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Weyl's Theorem for Algebraically Totally K - Quasi – Paranormal Operators

N. Sivakumar[1], D. Senthilkumar[2] and P. Maheswari Naik[3*]
[1]Department of Mathematics, Hindusthan College of Arts and Science, Coimbatore - 641028, TamilNadu, India.
[2]Post Graduate and Research Department of Mathematics, Governmant Arts College (Autonomous), Coimbatore - 641 018, TamilNadu, India.
[3]Department of Mathematics, Sri Ramakrishna Engineering College, Vattamalaipalyam, Coimbatore - 641 022, TamilNadu, India.*

Abstract: *An operator $T \in B(H)$ is said to be k - quasi - paranormal operator if $\left\|T^{k+1}x\right\|^2 \le \left\|T^{k+2}x\right\| \left\|T^k x\right\|$ for every $x \in H$, k is a natural number. This class of operators contains the class of paranormal operators and the class of quasi - class A operators. Let T or T^* be an algebraically k - quasi - paranormal operator acting on Hilbert space. Using Local Spectral Theory, we prove (i)Weyl's theorem holds for f(T) for every $f \in H(\sigma(T))$; (ii) a-Browder's theorem holds for f (S) for every $S \prec T$ and $f \in H(\sigma(S))$; (iii) the spectral mapping theorem holds for the Weyl spectrum of T and for the essential approximate point spectrum of T.*

Keywords: *Paranormal operator, Weyl's theorem, k - quasi - paranormal operator, Riesz idempotent, generalized a - Weyl's theorem, B - Fredholm, B - Weyl, generalized Weyl's theorem, SVEP.*

I. Introduction

Let $B(H)$ and $B_0(H)$ denotes the algebra of all bounded linear operators and the ideal of compact operators acting on an infinite dimensional separable Hilbert space H.

The following facts follows from some well known facts about paranormal operators.

(i) If T is paranormal and $M \subseteq H$ is invariant under T then $T|_M$ is paranormal.

(ii) Every quasinilpotent paranormal operator is a zero operator.

(iii) T is paranormal if and only if $T^{2*}T^2 - 2\lambda T^*T + \lambda^2 \ge 0$ for all $\lambda > 0$.

(iv) If T is paranormal and invertible, then T^{-1} is paranormal.

If $T \in B(H)$, we shall write N(T) and R(T) for the null space and the range of T, respectively. Also, let $\sigma(T)$ and $\sigma_a(T)$ denote the spectrum and the approximate point spectrum of T, respectively. Let $\sigma_p(T)$, $\pi(T)$, E(T) denotes the point spectrum of T, the set of poles of the resolvent of T, the set of all eigenvalues of T which are isolated in $\sigma(T)$, respectively.

The ascent (length of the null chain) of an operator $T \in B(H)$ is the smallest non negative integer p := p(T) such that $T^{-p}(0) = T^{-(p+1)}(0)$. If there is no such integer, i.e., $T^{-p}(0) \neq T^{-(p+1)}(0)$ for all p, then set p(T) = ∞. The descent (length of the image chain) of T is defined as the smallest non negative integer q := q(T) such that $T^q(H) = T^{(q+1)}(H)$. If there is no such integer, ie., $T^q(H) \neq T^{(q+1)}(H)$ for all q, then set q(T) = ∞. It is well known that if p(T) and q(T) are both finite then they are equal [14, Proposition 38.6].

An operator T is called Fredholm if R(T) is closed, $\alpha(T)$ = dim N(T) < ∞ and $\beta(T)$ = dim H / R(T) < ∞. Moreover if i(T) = $\alpha(T) - \beta(T)$ = 0, then T is called Weyl. The essential spectrum $\sigma_e(T)$ and the Weyl $\sigma_w(T)$ are defined by

$$\sigma_e(T) = \{ \lambda \in \mathbf{C} : T - \lambda \text{ is not Fredholm} \}$$

and

$$\sigma_w(T) = \{ \lambda \in \mathbf{C} : T - \lambda \text{ is not Weyl} \}$$

respectively.

The Browder spectrum $\sigma_b(T)$ is defined as

$$\sigma_b(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\}.$$

It is known that $\sigma_e(T) \subset \sigma_w(T) \subset \sigma_b(T) = \sigma_e(T) \cup \text{acc } \sigma(T)$ where we write acc K for the set of all accumulation points of $K \subset \mathbb{C}$. If we write iso K = K \ acc K, then we let

$$\pi_{00}(T) = \{ \lambda \in \text{iso } \sigma(T) : 0 < \alpha(T-\lambda) < \infty\}.$$
$$\pi_{00}^a(T) = \{ \lambda \in \text{iso } \sigma_a(T) : 0 < \alpha(T-\lambda) < \infty\}.$$

and

$$p_{00}(T) = \sigma(T) \setminus \sigma_b(T).$$

We say that Weyl's theorem holds for T if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$$

and Browder's theorem holds for T if

$$\sigma(T) \setminus \sigma_w(T) = \pi_0(T).$$

An operator $T \in B(H)$ is called upper semi - Fredholm if it has closed range and finite dimensional null space and is called lower semi - Fredholm if it has closed range and its range has finite co - dimension. If $T \in B(H)$ is either upper or lower semi - Fredholm, then T is called semi - Fredholm. For $T \in B(H)$ and a non negative integer n define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ to $R(T^n)$ (in particular $T_0 = T$). If for some integer n the range $R(T^n)$ is closed and T_n is upper (resp. lower) semi - Fredholm, then T is called upper (resp. lower) semi - B - Fredholm.

