δ – Closed Sets in Ideal Topological Spaces

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Abstract: In this paper we introduce the notion of δ–closed sets and studied some of its basic properties and characterizations. It shows this class lies between δ–closed sets and g–closed sets in particularly lies between δ–I–closed sets and g–closed sets. This new class of sets is independent of closed sets, semi closed and α–closed sets. Also we discuss the relationship with some of the known closed sets.

Keywords and Phrases: δ–closed, δ–open.

I. Introduction

A nonempty collection of subsets of X in a topological space (X, τ) is said to be an ideal I if it satisfies (i) A∈I and B⊆A implies B∈I and (ii) A∈I and B⊆I implies A∪B∈I. A topological space (X, τ) with an ideal I is called an ideal topological space or simply ideal space. If P(X) is the set of all subsets of X, a set operator (.)*: P(X)→P(X) is called a Kuratowski closure operator cl*(.) for a topology τ*(I, τ), called the *–topology, finer than τ is defined by cl*(A)=A∪A*(I, τ) [25], Levine [13], Velicko [27] introduced the notions of generalized closed (briefly g–closed) and δ–closed sets respectively and studied their basic properties. The notion of δ–closed sets was first introduced by Dontchev [6] in 1999. Navaneetha Krishnan and Joseph [20] further investigated and characterized g–closed sets. Julian Dontchev and Maximilian Ganster [5], Yuksel, Acikgoz and Noiri [28] introduced and studied the notions of δ–generalized closed (briefly δg–closed) and δ–I–closed sets respectively. The purpose of this paper is to define a new class of sets called δ–closed sets and also study some basic properties and characterizations.

II. Preliminaries

Definition 2.1. A subset A of a topological space (X, τ) is called a
(i) Semi-open set [12] if A⊆cl(int(A))
(ii) Pre-open set [17] if A⊆int(cl(A))
(iii) α–open set [24] if A⊆int(cl(int(A))
(iv) Regular open set [24] if A=cl(int(A))

The complement of a semi-open (resp. pre-open, α–open, regular open) set is called Semi-closed (resp. pre-closed, α–closed, regular closed). The semi-closure (resp. pre closure, α–closure) of a subset A of (X, τ) is the intersection of all semi-closed (resp. pre-closed α-closed) sets containing A and is denoted by scl(A) (resp. pcl(A), αcl(A)). The intersection of all semi-open sets of (X, τ) contains A is called semi-kernel of A and is denoted by ker(A).

Definition 2.2. [28]. Let (X, τ, I) be an ideal topological space. A a subset of X and x is a point of X. Then
i) x is called a δ–I–cluster points of A if A∩(int cl*(U))≠ϕ for each open neighbourhood U of x.
ii) the family of all δ-I-cluster points of A is called the δ-I-closure of A and is denoted by [A]δ,δ.
iii) A subset A is said to be δ-I-closed if [A]δ,δ=A. The complement of a δ-I-closed set of X is said to be δ-I-open.

Remarks 2.3. From the definition [ ] we can write [A]δ,δ= {x∈X : (int cl*(U)) ∩ A ≠ ϕ, for each U∈τ(x)}

Notation 2.4. Throughout this paper we use the notation σcl(A)=[A]δ,δ.

Lemma 2.5. [28] Let A and B be subsets of an ideal topological space (X, τ, I). Then, the following properties hold.
(i) A⊆σcl(A)
(ii) If A ⊆ B, then σcl(A) ⊆ σcl(B)

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Theor

Example

open in (X, \tau). In this section we introduce a (iv) (xii) (xv)

Definition \tau_{\delta,1} = \{ A \subseteq X \mid A \text{ is } \delta-I \text{-open set of } (X, \tau, I) \}. Then \tau_{\delta,1} is a topology such that \tau_{\subset \tau_{\delta,1} \subset \tau}

Remark 2.7. [28] \tau_{\delta}, (resp. \tau_{\delta,1}) is the topology formed by the family of \delta-open sets (resp. \delta-I - open sets).

Lemma 2.8. Let (X, \tau, I) be an ideal topological space and A \subseteq X. Then \sigma cl(A) = \{ x \in X : int (cl^*(U)) \cap A \neq \emptyset, U \in \tau(x) \} is closed.

Definition 2.9. Let (X, \tau) be a topological space. A subset A of X is said to be

(i) a g-closed set [13] if \text{cl}(A) \subseteq U whenever A \subseteq U and U is open in (X, \tau).

(ii) a generalized semi-closed (briefly gs-closed) set [3] if \text{cl}(A) \subseteq U whenever A \subseteq U and U is open in (X, \tau).

(iii) a Semi-generalized closed (briefly sg-closed) set [4] if \text{cl}(A) \subseteq U whenever A \subseteq U and U is semi-open set in (X, \tau).

(iv) \alpha - general closed (briefly \alpha g-closed) set [15] if \text{cl}(A) \subseteq U whenever A \subseteq U and U is open in (X, \tau).

(v) a \alpha - closed (briefly \alpha g-closed) set [14] if \text{cl}(A) \subseteq U whenever A \subseteq U and U is \alpha-open in (X, \tau).

(vi) a \delta-closed set [27] if A = \text{cl}_g(A), where \delta cl(A) = \text{cl}_g(A) = \{ x \in X : (\text{int cl}(U) \cap A \neq \emptyset, U \in \tau \text{ and } x \in U \}.

(vii) a \delta - closed set (briefly \delta g-closed) set [5] if \text{cl}_g(A) \subseteq U whenever A \subseteq U and U is open.

(viii) a \alpha - \hat{g} - closed (briefly \alpha \hat{g} - closed) set [1] if \text{cl}(A) \subseteq U whenever A \subseteq U and U is a \hat{g} - open set in (X, \tau).

(ix) a \delta - \hat{g} - closed set [11] if \text{cl}(A) \subseteq U whenever A \subseteq U and U is \hat{g} - open set in (X, \tau).

(x) a \theta-closed set [27] if \delta cl(A) \subseteq U whenever A \subseteq U and U is semi-open set in (X, \tau), where \delta cl(A) = \text{cl}_s(A).

