Hardy-Steklov operator on two exponent Lorentz spaces for non-decreasing functions

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Abstract: In this paper, we obtain the characterization on pair of weights \( v \) and \( w \) so that the Hardy-Steklov operator \( \int_{a(x)}^{b(x)} f(t)dt \) is bounded from \( L^p_v (0, \infty) \) to \( L^q_w (0, \infty) \) for \( 0 < p, q, r, s < \infty \).

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I. Introduction

By a weight function \( u \) defined on \((0, \infty)\) we mean a non-negative locally integrable measurable function. We take \( M_0^+ = M_0^+ ((0, \infty), u(x)dx) \) to be the set of functions which are measurable, non-negative and finite a.e. on \((0, \infty)\) with respect to the measure \( u(x)dx \). Then the distribution function \( \tilde{\lambda}^u_f (t) \) of \( f \in M_0^+ \) is given by

\[
\tilde{\lambda}^u_f (t) := \int_{\{x \in (0, \infty) : f(x) \geq t\}} u(x)dx, \quad t \geq 0.
\]

The non-increasing rearrangement \( f_u^* \) of \( f \) with respect to \( du(x) \) is defined as

\[
f_u^*(y) := \inf \{ t : \tilde{\lambda}^u_f (t) \leq y \}, \quad y \geq 0.
\]

For \( 0 < p < \infty, \ 0 < q \leq \infty \), the two exponent Lorentz spaces \( L^{p,q}_v (0, \infty) \) consist of \( f \in M_0^+ \) for which

\[
\|f\|_{L^{p,q}_v} := \left\{ \left( \int_0^\infty \left( \int_0^t f^p_u(x) \frac{dx}{x} \right)^{\frac{q}{p}} \left( \frac{dt}{t} \right)^{\frac{q}{q-p}} \right)^{\frac{1}{q}} \right\}^{\frac{1}{q}}, \quad 0 < q < \infty,
\]

\[
\|f\|_{L^{p,q}_v} := \sup_{t > 0} \left( \int_0^t f^p_u(x) \frac{dx}{x} \right)^{\frac{q}{p}} \left( \frac{dt}{t} \right)^{\frac{q}{q-p}}, \quad q = \infty
\]

is finite.

In this paper, we characterize the weights \( v \) and \( w \) for which a constant \( C > 0 \) exists such that

\[
\|Tf\|_{L^{r,s}_w} \leq C \|f\|_{L^{p,q}_v}, \quad f \geq 0
\]

where \( T \) is the Hardy-Steklov operator defined as

\[
(Tf)(x) = \int_{a(x)}^{b(x)} f(t)dt.
\]
The functions \( a = a(x) \) and \( b = b(x) \) in (3) are strictly increasing and differentiable on \( (0, \infty) \).
Also, they satisfy
\[
a(0) = b(0) = 0; \quad a(\infty) = b(\infty) = \infty \quad \text{and} \quad a(x) < b(x) \quad \text{for} \quad 0 < x < \infty.
\]

Clearly, \( a^{-1} \) and \( b^{-1} \) exist, and are strictly increasing and differentiable. The constant \( C \) attains different bounds for different appearances.

II. Lemmas

**Lemma 1.** We have
\[
\|f\|_{L^p_{[0,\infty)}}^p = \left( \int_0^\infty s^{p-1} [\lambda_f^v(t)]^{slr} \, dt \right)^{\frac{p}{s}}, \quad 0 < s < \infty
\]  
\[
s = \frac{r}{s}, \quad 0 < r < \infty, \quad \text{and} \quad \sup_{t > 0} t [\lambda_f^v(t)]^{slr}, \quad s = \infty.
\]

**Proof.** Applying the change of variable \( y = \lambda_f^v(t) \) to the R.H.S. of (1) and integrating by parts we get the lemma. \( \Box \)

**Lemma 2.** If \( f \) is nonnegative and non-decreasing, then
\[
\|f\|_{L^p_s}^s = \frac{s}{r} \int_0^\infty f^s(x) \left( \int_0^\infty v(t) \, dt \right)^{\frac{s-1}{s}} v(x) \, dx.
\]

**Proof.** We obtain the above equality by evaluating the two iterated integrals of \( st^{s-1} \left( \frac{s}{r} \right) h^{-\frac{r-1}{r}} (x) v(x) \)
over the set \( \{ (x,t); 0 < t < f(x), 0 < x \} \), so that we have
\[
\int_0^\infty \int_0^f f(x) \left( \frac{s}{r} \right) h^{-\frac{r-1}{r}} (x) v(x) \, dx \, dt = \int_0^\infty \int_0^\infty st^{s-1} \left( \frac{s}{r} \right) h^{-\frac{r-1}{r}} (x) v(x) \, dx \, dt,
\]
where \( x(t) = \sup \{ x : f(x) \leq t \} \) for a fixed \( t \), and \( h(x) = \int_0^\infty v(t) \, dt \).

