IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 11, Issue 1 Ver. V (Jan - Feb. 2015), PP 58-60
www.iosrjournals.org

Maxwell Stress Tensor In Hydrodynamics

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Abstract: *Analogous to Maxwell stress tensor in electric and magnetic fields, a stress tensor is defined in a vorticity field. Thus by treating vortices as physical structures, it is possible to study the forces on a surface element in it. Based on this the force between vortex lines, the pressure and the shearing stress that deform the volume element can also be defined.*

Keywords: *Enstrophy, Maxwell stress tensor, static pressure, vorticity equation, vorticity stress tensor.*

I. Introduction

Though the electromagnetic theory was developed more than a century back, the practice of calculating forces on magnetic media remained ambiguous (Casperson, 2002)[1]. In particular, the application of Maxwell stress tensor gave way to other methods based on energy variation principle (Shwarz, 1963) [2](Sher, 1968) [3] (Pohl and Crane, 1972)[4] and effective dipole or multipole method (Wang, 1996)[5]. The method was revived by Washizu and Jones (1996) [6], Sauer and Schlogl (1985) [7] etc.

Wang, Wang and Gascoyne (1997)[8] explored the application of the Maxwell Stress Tensor (MST) method to dielectrophoresis (DEP) and electrorotation (ROT) studies. By integrating Maxwell stress tensor over the surface of a dielectric particle, they derived the general expressions for DEP and ROT generated by electric fields of arbitrary configurations. As they claim, this was the first time that, such complete expressions were derived from the first principles using MST formalism.

The analogy between the electromagnetic and fluid dynamic equations was first noted by Maxwell (1861) [9]. He suggested that the vector potential **A** of the magnetic induction **B** represents some kind of a fluid velocity field. This was interpreted in the light of Fizeau's experiment by Cook, Fearn and Milonni (1995)[10]. The analogy between Navier Stoke's equation and Maxwell's equations was used by Marmanis (1997-1998) [11] in the development of his metafluid dynamics for the study of turbulence. This marked the beginning of introducing new flow parameters (Lamb vectors). To complete the analogy between electromagnetic and fluid dynamic equation, Scofield and Huq (2008, 2009, 2010)[12] [13] [14] introduced the concept of vortex field and developed a uniform theory for electro dynamic, fluid dynamic and gravitational fields. A main source of interest in the study of the analogy was the theoretical developments in ferrofluids following its synthesis in 1960's." The Maxwell Stress Tensor and the forces in magnetic liquids" by Klaus Stierstadt and Mario Liu (2014)[15] can be considered as the present state of the theory. In this paper we explore yet another analogy between Maxwell stress tensor in electromagnetic continuum and a stress in incompressible inviscid fluids whose origin is vorticity.

II. Vorticity Stress Tensor

In the case of incompressible flow of an inviscid fluid, the vorticity field is frozen-in and satisfies the equation:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\boldsymbol{\vartheta} \times \boldsymbol{\omega}) \tag{1}$$

Here $\vec{\vartheta}$ is the velocity and $\boldsymbol{\omega} = \nabla \times \boldsymbol{\vartheta}$ is the vorticity. This equation admits the solution given by:

$$\omega_i(X, t) = \omega_j(x, 0)\frac{\partial X_i}{\partial x_j} \tag{2}$$

This is the well-known Cauchy's equation, which relates the current vorticity (at Xi) to the initial velocity (at xi), and thus it establishes a topological equivalence between them. Corresponding to the magnetic energy, the energy associated with the vorticity field is defined by $\frac{1}{2}\omega^2$. Thus the rate of change of energy is:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\omega^2}{2}\right) &= \boldsymbol{\omega} \cdot \frac{\partial \boldsymbol{\omega}}{\partial t} \\ &= \boldsymbol{\omega} \cdot [\nabla \times (\boldsymbol{\vartheta} \times \boldsymbol{\omega})] \qquad \text{[from (1)]} \\ &= -\boldsymbol{\vartheta} \cdot [(\nabla \times \boldsymbol{\omega}) \times \boldsymbol{\omega}] + \nabla \cdot [(\boldsymbol{\vartheta} \times \boldsymbol{\omega}) \times \boldsymbol{\omega}] \\ &= -\boldsymbol{\vartheta} \cdot (\mathbf{f} \times \boldsymbol{\omega}) + \nabla \cdot [(\boldsymbol{\vartheta} \times \boldsymbol{\omega}) \times \boldsymbol{\omega}] \qquad (3)\end{aligned}$$