(xi) a \theta-g-closed set [7] if A = \text{cl}(A) where \text{cl}_g(A) = \text{cl}_g(A) = \{ x \in X : (\text{int cl}(U) \cap A \neq \emptyset, U \in \tau \text{ and } x \in U \}.

(xii) a \wedge - set [16, 18] if A' = A where A' = \cap \{ U \in \tau : A \subseteq U \}.

(xiii) a weakly generalized closed (briefly wg-closed) set [19] if \text{cl}(\text{int}(A)) \subseteq U whenever A \subseteq U and U is open in X.

(xiv) a Regular weakly generalized closed (briefly Rwg-closed) set [9] if \text{cl}(\text{int}(A)) \subseteq U whenever A \subseteq U and U is regular open set in X.

(xv) a \hat{g} - (or \omega) - closed set [26] if \text{cl}(A) \subseteq U whenever A \subseteq U and U is semi-open set in (X, \tau). The complement of \hat{g} - (or \omega) - closed set is \hat{g} - (or \omega) - open.

Definition 2.10. Let (X, \tau, I) be an ideal space. A subset A of X is said to be

(i) Ig - closed set [6] if A^* \subseteq U whenever A \subseteq U and U is open in X.

(ii) \theta - I - closed set [2] if \text{cl}^*_{\emptyset} (A) = A where \text{cl}^*_{\emptyset} (A) = \{ x \in X : \text{cl}^*(U) \cap A \neq \emptyset \text{ for all } U \in \tau (x) \}.

(iii) I-g-closed set [23] if A^* \subseteq U whenever A \subseteq U and U is \hat{g} - open.

(iv) R - I - open set [28] if Int (cl^*(A)) = A.

III. \delta - closed set

In this section we introduce and study a new class of sets known as \delta - closed sets in ideal topological spaces.

Definition 3.1. A subset A of an ideal space (X, \tau, I) is called \delta - closed if \text{cl}(A) \subseteq U whenever A \subseteq U and U is open in (X, \tau, I). The complement of \delta - closed set in (X, \tau, I), is called \delta - open set in (X, \tau, I).

Example 3.2. Let X = \{ a, b, c, d \}, \tau = \{ X, \phi, \{ a \}, \{ b \}, \{ a, b \}, \{ a, b, d \} \text{ and } I = \{ \phi, \{ c \}, \{ d \}, \{ c, d \} \}. Let A = \{ a, b, c \}, then A is \delta - closed set.

Theorem 3.3. Every \delta - closed set is \hat{g} - closed.

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\[\hat{\delta} \quad \text{Closed Sets in Ideal Topological Spaces}\]

**Proof:** Let \(A\) be any \(\delta\)-closed set and \(U\) be any open set in \((X, \tau, I)\) such that \(A \subseteq U\). Since \(A\) is \(\delta\)-closed and \(\sigma \text{cl}(A) \subseteq \text{cl}_\delta(A)\). Therefore \(\sigma \text{cl}(A) \subseteq U\). Thus \(A\) is \(\hat{\delta}\)-closed.

**Remark 3.4.** The converse is not always true from the following example.

**Example 3.5.** Let \(X = \{a, b, c, d\}, \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \) and \(I = \{\phi, \{c\}, \{d\}, \{c,d\}\}\). Let \(A = \{a, b, d\}\). Then \(A\) is \(\hat{\delta}\)-closed set but not \(\delta\)-closed.

**Theorem 3.6.** Every \(\delta\)-I-closed set is \(\hat{\delta}\)-closed.

**Proof:** Let \(A\) be any \(\delta\)-I-closed set and \(U\) be any open set in \((X, \tau, I)\) such that \(A \subseteq U\). Since \(A\) is \(\delta\)-I-closed, \(\sigma \text{cl}(A) = A \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open. Therefore \(A\) is \(\hat{\delta}\)-closed.

**Remark 3.7.** The reversible implication is not always possible from the following example.

**Example 3.8.** Let \(X = \{a, b, c, d\}, \tau = \{X, \phi, \{b\}, \{b, c\}\} \) and \(I = \{\phi\}\). Let \(A = \{a, b, c\}\). Then \(A\) is \(\hat{\delta}\)-closed set but not \(\delta\)-I-closed.

**Theorem 3.9.** Every \(\delta\)-closed set is \(\hat{\delta}\)-closed.

**Proof:** Let \(A\) be any \(\delta\)-closed set and \(U\) be any open set. Such that \(A \subseteq U\). Then \(\sigma \text{cl}(A) \subseteq \text{cl}(\sigma \text{cl}(A)) \subseteq U\). Hence \(A\) is \(\hat{\delta}\)-closed.

**Remark 3.10.** \(\hat{\delta}\)-closed set need not be \(\delta\)-closed as shown in the following example.

**Example 3.11.** Let \(X = \{a, b, c, d\}, \tau = \{X, \phi, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\} \) and \(I = \{\phi, \{b\}\}\). Let \(A = \{a, c, d\}\). Then \(A\) is \(\hat{\delta}\)-closed but not \(\delta\)-closed.

**Theorem 3.12.** Every \(\delta\)-\(\hat{\delta}\)-closed set is \(\hat{\delta}\)-closed.

**Proof:** Let \(A\) be any \(\delta\)-\(\hat{\delta}\)-closed set and \(U\) be any open set containing \(A\). Since every open set is \(\delta\)-closed, \(\sigma \text{cl}(A) \subseteq \text{cl}(\sigma \text{cl}(A)) \subseteq U\). Hence \(A\) is \(\hat{\delta}\)-closed.

**Remark 3.13.** \(\hat{\delta}\)-closed set is not always a \(\delta\)-\(\hat{\delta}\)-closed as shown in the following example.

**Example 3.14.** Let \(X = \{a, b, c, d\}, \tau = \{X, \phi, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\} \) and \(I = \{\phi, \{b\}\}\). Let \(A = \{a, d\}\). Then \(A\) is \(\hat{\delta}\)-closed but not \(\delta\)-\(\hat{\delta}\)-closed.