The essential approximate point spectrum $\sigma_{ea}(T)$ and the Browder approximate point spectrum $\sigma_{ab}(T)$ of T are defined by

$$\sigma_{ea}(T) = \bigcap \{\sigma_a(T+K) : K \in K(H)\}$$
$$\sigma_{ab}(T) = \bigcap \{\sigma_a(T+K) : TK = KT \text{ and } K \in K(H)\}.$$

The semigroup $\Phi_{\pm}(H) = \{T \in \Phi_+(H) \ \text{ind}(T) \leq 0\}$ was introduced in [20]. It is well known that

$$\sigma_{ea}(T) = \{ \lambda \in \mathbb{C} : T - \lambda \notin \Phi_{\pm}(H)\} \text{ [20]}$$

and

$$\sigma_{ab}(T) = \sigma_{ea}(T) \cup \{\text{limit points of } \sigma_a(T)\} \text{ [22]}.$$

Evidently, $\sigma_{ea}(T) \subseteq \sigma_{ab}(T)$.

We say that an operator T has the single valued extension property at λ (abbreviated SVEP at λ) if for every open set U containing λ the only analytic function $f : U \to H$ which satisfies the equation

$$(T - \lambda) f(\lambda) = 0$$

is the constant function $f \equiv 0$ on U. An operator T has SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

We say that Generalized Weyl's theorem holds for T if (in symbols, $T \in gw$) if

$$\sigma(T) \setminus \sigma_{BW}(T) = E(T).$$

We say that Generalized Browder's theorem holds for T if (in symbols, $T \in gB$) if

$$\sigma(T) \setminus \sigma_{BW}(T) = \pi(T).$$

We say that Generalized a - Weyl's theorem holds for T if (in symbols, $T \in gaW$) if

$$\sigma_a(T) \setminus \sigma_{B_{ea}}(T) = \pi_0^a(T).$$

We say that Generalized a - Browder's theorem holds for T if (in symbols, T $\in gaB$) if

$$\sigma_a(T) \setminus \sigma_{B_{ea}}(T) = p_0^a(T).$$

We say that a - Weyl's theorem holds for T if (in symbols, T $\in aw$) if

$$\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}^a(T)..$$

We say that a - Browder's theorem holds for T if (in symbols, T $\in aB$) if

$$\sigma_{ea}(T) = \sigma_{ab}(T).$$

In [25], H. Weyl proved that Weyl's theorem holds for hermitian operators. Weyl's theorem has been extended from hermitian operators to hyponormal operators [7], algebraically hyponormal operators [13], p - hyponormal operators [6] and algebraically p - hyponormal operators [10]. More generally, M. Berkani investigated generalized Weyl's theorem which extends Weyl's theorem, and proved that generalized Weyl's theorem holds for hyponormal operators [2, 3, 4]. In a recent paper [18] the author showed that generalized Weyl's theorem holds for (p, k) - quasihyponormal operators. Recently, X. Cao, M. Guo and B. Meng [5] proved Weyl type theorems for p - hyponormal operators.

In this paper, we prove that (i)Weyl's theorem holds for f(T) for every $f \in H(\sigma(T))$; (ii) a-Browder's theorem holds for f (S) for every $S \prec T$ and $f \in H(\sigma(S))$; (iii) the spectral mapping theorem holds for the Weyl spectrum of T and for the essential approximate point spectrum of T.

II. Weyl's Theorem For Algebraically K - Quasi - Paranormal Operators

Salah Mecheri [19] has introduced k - quasi - paranormal operators and has proved many interesting properties of it.

Definition 2.1 [19] An operator $T \in B(H)$ is said to be k - quasi - paranormal operator if $\left\|T^{k+1}x\right\|^2 \le \left\|T^{k+2}x\right\| \left\|T^k x\right\|$ for every $x \in H$, k is a natural number.

Definition 2.2 [19] An operator T is called algebraically k - quasi - paranormal if there exists a nonconstant complex polynomial s such that s(T) belongs to k - quasi - paranormal.

Lemma 2.3 [19] (1) Let $T \in B(H)$ be a k - quasi - paranormal, the range of T^k be not dense and

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

on $H = \overline{ran(T^k)} \oplus \ker(T^{*k})$. Then T_1 is paranormal, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$.

(2) Let M be a closed T - invariant subspace of H. Then the restriction $T|_M$ of a k - quasi - paranormal operator T to M is a k - quasi - paranormal.

Lemma 2.4 [19] Let $T \in B(H)$ be a k - quasi - paranormal operator. Then T has Bishop's property (β), i.e., if $f_n(z)$ is analytic on D and (T - z) $f_n(z) \to 0$ uniformly on each compact subset of D, then $f_n(z) \to 0$ uniformly on each compact subset of D. Hence T has the single valued extension property.

The following facts follows from the definition and some well known facts about k - quasi - paranormal operators [23, 9] :

Lemma 2.5 (i) If $T \in B(H)$ is algebraically k - quasi - paranormal, then so is $T - \lambda$ for each $\lambda \in \mathbb{C}$.

