obtain some properties of this polynomial.*

Keywords*: square of wheel, total domination set, total domination polynomial*

I. Introduction

Let G = (V,E) be a simple connected graph. A set S ⊆ V is a dominating set of G, if every vertex in V-S is adjacent to atleast one vertex in S. A set S ⊆ V is total dominating set if every vertex of the graph is adjacent to an element of S. The total domination number of a graph G is the minimum cardinality of a total dominating set in G, and it is denoted by $\Upsilon_t(G)$. Obviously $\Upsilon(G) < |V|$. The square of a simple connected graph G is a graph with same set of vertices of G and an edge between two vertices if and only if there is a path of length at most 2 between them. It is denoted by G^2. We use the notation $\lfloor x \rfloor$ for the largest integer less than or equal to x and $\lceil x \rceil$ for the smallest integer greater than or equal to x. Also we denote the set {1,2,…………,n} by [n], throughout this paper.

In this paper, we study the concept of total dominating sets and total domination polynomials of square of wheels W_n^2, $n \geq 3$. Let $D_t(W_n^2,i)$ be the total dominating set of W_n^2 with cardinality i. Let $d_t(W_n^2,i) = |D_t(W_n^2,i)$. The total domination polynomial of W_n^2 is $D_t(W_n^2,x) = \sum_{i=\Upsilon_t(W_n^2)}^{n+1} d_t(W_n^2,i)x^i$.

Definition 1.1

The square of a wheel W_n is a graph with same set of vertices as W_n and an edge between two vertices if and only if there is a path of length atmost 2 between them. It is denoted by W_n^2.

Definition 1.2

Let W_n^2, $n \geq 3$ be the square of wheel with n+1 vertices. Let $V(W_n^2) = \{v_0, v_1, v_2, \ldots\ldots. v_n\}$ and $E(W_n^2) = \{(v_0, v_i);i=1,2\ldots\ldots\ldots,n\} \cup \{(v_i, v_{i+1});i=1,2,\ldots.\ n-1\} \cup \{(v_i, v_{i+2});i=1,2,\ldots\ldots\ldots,n-2\} \cup (v_n, v_1), (v_{n-1}, v_1), (v_n, v_2)\}$. Let $D_t(W_n^2,i)$ be the family of total dominating sets of W_n^2 with cardinality i and let $d_t(W_n^2,i) = |D_t(W_n^2,i)|$. Then the total dominating polynomial $D_t(W_n^2,x)$ of W_n^2 is defined as $d_t(W_n^2,x) = \sum_{i=\Upsilon_t(W_n^2)}^{n} d_t(W_n^2,i)x^i$.

Notation 1.3

We categorize the total dominating sets of W_n^2 into two classes, the total dominating sets containing the vertex v_0 and the total dominating sets not containing the vertex v_0, where v_0 denotes the centre of the wheel.. Let $D_t^0(W_n^2,i)$ be the collection of total dominating sets of W_n^2 containing the vertex v_0 with cardinality i. Let $D_t^1(W_n^2,i)$ be the collection of total dominating sets of W_n^2 not containing the vertex v_0 with cardinality i. The total dominating sets of W_n^2 not containing v_0 with cardinality i is same as the total dominating sets of the square of cycle C_n^2 with cardinality i. Hence $D_t^1(W_n^2,i) = D_t(W_n^2,i)$.

Let $d_t^0(W_n^2,i) = |D_t^0(W_n^2,i)|$ and $d_t^1(W_n^2,i) = |D_t^1(W_n^2,i)|$

So $d_t^1(W_n^2,i) = d_t(C_n^2,i)$

But in general $d_t(W_n^2,i) = d_t^0(W_n^2,i) + d_t^1(W_n^2,i)$

That is $d_t(W_n^2,i) = d_t^0(W_n^2,i) + d_t(C_n^2,i)$

Lemma 1.4

For w_n^2, $n \geq 3$ with $|V(w_n^2)| = n+1$, then the total domination number is $\Upsilon_t(w_n^2) = 2$

Proof:
In the graph W_n^2, a single vertex covers all the remaining vertices of W_n^2.
By the definition of total domination, every vertex in total dominating set S is adjacent to another vertex of S.
Therefore $\Upsilon_t(w_n^2) = 2$.
Remark 1.5
We have $\Upsilon_t(w_n^2) = 2$
Therefore $D_t(W_n^2,i) \neq \varphi$ if $2 \leq i \leq n+1$.
That is $d_t(W_n^2,i) \neq 0$ if $2 \leq i \leq n+1$.