**Theorem 3.15.** Every \(\theta\)-closed set is \(\hat{\delta}\)-closed.

**Proof:** Let \(A\) be any \(\theta\)-closed set and \(U\) be any open set containing \(A\). Since \(A\) is \(\theta\)-closed and \(\sigma \text{cl}(A) \subseteq \text{cl}(\sigma \text{cl}(A))\), \(\sigma \text{cl}(A) \subseteq U\), whenever \(A \subseteq U\) and \(U\) is open. Therefore \(A\) is \(\hat{\delta}\)-closed.

**Remark 3.16.** \(\hat{\delta}\)-closed set is not always a \(\theta\)-closed set as it can be seen in the following example.

**Example 3.17.** Let \(X = \{a, b, c, d\}, \tau = \{X, \phi, \{a\}, \{b, c\}, \{a, b, c\}\} \) and \(I = \{\phi, \{b\}\}\). Let \(A = \{b, c, d\}\). Then \(A\) is \(\hat{\delta}\)-closed set but not \(\theta\)-closed.

**Theorem 3.18.** Every \(\theta\)-\(g\)-closed set is \(\hat{\delta}\)-closed.
Proof: Let A be any 0-g-closed set and U be any open set containing A. Then cl_0(A) \subseteq U. Since \sigma cl(A) \subseteq cl_{0}(A). A is \hat{\delta} -closed.

Remark 3.19 A \hat{\delta} -closed set is not always a 0-g-closed set as shown in the following example.

Example 3.20. Let X = \{a, b, c, d\}, \tau = (X, \phi, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}) and I = \{\phi, \{b\}\}. Let A = \{d\}. Then A is \hat{\delta} -closed set but not 0-g-closed.

Theorem 3.21. Every 0-I-closed set is \hat{\delta} -closed.

Proof: Let A be any 0-I-closed set and U be any open set such that A \subseteq U. Since \sigma cl(A) \subseteq cl^*_{0}(A), A is \hat{\delta} -closed.

Remark 3.22. The reversible implication is not always true from the following example.

Example 3.23. Let X = \{a, b, c, d\}, \tau = (X, \phi, \{a\}, \{b, c\}, \{a, b, c\}) and I = \{\phi, \{b\}\}. Let A = \{a, c, d\}. Then A is \hat{\delta} -closed but not 0-I-closed.

Theorem 3.24. In an Ideal Space (X, \tau, I), every \hat{\delta} -closed set is (i) g-closed (ii) Ig-closed (iii) gs-closed (iv) \alpha g-closed (v) wg-closed (vi) Rwg-closed.

Proof: (i) Suppose that A is a \hat{\delta} -closed set and U be any open set such that A \subseteq U. By hypothesis \sigma cl(A) \subseteq U. Then cl(A) \subseteq U and hence A is g-closed.

(ii) Since every g-closed set is Ig-closed set in (X, \tau, I). It holds

(iii) It is true that scl(A) \subseteq \sigma cl(A) for every subset A of (X, \tau, I).

(iv) It is true that \sigma cl(A) \subseteq \sigma cl(A) for every subset A of (X, \tau, I).

(v) Since cl(int(A)) \subseteq \sigma cl(A). It holds

(vi) Proof follows from the fact that cl(int(A)) \subseteq \sigma cl(A) and every regular open set is open.

Remark 3.25. The following examples reveals that the reversible implications of (i), (ii), (iii), (iv), (v), (vi) in Theorem 3.24 are not true in general.

Example 3.26. Let X = \{a, b, c, d\}, \tau = (X, \phi, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}) and I = \{\phi, \{a\}, \{c\}, \{a, c\}\}. Let A = \{a, d\}. Then A is g-closed set but not \hat{\delta} -closed.

Example 3.27. Let X = \{a, b, c, d\}, \tau = (X, \phi, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}) and I = \{\phi, \{b\}\}. Let A = \{b, c\}. Then A is Ig-closed set but not \hat{\delta} -closed.

Example 3.28. Let X = \{a, b, c, d\}, \tau = (X, \phi, \{b\}, \{c\}, \{b, c\}, \{a, c\}) and I = \{\phi, \{b\}, \{c\}, \{b, c\}\}. Let A = \{c, d\}. Then A is gs-closed set but not \hat{\delta} -closed.

Example 3.29. Let X = \{a, b, c, d\}, \tau = (X, \phi, \{a\}, \{b\}, \{a, b\}) and I = \{\phi, \{b\}\}. Let A = \{a\}. Then A is \alpha g-closed but not \hat{\delta} -closed.

Example 3.30. Let X = \{a, b, c, d\}, \tau = (X, \phi, \{a\}, \{b\}, \{a, b\}) and I = \{\phi, \{b\}\}. Let A = \{b\}. Then A is wg-closed but not \hat{\delta} -closed.

Example 3.31. Let X = \{a, b, c, d\}, \tau = (X, \phi, \{a\}, \{b\}, \{a, b\}) and I = \{\phi, \{b\}\}. Let A = \{b\}. Then A is Rwg-closed but not \hat{\delta} -closed.
Remark 3.32. The following examples shows that, in an Ideal space \( \hat{\delta} \) -closed set is independent of (i) closed (ii) sg-closed (iii) go-closed (iv) \( \alpha \hat{g} \) -closed.

Example 3.33. Let \( X = \{a, b, c, d\}, \tau = \{X, \phi, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\} \) and \( I = \{\phi, \{a\}, \{c\}, \{a, c\}\} \). Let \( A = \{b, c, d\} \) is \( \hat{\delta} \) -closed but not closed. Let \( B = \{a, d\} \). Then \( B \) is closed but not \( \hat{\delta} \) -closed.

Example 3.34. Let \( X = \{a, b, c, d\}, \tau = \{X, \phi, \{a\}, \{b, c\}, \{a, b, c\}\} \) and \( I = \{\phi, \{b\}\} \). Let \( A = \{b, c\} \). Then \( A \) is sg-closed but not sg-closed.

Example 3.35. Let \( X = \{a, b, c\}, \tau = \{X, \phi, \{a\}\} \) and \( I = \{\phi, \{a\}\} \). Let \( A = \{a, b\} \). Then \( A \) is \( \hat{\delta} \) -closed but not sg-closed.

Example 3.36. Let \( X = \{a, b, c, d\}, \tau = \{X, \phi, \{b\}, \{c\}, \{b, c\}\} \) and \( I = \{\phi, \{d\}\} \). Let \( A = \{a, b, c\} \). The \( A \) is \( \hat{\delta} \) -closed but not go-closed.