(ii) If $T \in B(H)$ is algebraically k - quasi - paranormal and M is a closed T - invariant subspace of H, then $T|_M$ is algebraically k - quasi - paranormal.

(iii) If T is algebraically k - quasi - paranormal, then T has SVEP.

(iv) Suppose T does not have dense range. Then we have:

T is k - quasi - paranormal $\Leftrightarrow$ $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$

on $H = \overline{ran(T^k)} \oplus \ker(T^{*k})$ where T_1 is paranormal operator.
In general, the following implications hold:

paranormal $\Rightarrow$ k - quasi - paranormal $\Rightarrow$ algebraically k - quasi - paranormal.

Proposition 2.6 [24] Suppose that T is algebraically k - quasi - paranormal. Then T has Bishop's property (β).

Corollary 2.7 [24] Suppose T is algebraically k - quasi - paranormal. Then T has SVEP.

Lemma 2.8 [24] Let $T \in B(H)$ be a quasinilpotent algebraically k - quasi - paranormal operator. Then T is nilpotent.

An operator $T \in B(H)$ is called isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T and an operator $T \in B(H)$ is called polaroid if iso $\sigma(T) \subseteq \pi_0(T)$. In general, if T is polaroid then it is isoloid. However, the converse is not true. Consider the following example. Let $T \in B(l_2)$ be defined by

$$T(x_1, x_2, x_3, ...) = \left(\frac{1}{2}x_2, \frac{1}{3}x_3, ...\right)$$

Then T is a compact quasinilpotent operator with $\alpha(T) = 1$, and so T is isoloid. However, since a(T) = ∞, T is not polaroid. It is well known that every algebraically paranormal operator is isoloid. We now extend this result to algebraically k - quasi - paranormal operators.

Lemma 2.9 Let $T \in B(H)$ be an algebraically k - quasi - paranormal operator. Then T is polaroid.

Proof: Suppose T is algebraically k - quasi - paranormal operator. Then p(T) is k - quasi - paranormal operator for some nonconstant polynomial p. Let $\lambda \in$ iso $\sigma(T)$. Using the spectral projection $P = \frac{1}{2\pi i}\int_{\partial D}(\mu - T)^{-1}d\mu$, where D is a closed disk of center λ which contains no other points of $\sigma(T)$, we can represent T as the direct sum

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$$

where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T) \setminus \{\lambda\}$.

Since T_1 is algebraically k - quasi - paranormal operator, $T_1 - \lambda$ is algebraically k - quasi - paranormal operator. But $\sigma(T_1 - \lambda) = \{0\}$, it follows from Lemma 2.5 that $T_1 - \lambda$ is nilpotent. Therefore $T_1 - \lambda$ has finite ascent and descent. On the other hand, since $T_2 - \lambda$ is invertible, clearly it has finite ascent and descent. Therefore T - λ has finite ascent and descent, and hence λ is a pole of the resolvent of T. Thus $\lambda \in$ iso $\sigma(T)$ implies $\lambda \in \pi_0(T)$, and so iso $\sigma(T) \subseteq \pi_0(T)$. Hence T is polaroid.

Let $H(\sigma(T))$ is the space of functions analytic in an open neighborhood of $\sigma(T)$.

Theorem 2.10 Suppose T or T^* is algebraically k - quasi - paranormal operator. Then f (T) $\in$ W for every f $\in H(\sigma(T))$.

Proof : Suppose T is algebraically quasi-paranormal. We first show that T $\in$W. Suppose $\lambda \in \sigma(T) \setminus \sigma_w(T)$. Then T - λ is Weyl but not invertible. We claim that $\lambda \in \partial\sigma(T)$. Assume that λ is an interior point of $\sigma(T)$. Then there exist a neighbourhood U of λ, such that dim N(T - λ) > 0 for all $\lambda \in$ U. It follows from [12, Theorem 10] that T doesnot have single valued extension property [SVEP]. On the other hand, since p(T) is k- quasi - paranormal operator for some non constant polynomial p, it follows from Lemma 2.5, that T has SVEP. It is a contradiction, Therefore $\lambda \in \partial\sigma(T)$ and it follows from the punctured neighbourhood theorem that $\lambda \in \pi_{00}(T)$.

Conversely suppose that $\lambda \in \pi_{00}(T)$. Using the Spectral Projection $P = \frac{1}{2\pi i}\int_{\partial D}(\mu - T)^{-1}d\mu$ where D is the closed disk centered at λ which contains no other point of $\sigma(T)$.

We can represent T as the direct sum $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T) \setminus \{\lambda\}$.

Since $\sigma(T_1) = \{\lambda\}$, $T_1 - \lambda$ is quasinilpotent. But T is algebraically k - quasi - paranormal, hence T_1 is also algebraically k - quasi - paranormal. It follows from Lemma 2.8 that $T_1 - \lambda$ is nilpotent. Since $\lambda \in \pi_{00}(T)$, $T_1 - \lambda$ is a finite dimensional operator. Therefore $T_1 - \lambda$ Weyl. Since $T_2 - \lambda$ is invertible, T - λ is Weyl. Thus T $\in$ W.