Lemma 1.6
Let W_n^2 ,$n \geq 3$ be the square of wheel with $|V(W_n^2)| = n+1$. Then we have
(i) $D_t(W_n^2,i) = \varphi$ if $i > n+1$.
(ii) $D_t(W_n^2,x)$ is a multiple of x^2.
(iii) $D_t(W_n^2,x)$ is a strictly increasing function on $[0,\infty)$.

Proof of (i)
Since W_n^2 has n+1 vertices, there is only one way to choose all these vertices.
Therefore $d_t(W_n^2,n+1) = 1$.
Out of these n+1 vertices, every combination of n vertices can dominate totally only if $\delta(W_n^2) > 1$. Therefore $d_t(W_n^2,n) = n+1$ if $\delta(W_n^2) > 1$.
Therefore $D_t(W_n^2, i) = \varphi$ if $i < \Upsilon_t(W_n^2)$ and $D_t(W_n^2, n+k) = \varphi$, k = 2,3,………..
Thus $d_t(W_n^2, i) = 0$ for $i < \Upsilon_t(W_n^2)$ and $d_t(W_n^2, n+i) = 0$,for i = 2,3,………. ■
Proof of (ii)
A single vertex of W_n^2 cannot totally dominate all the vertices of W_n^2,$n \geq 3$.So the set of all vertices of W_n^2 is totally dominated by atleast two of the vertices of W_n^2 . Hence the total domination polynomial has no constant term as well as first degree term. ■
Proof of (iii)
By the definition of total domination, every vertex of W_n^2 is adjacent to an element of total dominating set.
That is $D_t(W_n^2,x) = \sum_{i=\Upsilon_t(W_n^2)}^{n+1} \left| D_t(W_n^2, i) \right| x^i$
Therefore $D_t(W_n^2,x)$ is a strictly increasing function on $[0, \infty)$. ■

Theorem 1.7
If $D_t(W_n^2,i)$ and $D_t(C_n^2,i)$ are the collection of total dominating sets of W_n^2 and C_n^2 respectively with cardinality I, where $i > \lceil \frac{n}{5} \rceil + 1$ then $d_t(W_n^2,i) = nc_{i-1} + d_t(C_n^2,i)$
Proof:
We have $d_t(W_n^2,i) = d_t^0(W_n^2,i) + d_t^1(W_n^2,i)$
$= d_t^0(W_n^2,i) + d_t(C_n^2,i)$
The number of total dominating sets of W_n^2 containing the vertex v_0 with cardinality i is the number of ways to choose i-1 vertices from the vertices 1,2,3,………,n. Therefore $d_t^0(W_n^2,i) = nc_{i-1}$. Therefore $d_t(W_n^2,i) = nc_{i-1} + d_t(C_n^2,i)$.

Lemma 1.8
. Let W_n^2 ,$n \geq 3$ be the square of path with $|V(W_n^2)| = n+1$. Suppose that $D_t(W_n^2,i) \neq \varphi$, then we have
(i) $D_t(W_{n-2}^2,i-1) = \varphi$ and $D_t(W_{n-1}^2, i-1) \neq \varphi$ if and only if i =n +1.
(ii) $D_t(W_{n-1}^2, i-1) \neq \varphi$, $D_t(W_{n-2}^2, i-1) \neq \varphi$ and $D_t(W_{n-3}^2,i-1) = \varphi$ if only if i = n.
Proof of (i)
Suppose, $D_t(W_{n-2}^2,i-1) = \varphi$ and $D_t(W_{n-1}^2, i-1) \neq \varphi$
$\Longrightarrow d_t(W_{n-2}^2,i-1) = 0$ and $d_t(W_{n-1}^2,i-1) \neq 0$
$\Longrightarrow i-1 < \Upsilon_t(W_n^2)$ or $i-1 > n-1$ and $\Upsilon_t(W_n^2) \leq i-1 \leq n$
If $i-1 < \Upsilon_t(W_n^2) \Rightarrow i-1 < i < \Upsilon_t(W_n^2) \Rightarrow i < \Upsilon_t(W_n^2)$
$\Rightarrow d_t(W_n^2,i) = 0$ which is a contradiction, since $d_t(W_n^2,i) \neq 0$, so $i-1 < \Upsilon_t(W_n^2)$ is not possible. Therefore $i-1 > n-1$.
$\Rightarrow i > n \Rightarrow i \geq n+1$ ——————— (1)
Also , we have $i-1 \leq n$
$\Rightarrow i \leq n+1$ ——————— (2)
From (1) and (2) we get i= n+1.

Conversely, if i= n+1, then
$D_t(W_{n-2}^2,i\text{-}1) = D_t(W_{n-2}^2,n) = \varphi$ and
$D_t(W_{n-1}^2, i\text{-}1) = D_t(W_{n-1}^2, n) \neq \varphi$.