Example 3.37. Let \( X = \{a, b, c, d\}, \tau = \{X, \phi, \{c\}, \{d\}\} \) and \( I = \{\phi, \{c\}, \{d\}, \{c, d\}\} \). Let \( A = \{d\} \). Then \( A \) is go-closed but not \( \hat{\delta} \) -closed set.

Example 3.38. Let \( X = \{a, b, c, d\}, \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \) and \( I = \{\phi, \{b\}\} \). Let \( A = \{a\} \). Then \( A \) is \( \alpha \hat{g} \) -closed but not \( \alpha \hat{G} \) -closed.

Example 3.39. Let \( X = \{a, b, c\}, \tau = \{X, \phi, \{a\}\} \) and \( I = \{\phi, \{a\}\} \). Let \( A = \{a, b\} \). Then \( A \) is \( \hat{\delta} \) -closed but not \( \alpha \hat{g} \) -closed.

Example 3.40. Let \( X = \{a, b, c, d\}, \tau = \{X, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\} \) and \( I = \{\phi, \{a\}, \{b\}, \{a, b\}\} \). Let \( A = \{a, b\} \). Then \( A \) is \( I_{\hat{\delta}} \) closed set but not \( \hat{\delta} \) -closed.

Example 3.41. Let \( X = \{a, b, c, d\}, \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}\} \) and \( I = \{\phi, \{c\}, \{d\}, \{c, d\}\} \). Let \( A = \{a, b, c\} \). Then \( A \) is \( \hat{\delta} \) -closed but not \( I_{\hat{\delta}} \) closed.

Theorem 3.42. Let \( (X, \tau, I) \) be an ideal space and \( A \) a subset of \( X \). Then \( \sigmacl(A) = \{x \in X / (\text{int}(\text{cl}^*(U))) \cap A \neq \phi, \) for all \( U \in \tau(x)\} \) is closed.

Proof: If \( x \in \text{cl} (\sigmacl(A)) \) and \( U \in \tau(x) \), then \( U \cap \sigmacl(A) \neq \phi \). Then \( y \in U \cap \sigmacl(A) \) for some \( y \in X \). Since \( U \in \tau(y) \) and \( y \in \sigmacl(A) \), from the definition of \( \sigmacl(A) \) we have \( \text{int}(\text{cl}^*(U)) \cap A \neq \phi \). Therefore \( x \in \sigmacl(A) \). So \( \text{cl}(\sigmacl(A)) \subset \sigmacl(A) \) and hence \( \sigmacl(A) \) is closed.

IV. Characterizations

Theorem 4.1. If \( A \) is a subset of an ideal space \( (X, \tau, I) \), then the following are equivalent.

(a) \( A \) is \( \hat{\delta} \) -closed
(b) For all \( x \in \sigmacl(A), \text{cl}(\{x\}) \cap A \neq \phi \)
(c) \( \sigmacl(A) - A \) contains no non-empty closed set.

Proof: (a) \( \Rightarrow \) (b) Suppose \( x \in \sigmacl(A) \). If \( \text{cl}(\{x\}) \cap A = \phi \), then \( A \subset X - \text{cl}(\{x\}) \). Since \( A \) is \( \hat{\delta} \) -closed, \( \sigmacl(A) \subset X - \text{cl}(\{x\}) \). It is a contradiction to the fact that \( x \in \sigmacl(A) \). This proves (b).

(b) \( \Rightarrow \) (c) Suppose \( F \subset \sigmacl(A) - A \), \( F \) is closed and \( x \in F \). Since \( F \subset X - A \) and \( F \) is closed \( \text{cl}(\{x\}) \cap A \subset \text{cl}(F) \cap A = F \cap A = \phi \). Since \( x \in \sigmacl(A) \), by (b), \( \text{cl}(\{x\}) \cap A \neq \phi \), a contradiction which proves (c).

(c) \( \Rightarrow \) (a). Let \( U \) be an open set containing \( A \). Since \( \sigmacl(A) \) is closed, \( \sigmacl(A) \cap (X-U) \) is closed and \( \sigmacl(A) \cap (X-U) \subset \sigmacl(A) - A \). By hypothesis \( \sigmacl(A) \cap (X-U) = \phi \) and hence \( \sigmacl(A) \subset U \). Thus \( A \) is \( \hat{\delta} \) -closed.

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If we put $I = \{\phi\}$ in the above Theorem 4.1, we get the Corollary 4.2 which gives the characterizations for $\delta g$-closed sets. If we put $I = P(X)$ in the above Theorem 4.1, we get corollary 4.3 which gives the characterizations for $g$-closed sets.

**Corollary 4.2.** If $A$ is subset of an Ideal topological space $(X, \tau, I)$. Then the following are equivalent.
(a) $A$ is $\delta g$-closed.
(b) For all $x \in \delta cl(A)$, $cl(\{x\}) \cap A \neq \phi$
(c) $\delta cl(A) - A$ contains no non-empty closed set.

**Corollary 4.3.** If $A$ is a subset of a topological space $(X, \tau)$. Then the following are equivalent.
(a) $A$ is $g$-closed
(b) For all $x \in cl(A)$, $cl(\{x\}) \cap A \neq \phi$
(c) $cl(A) - A$ contains no non-empty closed set.

If $I = \{\phi\}$, then $\delta cl(A) = \sigma cl(A)$ and hence $\hat{\delta}$-closed sets coincide with $\delta g$-closed sets. If $I = P(X)$, then $\sigma cl(A) = cl(A)$ and hence $\hat{\delta}$-closed sets coincide with $g$-closed sets.

**Corollary 4.4.** If $(X, \tau, I)$ is an ideal space and $A$ is a $\hat{\delta}$-closed set, then the following are equivalent.
(a) $A$ is a $\delta I$-closed set
(b) $\sigma cl(A) - A$ is a closed set

**Proof:** (a) $\Rightarrow$ (b). If $A$ is $\delta I$-closed set, then $\sigma cl(A) - A = \phi$ and so $\sigma cl(A) - A$ is closed.
(b) $\Rightarrow$ (c) If $\sigma cl(A) - A$ is closed, then $A$ is $\hat{\delta}$-closed, by theorem, $\sigma cl(A) - A = \phi$ and so $A$ is $\delta I$-closed.