Now we have to prove that $f(\sigma_w(T)) = \sigma_w(f(T))$, for every function f analytic in a neighborhood of $\sigma(T)$. Let f be a analytic function in a neighborhood of $\sigma(T)$. Since $\sigma_w(f(T)) \subseteq f(\sigma_w(T))$ with no restriction on T, it is sufficient to prove that $f(\sigma_w(T)) = \sigma_w(f(T))$. Assume that $\lambda \notin \sigma_w(f(T))$. Then $f(T)$ is Weyl and

$$f(T) - \lambda = c(T-\alpha_1)(T-\alpha_2)(T-\alpha_3).......(T-\alpha_n)g(T) \qquad (2.1)$$

where $c, \alpha_1, \alpha_2, ...\alpha_n \in \mathbb{C}$ and g(T) is invertible. Since the operators on the right hand side (2.1) commute, every T - α_i is Fredholm. Since T is algebraically k- quasi - paranormal operator, T has SVEP by Lemma 2.5. Therefore by [1, Corollary 3.19] i(T - α_i) ≥ 0 for each i = 1, 2, 3,...., n. Therefore $\lambda \notin f(\sigma_w(T))$, hence $f(\sigma_w(T)) = \sigma_w(f(T))$. It is known that if T is isoloid [1, Lemma 3.89] then

$$f(\sigma(T)) \setminus \pi_{00}(f(T)) = \sigma(f(T)) \setminus \pi_{00}(f(T))$$

for every analytic function in a neighborhood of $\sigma(T)$. Since T is isoloid by Lemma 2.9 and Weyl's theorem holds for f(T),

$$\sigma(f(T)) \setminus \pi_{00}(f(T)) = f(\sigma(T)) \setminus \pi_{00}(f(T)) = f(\sigma_w(T)) = \sigma_w(f(T))$$

Now suppose that T^* is algebraically k- quasi - paranormal operator. We first show that T $\in$ W. Suppose that $\lambda \in \sigma(T) \setminus \sigma_w(T)$. Observe that $\sigma(T^*) = \overline{\sigma(T)}$ and $\sigma_w(T^*) = \overline{\sigma_w(T)}$, so $\overline{\lambda} \in \sigma(T^*) \setminus \sigma_w(T^*)$. Since T^* $\in$ W, $\overline{\lambda} \in \pi_{00}(T^*)$. Therefore λ is an isolated point of $\sigma(T)$, and so $\lambda \in \pi_{00}(T)$. Conversely, suppose that $\lambda \in \pi_{00}(T)$. Then λ is an isolated point of $\sigma(T)$ and 0 < α (T - λ) < ∞. Since $\overline{\lambda}$ is an isolated point of $\sigma(T^*)$ and T^* is algebraically k- quasi - paranormal operator, it follows from Lemma 2.9 that $\lambda \in \pi(T^*)$. So $\lambda \in \pi(T)$, and hence T - λ is Weyl.

Consequently, $\lambda \in \sigma(T) \setminus \sigma_w(T)$. Thus T $\in$ W. Now we show that $f(\sigma_w(T)) = \sigma_w(f(T))$, for every function f analytic in a neighborhood of $\sigma(T)$. It is sufficient to show that $f(\sigma_w(T)) = \sigma_w(f(T))$. Suppose that $\lambda \in \sigma_w(f(T))$. Then f (T) - λ is Weyl. Since T^* is algebraically k- quasi - paranormal operator, it has SVEP. It follows from [1, Corollary 3.19] that
i(T - α_i) ≥ 0 for each i = 1, 2, 3,...., n.. Since

$$0 \leq \sum_{i=1}^{n} i(T - \alpha_i) = i(f(T) - \lambda) = 0,$$

T - α_i is Weyl for each i = 1, 2,...., n. Hence $\lambda \notin f(\sigma_w(T))$, and so $f(\sigma_w(T)) \subseteq \sigma_w(f(T))$. Thus $f(\sigma_w(T)) = \sigma_w(f(T))$ for each f $\in$ H($\sigma(T)$). Since T $\in$ W and T is isoloid, f (T) $\in$ W for every f $\in$ H($\sigma(T)$). This completes the proof.

Corollary 2.11 Suppose T or T^* is algebraically k- quasi - paranormal operator. Then $f(\sigma_w(T)) = \sigma_w(f(T))$ for each f $\in$ H($\sigma(T)$).

III. A - Weyl's Theorem For Algebraically K - Quasi - Paranormal Operators

In this section we show that the spectral mapping theorem holds for the essential approximate point spectrum for algebraically k - quasi - paranormal operators.

Theorem 3.1 Suppose T or T^* is algebraically k - quasi - paranormal operator. Then $f(\sigma_{ea}(T)) = \sigma_{ea}(f(T))$ for each f $\in$ H($\sigma(T)$).