Integrating with respect to \( t \) first, the L.H.S. of (6) gives us the R.H.S. of (5). Further
\[
\frac{s}{r} \int_0^\infty h^{-\frac{r-1}{r}} (x) v(x) \, dx = h(x(t))^{\frac{s}{r}} = \left( \int_{x(t)}^\infty v(s) \, ds \right)^{\frac{s}{r}} = \left[ v \left( x : f(x) > t \right) \right]^{\frac{s}{r}} = \left[ \lambda_f^v(t) \right]^{\frac{s}{r}}.
\]

Hence the lemma now follows in view of Lemma 1. \( \Box \)

III. Main Results

**Theorem 1.** Let \( 0 < p, q, r, s < \infty \) be such that \( 1 < q \leq s < \infty \). Let \( T_f \) be the Hardy-Steklov operator given in (3) with functions \( a \) and \( b \) satisfying the conditions given thereat. Also, we assume that \( a'(x) < b'(x) \) for \( x \in (0, \infty) \).

Then the inequality
\[
\left( \int_0^\infty \frac{s}{r} \left[ T_f(x) \right]^s \, x^{slr} \, dx \right)^{\frac{1}{ls}} \leq C \left( \int_0^\infty \frac{q}{p} \left[ f_r^v(x) \right]^q \, x^{qdr} \, dx \right)^{\frac{1}{lq}}
\]

where \( C \) is a constant. \( \Box \)
holds for all nonnegative non-decreasing functions $f$ if and only if

$$
\sup_{0 < s < x < \infty} \left( \frac{s}{r} \int_0^s \left( \int_y^\infty w(z)dz \right)^{\frac{q}{r}} w(y)dy \right) \frac{1}{s} \left( \int_0^s \int_0^x v(z)dz^p \right)^{\frac{1}{p}} v(y)dy \right) < \infty. \tag{8}
$$

**Proof.** Using differentiation under the integral sign, the condition $a'(x) < b'(x)$ for $x \in (0, \infty)$ ensures that $Tf$ is nonnegative and non-decreasing. Consequently, by Lemma 2, the inequality (7) is equivalent to

$$
\left( \int_0^s \left( \int_0^x w(z)dz \right)^{\frac{q}{r}} w(y)dy \right)^{\frac{1}{s}} \leq \frac{1}{s} \left( \int_0^s \int_0^x v(z)dz^p \right)^{\frac{1}{p}} v(x)dx \tag{9}
$$

where $W(x) = \frac{s}{r} \left( \int_0^x w(z)dz \right)^{\frac{q}{r}} w(x)$ and $V(x) = \frac{q}{p} \left( \int_x^\infty v(z)dz^p \right)^{\frac{1}{p}} v(x)$.

Thus it suffices to show that (9) holds if and only if (8) holds. The result now follows in view of Theorem 3.11 [2].

Similarly, in view of Theorem 2.5 [1], by making simple calculations, we may obtain the following:

**Theorem 2.** Let $0 < p, q, r, s, \infty$ be such that $0 < s < q, 1 < q < \infty$. Let $T$ be the Hardy-Steklov operator given in (3) with functions $a$ and $b$ satisfying the conditions given thereat. Also, we assume that $a'(x) < b'(x)$ for $x \in (0, \infty)$. Then the inequality (7) holds for all nonnegative non-decreasing functions $f$ if and only if

$$
\left( \int_0^\infty \left[ a^s(t) - b^s(t) \right]^{\frac{q}{p}} v(t)^{\frac{q}{p}} dx \right)^{\frac{r}{l}} \leq \frac{q}{p} \left( \int_0^\infty v(x)dy \right)^{\frac{q}{p}} v(x)dx \sigma(t)dt < \infty
$$

and

$$
\left( \int_0^\infty \left[ a^s(t) - b^s(t) \right]^{\frac{q}{p}} v(t)^{\frac{q}{p}} dx \right)^{\frac{r}{l}} \leq \frac{q}{p} \left( \int_0^\infty v(x)dy \right)^{\frac{q}{p}} v(x)dx \sigma(t)dt < \infty,
$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, $\frac{1}{l} = \frac{1}{q} - \frac{1}{s}$, and $\sigma$ is the normalizing function as defined in [3].

**Remark.** The condition $a'(x) < b'(x)$ for $x \in (0, \infty)$ cannot be relaxed since otherwise the monotonicity of $Tf$ would be on stake. For example, consider the functions

$$
a(x) = \begin{cases}
  \frac{\sqrt{x}}{10}, & 0 \leq x < 10 \\
  \frac{1}{10} - 9, & 10 \leq x < 20 \\
  \frac{\sqrt{x} + 9(\sqrt{2} - 1)}{10}, & x \geq 20
\end{cases}
$$

and
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\[ b(x) = \begin{cases} 
10\sqrt{10}x, & 0 \leq x < 10 \\
\frac{x}{\sqrt{10} + 99}, & 10 \leq x < 20 \\
10\sqrt{10}x + 99(\sqrt{2} - 1), & x \geq 20.
\end{cases} \]

Note that \( a \) and \( b \) satisfy all the aforementioned conditions, except that, we have \( a'(x) > b'(x) \) for \( 10 \leq x < 20 \).

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References