If we put $I = \{\phi\}$ in Corollary 4.4, we get Corollary 4.5. If we put $I = P(X)$ in the Corollary 4.4. Then we get Corollary 4.6.

**Corollary 4.5.** If $(X, \tau, I)$ be an Ideal Topological space and $A$ is a $\delta g$-closed set, then the following are equivalent.
(a) $A$ is a $\delta I$-closed set
(b) $\delta cl(A) - A$ is a closed set

**Corollary 4.6.** If $(X, \tau)$ is a topological space and $A$ is a $g$-closed set, then the following are equivalent.
(a) $A$ is closed set
(b) $cl(A) - A$ is a closed set

**Theorem 4.7.** Let $(X, \tau, I)$ be an ideal space. Then every subset of $X$ is $\hat{\delta}$-closed if and only if every open set is $\delta I$-closed.

**Proof:** Necessity - Suppose every subset of $X$ is $\hat{\delta}$-closed. If $U$ is open then $\delta I$-closed and so $\sigma cl(U) \subset U$. Hence $U$ is $\delta I$-closed.
Sufficiency - Suppose $A \subset U$ and $U$ is open. Since every open set is $\delta I$-closed, $\sigma cl(A) \subset \sigma cl(U) = U$ and so $A$ is $\hat{\delta}$-closed.

If we put $I = \{\phi\}$ in the above Theorem 4.7, we get Corollary 4.8. If we put $I = P(X)$ in the above Theorem 4.7, we get corollary 4.9.

**Corollary 4.8.** Let $(X, \tau)$ be a topological space. Then every subset of $X$ is $\delta g$-closed if and only if every open set is $\delta$-closed.

**Corollary 4.9.** Let $(X, \tau)$ be a topological space. Then every subset of $X$ is $g$-closed if and only if every open set is closed.

**Theorem 4.10.** If $A$ and $B$ are $\hat{\delta}$-closed sets in a topological space $(X, \tau, I)$, then $A \cup B$ is $\hat{\delta}$-closed set in $(X, \tau, I)$.
Proof: Suppose that \( A \cup B \subseteq U \) where \( U \) is any open set in \((X, \tau, I)\). Then \( A \subseteq U \) and \( B \subseteq U \). Since \( A \) and \( B \) are \( \delta \)-closed sets in \((X, \tau, I)\), \( \sigma cl(A) \subseteq U \) and \( \sigma cl(B) \subseteq U \). Always \( \sigma cl(A \cup B) = \sigma cl(A) \cup \sigma cl(B) \). Therefore, \( \sigma cl(A \cup B) \subseteq U \). Thus \( A \cup B \) is a \( \delta \)-closed set in \((X, \tau, I)\).

Remark 4.11. The following example shows that the intersection of two \( \delta \)-closed set is not always \( \delta \)-closed.

Example 4.12. In Example 3.29, let \( A = \{a, c\} \) and \( B = \{a, d\} \). Then \( A \) and \( B \) are \( \delta \)-closed sets, but \( A \cap B = \{a\} \) is not \( \delta \)-closed.

Theorem 4.13. Intersection of a \( \delta \)-closed set and a \( \delta \)-I-closed set is always \( \delta \)-closed.

Proof: Let \( A \) be a \( \delta \)-closed set and \( F \) be a \( \delta \)-I-closed set of an ideal space \((X, \tau, I)\). Suppose \( A \cap F \subseteq U \) and \( U \) is open set in \( X \). Then \( A \subseteq U \cap (X-F) \). Now, \( X-F \) is \( \delta \)-I-open and hence open. Therefore \( U \cap (X-F) \) is an open set containing \( A \). Since \( A \) is \( \delta \)-closed, \( \sigma cl(A) \subseteq U \cap (X-F) \). Therefore \( \sigma cl(A) \cap F \subseteq U \) which implies that \( \sigma cl(A \cap F) \subseteq U \). So \( A \cap F \) is \( \delta \)-closed.

If we put \( I = \{\phi\} \) in the above Theorem 4.13, we get Corollary 4.14 and if we put \( I = P(X) \) in Theorem 4.13, we get Corollary 4.15.

Corollary 4.14. Intersection of a \( \delta g \)-closed set and \( \delta \)-closed set is always \( \delta g \)-closed.

Corollary 4.15. Intersection of a \( g \)-closed set and a \( \delta \)-closed set is always \( g \)-closed set.

Theorem 4.16. A subset \( A \) of an ideal space \((X, \tau, I)\) is \( \delta \)-closed if and only if \( \sigma cl(A) \subseteq A^\wedge \).

Proof: Necessity - Suppose \( A \) is \( \delta \)-closed and \( x \in \sigma cl(A) \). If \( x \notin A^\wedge \), then there exists an open set \( U \) such that \( A \subseteq U \), but \( x \notin U \). Since \( A \) is \( \delta \)-closed, \( \sigma cl(A) \subseteq U \) and so \( x \notin \sigma cl(A) \), a contradiction. Therefore \( \sigma cl(A) \subseteq A^\wedge \).

Sufficiency - Suppose that \( \sigma cl(A) \subseteq A^\wedge \). If \( A \subseteq U \) and \( U \) is open, then \( A^\wedge \subseteq U \) and so \( \sigma cl(A) \subseteq U \). Therefore \( A \) is \( \delta \)-closed.

If we put \( I = \{\phi\} \) in the above Theorem 4.16, we get Corollary 4.17. If we put \( I = P(X) \) in Theorem 4.16, we get Corollary 4.18.

Corollary 4.17. A subset \( A \) of a space \((X, \tau)\) is \( \delta g \)-closed if and only if \( \delta cl(A) \subseteq A^\wedge \).

Corollary 4.18. A subset \( A \) of a space \((X, \tau)\) is \( g \)-closed if and only if \( cl(A) \subseteq A^\wedge \).

Theorem 4.19. Let \( A \) be a \( \wedge \)-set of an ideal space \((X, \tau, I)\). Then \( A \) is \( \delta \)-closed, if and only if \( A \) is \( \delta \)-I-closed.