Proof : Suppose first that T is algebraically k- quasi - paranormal and let f $\in$ H($\sigma(T)$). It suffices to show that $f(\sigma_{ea}(T)) = \sigma_{ea}(f(T))$. Assume that $\lambda \notin \sigma_{ea}(f(T))$. Then f(T) - $\lambda \in \Phi_+^-(H)$ and

$$f(T) - \lambda = c(T-\alpha_1)(T-\alpha_2)(T-\alpha_3).......(T-\alpha_n)g(T) \qquad (3.1)$$

where $c, \alpha_1, \alpha_2, ...\alpha_n \in \mathbb{C}$ and g(T) is invertible. Since the operators on the right hand side (3.1) commute, every T - α_i is Fredholm. Since T is algebraically k- quasi - paranormal operator, T has SVEP by Lemma 2.5. Therefore by [1, Corollary 3.19] i(T - α_i) ≤ 0 for each i = 1, 2, 3,...., n. Therefore $\lambda \notin f(\sigma_{ea}(T))$, hence $f(\sigma_{ea}(T)) = \sigma_{ea}(f(T))$. Suppose now that T^* is algebraically k- quasi - paranormal operator, it has SVEP. It follows from [1, Corollary 3.19] that
i(T - α_i) ≥ 0 for each i = 1, 2, 3,...., n.. Since

$$0 \leq \sum_{i=1}^{n} i(T - \alpha_i) = i(f(T) - \lambda) = 0,$$

T - α_i is Weyl for each i = 1, 2,...., n. Hence $\lambda \notin f(\sigma_{ea}(T))$, and so $f(\sigma_{ea}(T)) \subseteq \sigma_{ea}(f(T))$. Thus $f(\sigma_{ea}(T)) = \sigma_{ea}(f(T))$ for each f $\in$ H($\sigma(T)$). This completes the proof.

An operator X $\in$ B(H) is called a quasiaffinity if it has trivial kernel and dense range. An operator S $\in$ B(H) is said to be a quasiaffine transform of T $\in$ B(H) (notation: S $\prec$ T) if there is a quasiaffinity X $\in$ B(H) such that XS = TX. If both S $\prec$ T and T $\prec$ S, then we say that S and T are quasisimilar. In general, we cannot expect that Weyl's theorem holds for operators having SVEP.

Theorem 3.2 Suppose T is algebraically k- quasi - paranormal operator and that S $\prec$ T. Then f (S) $\in$ aB for every f $\in$ H($\sigma(T)$).

Proof: Suppose T is algebraically k- quasi - paranormal and that S $\prec$ T. We first show that S has SVEP. Let U be any open set and let f : U $\rightarrow$ H be any analytic function such that (S - λ) f (λ) = 0 for all λ $\in$ U. Since S $\prec$ T, there exists a quasiaffinity X such that X S = T X. So X (S - λ) = (T - λ) X for all λ $\in$ U. Since (S - λ) f (λ) = 0 for all λ $\in$ U, 0 = X(S - λ) f (λ) = (T - λ) X f (λ) for all λ $\in$ U. But T is algebraically k - quasi - paranormal, hence T has SVEP. Therefore X f (λ) = 0 for all λ $\in$ U. Since X is a quasiaffinity, f (λ) = 0 for all λ $\in$ U. Therefore S has SVEP. Now we show that S $\in$ aB. It is well known that $\sigma_{ea}(S) \subseteq \sigma_{ab}(S)$. Conversely, suppose that $\lambda \in \sigma_a(S) \setminus \sigma_{ea}(S)$. Then S - $\lambda \in \Phi_+^-(H)$ and S - λ is not bounded below. Since S has SVEP and S - $\lambda \in \Phi_+^-(H)$, it follows from [1, Theorem 3.16] that a(S - λ) < ∞. Therefore by [21, Theorem 2.1], $\lambda \in \sigma_a(S) \setminus \sigma_{ab}(S)$.

Thus S $\in$ aB. Let f $\in$ H($\sigma(S)$)be arbitrary. Since S has SVEP, it follows from the proof of Theorem 3.1 that $\sigma_{ea}(f(S)) = f(\sigma_{ea}(S))$. Therefore

$$\sigma_{ab}(f(S)) = f(\sigma_{ab}(S)) = f(\sigma_{ea}(S)) = \sigma_{ea}(f(S)),$$

and hence f (S) $\in$ aB.

An operator T $\in$ B(H) is called a-isoloid if every isolated point of $\sigma_a(T)$ is an eigenvalue of T. Clearly, if Tis a-isoloid then it is isoloid.

Theorem 3.3 Suppose T^* is algebraically k- quasi - paranormal operator. Then f (T) $\in$ aW for every f $\in$ H($\sigma(T)$).

Proof : Suppose T^* is algebraically k- quasi - paranormal operator. We first show that T $\in$ aW. Suppose that $\lambda \in \sigma_a(S) \setminus \sigma_{ea}(S)$. Then T - λ is upper semi-Fredholm and i(T - λ) $\leq$ 0. Since T^* is algebraically k- quasi - paranormal, T^* has SVEP. Therefore by [1, Corollary 3.19] i(T - λ) $\geq$ 0, and hence T - λ is Weyl. Since T^* has SVEP, it follows from [12, Corollary 7] that $\sigma(T) = \sigma_a(T)$. Also, since T $\in$ W by Theorem 2.10, $\lambda \in \pi_{00}(T)$..