Proof of (ii)

Suppose,
$D_t(W_{n-1}^2, i\text{-}1) \neq \varphi$, $D_t(W_{n-2}^2, i\text{-}1) \neq \varphi$
Then $d_t(W_{n-1}^2, i\text{-}1) \neq 0$, $d_t(W_{n-2}^2, i\text{-}1) \neq 0$
$\Rightarrow \Upsilon_t(W_n^2) \leq$ i-1 $\leq$ n and $\Upsilon_t(W_n^2) \leq$ i-1 $\leq$ n-1
$\Rightarrow$ i-1 $\leq$ n-1
$\Rightarrow$ i $\leq$ n ________ (1)
Also we have $D_t(W_{n-3}^2,i\text{-}1) = \varphi$

Then $d_t(W_{n-3}^2,i\text{-}1) = 0$
$\Rightarrow$ i-1 < $\Upsilon_t(W_n^2)$ or i-1 > n-2
If i-1 < $\Upsilon_t(W_n^2)$ $\Rightarrow$i-1 < i < $\Upsilon_t(W_n^2)$ $\Rightarrow$ i < $\Upsilon_t(W_n^2)$
$\Rightarrow$ $d_t(W_n^2,i\text{-}1)$ = o which is a contradiction , since $d_t(W_n^2,i) \neq$ o.
Therefore i-1 < $\Upsilon_t(W_n^2)$ is not possible, so i-1 > n-2
$\Rightarrow$ i > n-1
$\Rightarrow$ i $\geq$ n ______________ (2)
From (1) and (2) we get i= n.
Conversely, if i= n, then
$D_t(W_{n-1}^2, i\text{-}1) = D_t(W_{n-1}^2, n\text{-}1) \neq \varphi$
$D_t(W_{n-2}^2, i\text{-}1) = D_t(W_{n-2}^2, n\text{-}1) \neq \varphi$ and
$D_t(W_{n-3}^2, i\text{-}1) = D_t(W_{n-3}^2, n\text{-}1) = \varphi$

Theorem 1.9

Let W_n^2 ,n $\geq$ 3be the square of wheel with $|V(W_n^2)|$ = n+1. Then the following properties hold for the coefficients of $D_t(W_n^2,x)$:

(i) For n $\geq$ 2, $d_t(W_n^2,n+1) = 1$
(ii) For n $\geq$ 2, $d_t(P_n^2,n) = n+1$
(iii) For n $\geq$ 3, $d_t(W_n^2,n\text{-}1) = (n+1)c_2$
(iv) For n $\geq$ 5, $d_t(W_n^2,n\text{-}2) = (n+1)c_3$
(v) For n $\geq$ 5, $d_t(W_n^2,n\text{-}3) = (n+1)c_4$
(vi) For n $\geq$ 7, $d_t(W_n^2,n\text{-}4) = (n+1)c_5$ - n

Proof of (i)

Since for any graph G with n+1 vertices, $d_t(G,n+1) = 1$, then
$d_t(W_n^2,n+1) = 1$.

Proof of (ii)

To prove $d_t(W_n^2,n) = n+1$, for n $\geq$ 2.
We have , $d_t(W_n^2,i) = nc_{i-1} + d_t(C_n^2,i)$
$d_t(W_n^2,n) = nc_{n-1} + d_t(C_n^2,n)$
= n+1
Therefore $d_t(W_n^2,n) = n+1$.

Proof of (iii)

To prove $d_t(W_n^2,n\text{-}1) = (n+1)c_2$, for n $\geq$ 3
We have , $d_t(W_n^2,i) = nc_{i-1} + d_t(C_n^2,i)$
$d_t(W_n^2,n\text{-}1) = nc_{n-2} + d_t(C_n^2,n\text{-}1)$
$= \frac{1}{2}n(n-1) + n$
$= \frac{1}{2}n(n-1 + 2)$
$= \frac{1}{2}n(n+1)$
$= (n+1)c_2$
$= (n+1)c_2$ -2, for n $\geq$ 5

Proof of (iv)

To prove $d_t(W_n^2,n\text{-}2) = (n+1)c_3$ for every n $\geq$ 5
We have , $d_t(W_n^2,i) = nc_{i-1} + d_t(C_n^2,i)$
$d_t(W_n^2,n\text{-}2) = nc_{n-3} + d_t(C_n^2,n\text{-}2)$

$= \frac{1}{6}[n(n\text{-}1)(n\text{-}2)] - 2(n\text{-}2) + \frac{1}{2}n(n-1)$

$= \frac{1}{6}n(n\text{-}1)[n\text{-}2+3]$

$= \frac{1}{6}n(n\text{-}1)(n+1)$

$= \frac{1}{6}(n+1)n(n\text{-}1)$

$= (n+1)c_3$

$d_t(W_n^2, n\text{-}2) = (n+1)c_3$ for every $n \geq 5$

Proof of (v)

To prove $d_t(W_n^2, n\text{-}3) = (n+1)c_4$, for $n \geq 5$.