Proof: Necessity - Suppose \( A \) is \( \delta \)-closed. By Theorem 4.16, \( \sigma cl(A) = A^\wedge = A \), since \( A \) is a \( \wedge \)-set. Therefore, \( A \) is \( \delta \)-I-closed.

Sufficiency - Proof follows from the fact that every \( \delta \)-I-closed set is \( \delta \)-closed.

If we put \( I = \{\phi\} \) in Theorem 4.19, we get Corollary 4.20. If we put \( I = P(X) \) in Theorem 4.19, we get Corollary 4.21.

Corollary 4.20. Let \( A \) be a \( \wedge \)-set of a space \((X, \tau)\). Then \( A \) is \( \delta g \)-closed if and only if \( A \) is \( \delta \)-closed.

Corollary 4.21. Let \( A \) be a \( \wedge \)-set of a space \((X, \tau)\). Then \( A \) is \( g \)-closed if and only if \( A \) is \( \delta \)-closed.

Theorem 4.22. Let \((X, \tau, I)\) be an ideal space and \( A \subseteq X \). If \( A^\wedge \) is \( \delta \)-closed, then \( A \) is also \( \delta \)-closed.
Proof: Suppose that $A^\omega$ is a $\hat{\delta}$-closed set. If $A \subset U$ and $U$ is open then $A^\omega \subset U$. Since $A^\omega$ is $\hat{\delta}$-closed, $\sigma cl(A^\omega) \subset U$. But $cl(A) \subset \sigma cl(A^\omega)$. Therefore, $A$ is $\hat{\delta}$-closed.

If we put $I = \{\phi\}$ in Theorem 4.22, we get Corollary 4.23. If we put $I = P(X)$ in Theorem 4.22, we get Corollary 4.24.

Corollary 4.23. Let $(X, \tau)$ be a topological space and $A \subset X$. If $A^\omega$ is $\delta g$-closed, then $A$ is also $\delta g$-closed.

Corollary 4.24. Let $(X, \tau)$ be a topological space and $A \subset X$. If $A^\omega$ is $g$-closed then $A$ is also $g$-closed.

Theorem 4.25. Let $(X, \tau, I)$ be an ideal space. If $A$ is a $\hat{\delta}$-closed subset of $X$ and $A \subset B \subset \sigma cl(A)$, then $B$ is also $\hat{\delta}$-closed.

Proof: $\sigma cl(B) = B \subset \sigma cl(A) - A$, and since $\sigma cl(A) - A$ has no non-empty closed subset, $\sigma cl(B) - B$ also has no non-empty closed subset. Then by Theorem 4.1, $B$ is $\hat{\delta}$-closed.

If we put $I = \{\phi\}$ in the above Theorem 4.25, we get Corollary 4.26. If we put $I = P(X)$ in the above Theorem 4.25, we get Corollary 4.27.

Corollary 4.26. Let $(X, \tau)$ be a topological space. If $A$ is a $\delta g$-closed subset of $X$ and $A \subset B \subset \delta cl(A)$, then $B$ is also $\delta g$-closed.

Corollary 4.27. Let $(X, \tau)$ be a space. If $A$ is a $g$-closed subset of $X$ and $A \subset B \subset cl(A)$, then $B$ is also $g$-closed.

Theorem 4.28. Let $(X, \tau, I)$ be an ideal space and $A \subset U$. Then the following are equivalent.

a) $A$ is $\hat{\delta}$-closed
b) $A \cup (X - \sigma cl(A))$ is $\hat{\delta}$-closed
c) $\sigma cl(A) - A$ is $\hat{\delta}$-open

Proof: (a) $\Rightarrow$ (b) Suppose $A$ is $\hat{\delta}$-closed. If $U$ is any open set such that $A \cup (X - \sigma cl(A)) \subset U$, then $X - U \subset (A \cup (X - \sigma cl(A))) = \sigma cl(A) - A$. Since $A$ is $\hat{\delta}$-closed, by Theorem 4.1, it follows that $X - U = \phi$ and so $X = U$.

Since $X$ is the only open set containing $A \cup (X - \sigma cl(A))$, $A \cup (X - \sigma cl(A))$ is $\hat{\delta}$-closed.

(b) $\Rightarrow$ (a) Suppose $A \cup (X - \sigma cl(A))$ is $\hat{\delta}$-closed. If $F$ is a closed set contained in $\sigma cl(A) - A$, then $A \cup (X - \sigma cl(A)) \subset X - F$ and $X - F$ is open. Therefore $\sigma cl(A \cup X - \sigma cl(A)) \subset X - F$, which implies that $\sigma cl(A) \cup \sigma cl(X - \sigma cl(A)) \subset X - F$ and so $X \subset X - F$ it follows that $F = \phi$. Hence $A$ is $\hat{\delta}$-closed.

The equivalence of (b) and (c) follows from the fact $X - (\sigma cl(A) - A) = A \cup (X - \sigma cl(A))$.

If we put $I = \{\phi\}$ in the above Theorem 4.28, we get Corollary 4.29. If we put $I = P(X)$ in the above Theorem 4.28, we get Corollary 4.30.

Corollary 4.29. Let $(X, \tau)$ be a topological space, and $A \subset U$. Then the following are equivalent.

a) $A$ is $\delta g$-closed
b) $A \cup (X - \delta cl(A))$ is $\sigma g$-closed
c) $\delta cl(A) - A$ is $\delta g$-open

Corollary 4.30. Let $(X, \tau)$ be a space and $A \subset U$. Then the following are equivalent.

a) $A$ is $g$-closed
b) $A \cup (X - cl(A))$ is $g$-closed
c) $cl(A) - A$ is $g$-open.

Theorem 4.31. For an ideal space $(X, \tau, I)$, the following are equivalent.

a) Every $\hat{\delta}$-closed set is $\delta I$-closed
b) Every singleton of $X$ is closed or $\delta I$-open.

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Proof : (a) ⇒ (b) Let \( x \in X \). If \( \{ x \} \) is not closed, then \( A = X - \{ x \} \) is not open and then \( A \) is trivially \( \delta \)-closed. Since the only open set containing \( A \) is \( X \). Therefore by (A), \( A \) is \( \delta \)-I-closed. Hence \( \{ x \} \) is \( \delta \)-I-open.