Conversely, suppose that $\lambda \in \pi_{00}(T)$. Since T^* has SVEP, $\sigma(T) = \sigma_a(T)$. Therefore λ is an isolated point of $\sigma(T)$, and hence $\overline{\lambda}$ is an isolated point of $\sigma(T^*)$. But T^* is algebraically k- quasi - paranormal, hence by Lemma 2.9 that $\overline{\lambda} \in \pi(T^*)$. Therefore $\lambda \in \pi(T)$, and hence T - λ is Weyl. So $\lambda \in \sigma_a(T) \setminus \sigma_{ea}(T)$. Thus T $\in$ aW. Now we show that T is a-isoloid. Let λ be an isolated point of $\sigma_a(T)$. Since T^* has SVEP, λ is an isolated point of $\sigma(T)$. But T^* is polaroid, hence T is also polaroid. Therefore it is isoloid, and hence $\lambda \in \sigma_p(T)$. Thus T is a-isoloid.

Finally, we shall show that f (T) $\in$ aW for every f $\in$ H($\sigma(T)$). Let f $\in$ H($\sigma(T)$). Since T $\in$ aW, $\sigma_{ea}(T) = \sigma_{ab}(T)$. It follows from Theorem 3.1 that

$$\sigma_{ab}(f(T)) = f(\sigma_{ab}(T)) = f(\sigma_{ea}(T)) = \sigma_{ea}(f(T)),$$

and hence f (S) $\in$ aB. So $\sigma_a(f(T)) \setminus \sigma_{ea}(f(T)) \subseteq \pi_{00}^a(f(T))$..

Conversely, suppose $\lambda \in \pi_{00}^a(f(T))$. Then λ is an isolated point of $\sigma_a(f(T))$ and 0 < α (f (T) - λ) < ∞. Since λ is an isolated point of $f(\sigma_a(T))$, if $\alpha_i \in \sigma_a(T)$ then α_i is an isolated point of $\sigma_a(T)$ by (3.1). Since T is a-isoloid, 0 < α (T - α_i) < ∞ for each i = 1, 2,..., n. Since T $\in$ aW, T - α_i is upper semi-Fredholm and i(T - α_i) ≤ 0 for each i = 1, 2, ... , n. Therefore f (T) - λ is upper semi-Fredholm and i(f (T) - λ) = $\sum_{i=1}^{n} i(T - \alpha_i) \leq 0$. Hence $\lambda \in \sigma_a(f(T)) \setminus \sigma_{ea}(f(T))$, and so f (T) $\in$ aW for each f $\in$ H($\sigma(T)$). This completes the proof.

IV. On Totally K - Quasi - Paranormal Operators

Let T be a totally k - quasi - paranormal operator on a complex Hilbert space H. Let B(H) denote the algebra of all bounded linear operators acting on an infinite dimensional separable Hilbert space H. In this chapter we show that Weyl's theorem holds for Algebraically totally k- quasi - paranormal operator.

The conditionally totally posinormal was introduced by Bhagawati Prashad and Carlos Kubrusly [11]. In this section we focus on Weyl's theorem for algebraically totally k- quasi - paranormal operators.

Definition 4.1 An operator T is called totally k- quasi - paranormal operator, if the translate T - λ is k- quasi - paranormal operator for all $\lambda \in \mathbb{C}$.

In this section we study some properties of totally k- quasi - paranormal operator. The following Lemma summarizes the basic properties of such operators.

Lemma 4.2 If T is totally k- quasi - paranormal operator, then ker $T^k \subset$ ker T^{*k}, ker $T^k \subset$ ker T^{2k}, r(T) = ‖ T ‖, and T $|_M$ is a totally k- quasi - paranormal operator, where r(T) denotes the spectral radius of T and M is any invariant subspace for T.

Lemma 4.3 Every totally k- quasi - paranormal operator has the single valued extension property.

Proof : It is easy to prove that, by Lemma 4.2, T - λ has finite ascent for each λ. Hence T has the single valued extension property by [16].

Recall that an operator X $\in$ L(H, K) is called a quasiaffinity if it has trivial kernal and dense range. An operator S $\in$ L(H) is said to be quasiaffine transform of an operator T $\in$ L(K) if there is a quasiaffinity X $\in$ L(H, K) such that XS = TX.

If T has the single valued extension property, then for any x $\in$H there exists a unique maximal open set ρ T(x) ($\subset \rho$ (T)) and a unique H - valued analytic function f defined in ρ T(x) such that (T - λ)f(λ) = x, $\lambda \in \rho$ T(x). Moreover, if F is a closed set in $\mathbb{C}$ and $\sigma_T(x)$ = $\mathbb{C}$ ρ T(x), then

$$H_T(F) = \{x \in H : \sigma_T(x) \subset F\}$$

is a linear subspace of H [8].

Corollary 4.4 If T is totally k- quasi - paranormal operator, then

$$H_T(\lambda) = \{x \in H : \lim_{n\to\infty} \left\|(T-\lambda)^n x\right\|^{\frac{1}{n}} = 0\}$$

Proof : Since T has the single valued extension property by Lemma 4.3, the proof follows from [16].

Lemma 4.5 If T is totally k- quasi - paranormal operator, then it is isoloid.