We have , $d_t(W_n^2, i) = nc_{i-1} + d_t(C_n^2, i)$

$d_t(W_n^2, n\text{-}3) = nc_{n-4} + d_t(C_n^2, n\text{-}3)$

$= \frac{1}{24}[n(n\text{-}1)(n\text{-}2)(n\text{-}3)] - 2(n\text{-}2) + \frac{1}{6}n(n-1)(n\text{-}2)$

$= \frac{1}{24}n(n\text{-}1)(n\text{-}2)[n\text{-}3+4]$

$= \frac{1}{24}n(n\text{-}1)(n\text{-}2)(n+1)$

$= \frac{1}{24}(n+1)n(n\text{-}1)(n\text{-}2)$

$= (n+1)c_4$

$d_t(W_n^2, n\text{-}3) = (n+1)c_4$, for $n \geq 5$

Proof of (vi)

To prove $d_t(W_n^2, n\text{-}4) = (n+1)c_5 - n$, for $n \geq 7$.

We have , $d_t(W_n^2, i) = nc_{i-1} + d_t(C_n^2, i)$

$d_t(W_n^2, n\text{-}4) = nc_{n-5} + d_t(C_n^2, n\text{-}4)$

$= \frac{1}{120}[n(n\text{-}1)(n\text{-}2)(n\text{-}3)(n\text{-}4)] - 2(n\text{-}2) + \frac{1}{24}n(n-1)(n\text{-}2)(n\text{-}3) - n$

$= \frac{1}{120}n(n\text{-}1)(n\text{-}2)(n\text{-}3)[n\text{-}4+5] - n$

$= \frac{1}{120}n(n\text{-}1)(n\text{-}2)(n\text{-}3)(n+1) - n$

$= \frac{1}{120}(n+1)n(n\text{-}1)(n\text{-}2)(n\text{-}3) - n$

$= (n+1)c_5 - n$

$d_t(W_n^2, n\text{-}4) = (n+1)c_5 - n$, for $n \geq 7$.

Using theorem 1.7 and theorem 1.9, we obtain $d_t(W_n^2, i)$ for $1 \leq i \leq 11$ as shown in Table 1.1

Table1.1 $d_t(W_n^2, i)$, the number of total dominating sets of W_n^2 with cardinality i

i	2	3	4	5	6	7	8	9	10	11
n										
3	6	4	1							
4	10	10	5	1						
5	15	20	15	6	1					
6	1`8	35	35	21	7	1				
7	14	49	70	56	28	8	1			
8	8	52	118	126	84	36	9	1		
9	9	45	165	243	210	120	45	10	1	
10	10	55	201	403	452	330	165	55	11	1

Theorem 1.10

If $D_t(W_n^2, x)$ is the total dominating polynomial of square of wheel W_n^2 , then $D_t(W_n^2, x) = x[(1+x)^n - 1] + D_t(W_n^2, x)$

Proof:

We have, $D_t(W_n^2, x) = \sum_{i=2}^{n+1} d_t(W_n^2, i)x^i$

$= \sum_{i=2}^{\Upsilon_t(C_n^2)-1} d_t(W_n^2, i)x^i + \sum_{i=\Upsilon_t(C_n^2)}^{n+1} d_t(W_n^2, i)x^i$

$= \sum_{i=2}^{\Upsilon_t(C_n^2)-1} d_t^0(W_n^2, i)x^i + \sum_{i=\Upsilon_t(C_n^2)}^{n+1}\{nc_{i-1} + d_t(C_n^2, i)\}x^i$

$= \sum_{i=2}^{\Upsilon_t(C_n^2)-1} nc_{i-1}x^i + \sum_{i=\Upsilon_t(C_n^2)}^{n+1} nc_{i-1}x^i + \sum_{i=\Upsilon_t(C_n^2)}^{n+1} d_t(C_n^2, i)x^i$

$= \sum_{i=2}^{\Upsilon_t(C_n^2)-1} nc_{i-1}x^i + \sum_{i=\Upsilon_t(C_n^2)}^{n+1} d_t(C_n^2, i)x^i$

$= x\sum_{i=2}^{n+1} nc_{i-1}x^{i-1} + D_t(C_n^2, x)$

$D_t(W_n^2, x) = x[(1+x)^n - 1] + D_t(C_n^2, x)$