(b) ⇒ (a) Let \( A \) be a \( \delta \)-closed set and let \( x \in \text{cl}(A) \). We have the following cases.

Case (i). \( \{ x \} \) is closed. By Theorem 4.1, \( \text{cl}(A) - A \) does not contain a non-empty closed subset. This shows that \( \{ x \} \subseteq A \).

Case (ii) \( \{ x \} \) is \( \delta - I \)-open, then \( \{ x \} \cap A \neq \emptyset \). Hence \( \{ x \} \in A \).

Thus in both cases \( \{ x \} \subseteq A \) and so \( \text{cl}(A) \). That is, \( A \) is \( \delta \)-I-closed, which proves (a).

If we put \( I = \{ x \} \) in the above Theorem 4.31, we get Corollary 4.32. If we put \( I = \text{P}(x) \) in the above Theorem 4.31, we get Corollary 4.33.

Corollary 4.32. For a topological space \((X, \tau)\), the following are equivalent.

a. Every \( \delta g \)-closed set is \( \delta \)-closed.

b. Every singleton of \( X \) is closed or \( \delta \)-open.

Corollary 4.33. For a topological space \((X, \tau)\), the following are equivalent.

a. Every \( g \)-closed set is closed.

b. Every singleton of \( X \) is closed or open.

Theorem 4.34. Let \((X, \tau, I)\) be an ideal space and \( A \subseteq X \). Then \( A \) is \( \delta \)-closed if and only if \( A = F - N \), where \( F \) is \( \delta \)-I-closed and \( N \) contains no non empty closed set.

Proof: Necessity - If \( A \) is \( \delta \)-closed, then by Theorem 4.1, \( N = \text{cl}(A) - A \) contains no nonempty closed set. If \( F = \text{cl}(A) \), then \( F \) is \( \delta \)-I-closed, such that \( F - N = \text{cl}(A) - (\text{cl}(A) - A) = \text{cl}(A) \cap (X - (\text{cl}(A) - A)) = A \).

Sufficiency - Suppose \( A = F - N \), where \( F \) is \( \delta \)-I-closed and \( N \) contains no non-empty closed set. Let \( U \) be an open set such that \( A \subseteq U \). Then \( F - N \subseteq U \) which implies that \( F \cap (X - U) \subseteq N \). Now, \( A \subseteq F \) and \( F \) is \( \delta \)-I-closed implies that \( \text{cl}(A) \cap (X - U) \subseteq \text{cl}(F) \cap (X - U) \subseteq F \cap (X - U) \subseteq N \). Since \( \delta \)-I-closed sets are closed, \( \text{cl}(A) \cap (X - U) \) is closed. By hypothesis \( \text{cl}(A) \cap (X - U) = \emptyset \) and so \( \text{cl}(A) \subseteq U \) which implies that \( A \) is \( \delta \)-closed.

If we put \( I = \{ x \} \), in Theorem 4.34, we get Corollary 4.35, if we put \( I = \text{P}(x) \) in Theorem 4.34, we get Corollary 4.36.

Corollary 4.35. Let \((X, \tau, I)\) be a topological space and \( A \subseteq X \). Then \( A \) is \( \delta g \)-closed if and only if \( A = F - N \), where \( F \) is \( \delta \)-closed and \( N \) contains no nonempty closed set.

Corollary 4.36. Let \((X, \tau, I)\) be a topological space and \( A \subseteq X \). Then \( A \) is \( g \)-closed if and only if \( A = F - N \), where \( F \) is closed and \( N \) contains no nonempty closed set.

Theorem 4.37. Let \((X, \tau, I)\) be an ideal space and \( A \subseteq X \). If \( A \subseteq B \subseteq \text{cl}(A) \), then \( \text{cl}(A) = \text{cl}(B) \).

Proof: Since \( A \subseteq B \subseteq \text{cl}(A) \), then \( \text{cl}(A) \subseteq \text{cl}(B) \) and since \( B \subseteq \text{cl}(A) \), then \( \text{cl}(B) \subseteq \text{cl}(\text{cl}(A)) = \text{cl}(A) \). Therefore \( \text{cl}(A) = \text{cl}(B) \).

Theorem 4.38. If \((X, \tau, I)\) is an ideal space, then \( \text{cl}(A) \) is always \( \delta \)-closed for every subset \( A \) of \( X \).

Proof: Let \( \text{cl}(A) \subseteq U \) where \( U \) is open. Since \( \text{cl}(\text{cl}(A)) = \text{cl}(A) \) we have \( \text{cl}(\text{cl}(A)) \subseteq U \) whenever \( \text{cl}(A) \subseteq U \) and \( U \) is open. Hence \( \text{cl}(A) \) is \( \delta \)-closed.

Theorem 4.39. Let \((X, \tau, I)\) be an Ideal space. Then every \( \delta \)-closed open set is \( \delta \)-I-closed set.

Proof: Assume that \( A \) is \( \delta \)-closed and open set. Then \( \text{cl}(A) \subseteq A \) whenever \( A \subseteq A \) and \( A \) is open. Thus \( A \) is \( \delta \)-I-closed.
Theorem 4.40. If $A$ is both semi-open and pre-closed set in an ideal space $(X, \tau, I)$, then $A$ is $\hat{\delta}$-closed in $(X, \tau, I)$.

Proof: If is clear that if $A$ is both semi-open and pre-closed then $A$ is regular closed and hence it is $\delta$-closed in $(X, \tau, I)$ therefore it is $\hat{\delta}$-closed in $(X, \tau, I)$.

Corollary 4.41. If $A$ is both open and pre-closed set in an ideal space $(X, \tau, I)$, then $A$ is $\hat{\delta}$-closed in $(X, \tau, I)$.

Theorem 4.42. In an Ideal Space $(X, \tau, I)$, for each $x \in X$, either $\{x\}$ is closed or $\{x\}^c$ is $\hat{\delta}$-closed. That is $X = X_\circ \cup X_\delta$.