Proof : Since T has the translation invariance property, it suffices to show that if 0 $\in$ iso $\sigma(T)$, then 0 $\in \sigma_p(T)$. Choose ρ > 0 sufficiently small that 0 is the only point of $\sigma(T)$ contained in or on the circle $|\lambda| = \rho$. Define

$$E = \int_{|\lambda|=\rho} (\lambda I - T)^{-1} d\lambda .$$

Then E is the Riesz idempotent corresponding to 0. So M = E(H) is an invariant subspace for T, M $\neq$ {0}, and $\sigma(T|_M) = \{0\}$. Since $T|_M$ is also totally k- quasi - paranormal operator, $T|_M$ = 0. Therefore, T is not one-to-one. Thus 0 $\in \sigma_p(T)$.

Theorem 4.6 Weyl's theorem holds for any totally k- quasi - paranormal operator.

Proof : If T is totally k- quasi - paranormal operator, then it has the single valued extension property from Lemma 4.3. By [9, Theorem 2], it suffices to show that $H_T(\lambda)$ is finite dimensional for $\lambda \in \pi_{00}(T)$. If $\lambda \in \pi_{00}(T)$, then λ $\in$ iso $\sigma(T)$ and 0 < dim ker(T - λ) < ∞. Since ker(T - λ) is a reducing subspace for T - λ, write T - λ = 0 $\oplus$ (T_1 - λ), where 0 denotes the zero operator on ker(T - λ) and T_1 - λ = (T_1 - λ) $|_{(\ker(T - \lambda))^\perp}$ is injective. Therefore,

$$\sigma(T-\lambda) = \{0\} \cup \sigma(T_1 - \lambda)$$

If T_1 - λ is not invertible, 0 $\in$ $\sigma(T_1 - \lambda)$. Since $\sigma(T-\lambda) = \{0\} \cup \sigma(T_1 - \lambda)$, $\sigma(T-\lambda) = \sigma(T_1 - \lambda)$. Since $\lambda \in \pi_{00}(T)$, λ $\in$ iso $\sigma(T_1)$. Since T is totally k - quasi - paranormal, it is easy to show that T_1 is totally k- quasi - paranormal operator. Since T_1 is isoloid by Lemma 4.5, $\lambda \in \sigma_p(T_1)$. Therefore, ker($T_1$ - λ) $\neq \{0\}$. So we have a contradiction. Thus T_1 - λ is invertible. Therefore, (T - λ)((ker(T - λ))$^\perp$ = (ker(T - λ))$^\perp$. Thus (ker(T - λ))$^\perp$ $\subset$ ran(T - λ). Since ker(T - λ) $\subset$ ker(T - λ)* = (ran(T - λ))$^\perp$. Therefore, ran(T - λ) = (ker(T - λ))$^\perp$. Thus ran(T - λ) is closed. Since dim ker(T - λ) < ∞, T - λ is semi-Fredholm. By [15, Lemma 1], $H_T(\{\lambda\})$ is finite dimensional.

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V. Algebraically Totally k - Quasi - Paranormal Operators

Definition 5.1 An operator T $\in$ B(H) is called algebraically totally k- quasi - paranormal operator if there exists a nonconstant complex polynomial p such that p(T) is totally k- quasi - paranormal operator.

The following facts follow from the above Definition 5.1 and the well known facts of totally k- quasi - paranormal operator.

If T $\in$ B(H) is algebraically totally k- quasi - paranormal operator and M $\subseteq$ H is invariant under T, then (T $|_M$) is algebraically totally k- quasi - paranormal operator.

Lemma 5.2 If T $\in$ B(H) is algebraically totally k- quasi - paranormal operator and quasinilpotent, then T is nilpotent.

Proof : Suppose p(T) is totally k- quasi - paranormal operator for some nonconstant polynomial p. Since totally k- quasi - paranormal is translation-invariant, we may assume p(0) = 0. Thus we can write $p(\lambda)=a_0\lambda^m(\lambda-\lambda_1)(\lambda-\lambda_2).......(\lambda-\lambda_n)(m\neq 0,\ \lambda_i\neq 0)$ for every $(1\leq i\leq n)$. If T is quasinilpotent, then $\sigma(p(T))=p(\sigma(T))=p(\{0\})=\{0\}$, so that p(T) is also quasinilpotent. Since the only k- quasi - paranormal quasinilpotent operator is zero, it follows that

$$a_0T^m(T-\lambda_1I)(T-\lambda_2I).......(T-\lambda_nI)=0$$

since T - λ_iI is invertible for every $(1\leq i\leq n)$, we have $T^m = 0$.

Lemma 5.3 If T $\in$ B(H) is algebraically totally k- quasi - paranormal operator, then T is isoloid.

Proof : Suppose p(T) is totally k- quasi - paranormal for some nonconstant polynomial p. Let $\lambda\in\sigma(T)$. Then using the spectral decomposition, we can represent T as the direct sum T = $T_1 \oplus T_2$, where $\sigma(T_1)=\{\lambda\}$ and $\sigma(T_2)=\sigma(T)\setminus\{\lambda\}$. Note that T_1 - λ I is also algebraically totally k- quasi - paranormal operator. Since T_1 - λ I is quasinilpotent, by Lemma 5.2 , T_1 - λ I is nilpotent. Therefore $\lambda\in\pi_0(T)$. This shows that T is isoloid.

Theorem 5.4 Let T be an algebraically totally k- quasi - paranormal operator. Then T is polaroid.