Proof: Suppose that $\{x\}$ is not a closed set in $(X, \tau, I)$. Then $\{x\}^c$ is not an open set and the only open set containing $\{x\}^c$ is $X$. Therefore $\sigma cl(\{x\}^c) \subseteq X$ and hence $\{x\}^c$ is $\hat{\delta}$-closed set in $(X, \tau, I)$.

Theorem 4.43. In an Ideal Space $(X, \tau, I)$, $X_\circ \cap \sigma cl(A) \subseteq A^c$ for any subset $A$ of $(X, \tau, I)$, where $X_\circ = \{x \in X : \{x\} \subseteq \text{int}(\sigma cl(\{x\}^c))\}$.

Proof: Suppose that $x \in X_\circ \cap \sigma cl(A)$ and $x \notin A^c$. Since $x \in X_\circ$ and $x \notin X_\circ_i$ implies that $\text{int}(\sigma cl(\{x\}^c)) \neq \emptyset$. Since $x \in \sigma cl(A)$, $A \cap \text{int}(\sigma cl(\{x\}^c)) = \emptyset$ for any open set $U$ containing $x$. Choose $U = \text{int}(\sigma cl(\{x\}^c))$. Then $A \cap \text{int}(\sigma cl(\{x\}^c)) = \emptyset$.

Remark 4.48. Every maximal $\hat{\delta}$-closed set is a $\hat{\delta}$-closed set but not conversely as shown in the following example.

Example 4.47. Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{a, b\}\}$ and $I = \emptyset$. Let $A = \{a, b, c\}$. Then $A$ is maximal $\hat{\delta}$-closed set.

Example 4.49. In the above Example 4.47. $A = \{b, c\}$ is $\hat{\delta}$-closed but not maximal $\hat{\delta}$-closed.

Theorem 4.50. In an ideal space $(X, \tau, I)$, the following statements are true.

(i) Let $F$ be a maximal $\hat{\delta}$-closed set and $G$ be a $\hat{\delta}$-closed set. Then $F \cup G = X$ or $G \subseteq F$.

(ii) Let $F$ and $G$ be maximal $\hat{\delta}$-closed sets. Then $F \cup G = X$ or $F = G$.

Proof: (i) Let $F$ be a maximal $\hat{\delta}$-closed set and $G$ be a $\hat{\delta}$-closed set. If $F \cup G = X$, then there is nothing to prove. Assume that $F \cup G \neq X$. Now, $F \subseteq F \cup G$. By Theorem 4.10. $F \cup G$ is a $\hat{\delta}$-closed set. Since $F$ is a maximal $\hat{\delta}$-closed set, we have $F \cup G = X$ or $F \cup G = F$. Hence $F \cup G = F$ and so $G \subseteq F$. 

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(ii) Let $F$ and $G$ be maximal $\hat{\delta}$-closed sets. If $F \cup G = X$, then there is nothing to prove. Assume that $F \cup G \neq X$. Then by (i) $F \subseteq G$ and $G \subseteq F$ which implies that $F = G$.

**Theorem 4.51.** Let $(X, \tau, I)$ be an ideal space and $A$ a subset of $X$. Then $X - \sigma cl(X - A) = \sigma int(A)$.

**Theorem 4.53.** If $A$ is an $\hat{\delta}$-open set of an ideal space $(X, \tau, I)$ and $\sigma int(A) \subseteq B \subseteq A$. Then $B$ is also an $\hat{\delta}$-open set of $(X, \tau, I)$.

**Proof:** Suppose $F \subseteq B$ where $F$ is closed set of $(X, \tau, I)$. Then $F \subseteq A$. Since $A$ is $\hat{\delta}$-open, $F \subseteq \sigma int(A)$. Since $\sigma int(A) \subseteq \sigma int(B)$, we have $F \subseteq \sigma int(B)$. By the above Theorem 4.52, $B$ is $\hat{\delta}$-open.

**Theorem 4.52.** A subset $A$ of an ideal space $(X, \tau, I)$ is $\hat{\delta}$-open if and only if $F \subseteq \sigma int(A)$ whenever $F$ is closed and $F \subseteq A$.

**Proof:** Necessity - Suppose $A$ is $\hat{\delta}$-open and $F$ be a closed set contained in $A$. Then $X - A \subseteq X - F$ and hence $\sigma cl(X - A) \subseteq X - F$. Thus $F \subseteq X - \sigma cl(X - A) = \sigma int(A)$. Sufficiency - Suppose $X - A \subseteq U$ where $U$ is open. Then $X - U \subseteq A$ and $X - U$ is closed. Then $X - U \subseteq \sigma int(A)$ which implies $\sigma cl(X - A) \subseteq U$. Consequently $X - A$ is $\hat{\delta}$-closed and so $A$ is $\hat{\delta}$-open.

**Theorem 4.53.** Let $x$ be any point in an ideal topological space $(X, \tau, I)$. Then either $\operatorname{int}(\operatorname{cl}^*(\{x\})) = \emptyset$ or $\{x\} \subseteq \operatorname{int}(\operatorname{cl}^*(\{x\}))$ in $(X, \tau, I)$. Also $X = X_1 \cup X_2$, where $X_1 = \{x \in X : \operatorname{int}(\operatorname{cl}^*(\{x\})) = \emptyset\}$ and $X_2 = \{x \in X : \{x\} \subseteq \operatorname{int}(\operatorname{cl}^*(\{x\}))\}$.

**Proof.** Let $x$ be any point in an ideal space $(X, \tau, I)$.

**Case (i)** If $U \subseteq \operatorname{cl}^*(\{x\})$ for some $U \in \tau(x)$, then $x \in U \subseteq \operatorname{int}(\operatorname{cl}^*(\{x\}))$

**Case (ii)** If $U \nsubseteq \operatorname{cl}^*(\{x\})$ for all $U \in \tau(x)$. Let $V$ be any open set if $x \in V$ then $V \nsubseteq \operatorname{cl}^*(\{x\})$. If $x \notin V$, then for any $y \in V$, $\{x\} \cap V = \emptyset$. Therefore $y \notin \operatorname{cl}^*(\{x\})$. Therefore $V \nsubseteq \operatorname{cl}^*(\{x\})$ for any open set $V$. Therefore $\operatorname{int}(\operatorname{cl}^*(\{x\})) = \emptyset$.

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**References**