Proof : Let T be an algebraically totally k- quasi - paranormal operator. Then p(T) is totally k- quasi - paranormal for some non constant polynomial p. Let λ $\in$ iso $\sigma(T)$. Using the spectral projection $P=\frac{1}{2\pi i}\int_{\partial D}(\mu-T)^{-1}d\mu$ where D is the closed disk centered at μ which contains no other point of $\sigma(T)$.

We can represent T as the direct sum $T=\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ where $\sigma(T_1)=\{\lambda\}$ and $\sigma(T_2)=\sigma(T)\setminus\{\lambda\}$.

Since T_1 is algebraically k- quasi - paranormal operator and $\sigma(T_1)=\{\lambda\}$. But $\sigma(T_1)-\lambda I=0$ it follows from Lemma 5.2, that T_1 - λ I is nilpotent. Therefore T_1 - λ I has finite ascent and descent. On the other hand, since T_1 - λ I is invertible, clearly it has finite ascent and descent. Therefore T - λ I has finite ascent and descent. Therefore λ is a pole of the resolvent of T. Thus if λ $\in$ iso $\sigma(T)$ implies λ $\in\pi(T)$, and so iso($\sigma(T)$)$\subset\pi(T)$. Hence T is polaroid

Theorem 5.5 Let T^* $\in$ B(H) be an algebraically totally k- quasi - paranormal operator. Then T is a - isoloid.

Proof : Suppose T^* is algebraically totally k- quasi - paranormal operator. Since T^* has SVEP, then $\sigma(T)$ = $\sigma_a(T)$. Let $\lambda\in\sigma_a(T)=\sigma(T)$. But T^* is polaroid, hence T is also polaroid. Therefore it is isoloid, and hence $\lambda\in\sigma_p(T)$. Thus T is a - isoloid.

Theorem 5.6 Let T be an algebraically totally k- quasi - paranormal operator. Then T has SVEP.

Proof : First we show that if T is totally k- quasi - paranormal operator, then T has SVEP. Suppose that T is totally k- quasi - paranormal operator. If $\pi_0(T)=\phi$, then clearly T has SVEP. Suppose that $\pi_0(T)\neq\phi$, let $\Delta(T)=\lambda\in\pi_0(T):N(T-\lambda)\subseteq N(T^*-\overline{\lambda})$. Since T is totally k- quasi - paranormal operator and $\pi_0(T)=\phi,\Delta(T)=\phi$. Let M be the closed linear span of the subspaces N(T - λ) with $\lambda\in\Delta(T)$. Then M reduces T and we can write T as $T_1\oplus T_2$ on H = M $\oplus$ M$^{\perp}$. Clearly T_1 is normal and $\pi_0(T_2)=\phi$. Since T_1 and T_2 have both SVEP, T has SVEP. Suppose that T is algebraically totally k- quasi - paranormal operator. Then p(T) is totally k- quasi - paranormal operator for some non constant polynomial p. Since p(T) has SVEP, it follows from [17, Theorem 3.3.9] that T has SVEP.

Theorem 5.7 Weyl's theorem holds for algebraically totally k- quasi - paranormal operator.

Proof: Suppose that $\lambda\in\sigma(T)\setminus\sigma_w(T)$. Then T - λ is Weyl and not invertible, we claim that $\lambda\in\partial\sigma(T)$. Assume that λ is an interior point of $\sigma(T)$. Then there exist a neighbourhood U of λ, such that dim N(T - λ) > 0 for all $\lambda\in U$. It follows from [12, Theorem 10] that T doesnot have single valued extension property [SVEP]. On the other hand, since p(T) is k- quasi - paranormal operator for some non constant polynomial p, it follows from Theorem 5.6 . That T has SVEP. It is a contradiction, therefore $\lambda\in\partial\sigma(T)$.

Conversely suppose that $\lambda\in\pi_{00}(T)$. Using the Riesz idempotent $E_\lambda=\frac{1}{2\pi i}\int_{\partial D\lambda}(\mu-T)^{-1}d\mu$ where D is the closed disk centered at λ which contains no other point of $\sigma(T)$.

We can represent T as the direct sum $T=\begin{pmatrix}T_1 & 0\\ 0 & T_2\end{pmatrix}$ where $\sigma(T_1)=\{\lambda\}$ and $\sigma(T_2)=\sigma(T)\setminus\{\lambda\}$. Now we consider two cases

case (i): λ = 0

Here T_1 is algebraically k- quasi - paranormal operator and quasinilpotent. It follows from Lemma 5.2, that T_1 is nilpotent. We claim that dim R(E) < ∞. For if N(T_1) is infinite dimensional, then $0\notin\pi_{00}(T)$. It is contradiction. Therefore T_1 is a finite dimensional operator. So it follows that T_1 is Weyl. But since T_2 is invertible, we can conclude that T is Weyl. Therefore $0\in\sigma(T)\setminus\sigma_w(T)$.

case (ii): $\lambda\neq 0$.

By Lemma 5.3, that T_1 - λ is nilpotent. Since $\lambda\in\pi_{00}(T)$, T_1 - λ is a finite dimensional operator. So T_1 - λ is Weyl. Since T_2 - λ is invertible, T - λ is Weyl.

By case (i) and case (ii), Weyl's theorem holds for T. This complete the proof.

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