where $\mathbf{f} = \nabla \times \boldsymbol{\omega}$ is the flexion field.

If the vorticity is confined to a sub-domain of the fluid, the divergence term vanishes on integrating over the entire volume.

Thus we get:

$$\frac{dM}{dt} = -\int_V \boldsymbol{\vartheta} \cdot \mathbf{F}\, dV \tag{4}$$

Here $M = \int_V \frac{\omega^2}{2} dV$ is the total enstrophy and $\mathbf{F} = \mathbf{f} \times \boldsymbol{\omega}$ is a force analogous to Lorentz force.

The i^{th} component of this force $\mathbf{F}$ is:

$$F_i = \frac{\partial}{\partial x_j} \Pi_{ij} \quad \text{, where } \Pi_{ij} = \omega_i \omega_j - \frac{\omega^2}{2} \delta_{ij} \tag{5}$$

and δ_{ij} is the kronecker delta.

Thus analogous to the Maxwell Stress tensor associated to Lorentz force (Ferraro and Plumpton 1966) [16] we get Π_{ij} as the vorticity stress tensor. This tensor is related to the enstrophy in the same way, as magnetic stress tensor is associated to magnetic energy. If we consider a vortex, the normal component of this stress represents the tension in the line vortices and the terms $\omega_i \omega_j$ are the shearing forces between adjacent vortex lines of the filament whose limiting case is the line vortex.

Π_{ij} being a symmetric matrix can be diagonalized . Choosing the principal axis OX_1 in the direction of vorticity we get:

$$\Pi_{ij} = \begin{bmatrix} \frac{\omega^2}{2} & 0 & 0 \\ 0 & \frac{-\omega^2}{2} & 0 \\ 0 & 0 & \frac{-\omega^2}{2} \end{bmatrix} \tag{6}$$

Thus the principal stress tensor constitutes a tension $\frac{1}{2}\omega^2$ along the line vortex and an equal pressure normal to it.

We can rewrite this tensor as:

$$\Pi_{ij} = \begin{bmatrix} \omega^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{-\omega^2}{2} & 0 & 0 \\ 0 & \frac{-\omega^2}{2} & 0 \\ 0 & 0 & \frac{-\omega^2}{2} \end{bmatrix} \tag{7}$$

where the first matrix gives the effect of the force **F** and the second matrix represents the static pressure.

The above expressions for the vorticity stress tensor can be compared to the magnetic stress tensor as discussed by Steirstadt and Liu (2014)[17]. So the figures given by them apply to vortex stress tensor also. They consider electromagnetic stress tensor (EMST) and discuss magnetic stress tensor. What we find here is that a similar stress tensor exists in the case of vortex fields.

In the context of magneto hydrodynamics the analogy between the induction equation:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\boldsymbol{\vartheta} \times \mathbf{B}) + \lambda \nabla^2 \mathbf{B} \qquad (\nabla \cdot \mathbf{B} = 0) \tag{8}$$

and the vorticity equation:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\boldsymbol{\vartheta} \times \boldsymbol{\omega}) + \nu \nabla^2 \boldsymbol{\omega} \qquad (\nabla \cdot \boldsymbol{\omega} = 0) \tag{9}$$

for barotropic flow of a fluid was first pointed out by Elsasser (1946)[18]. In this equation λ is the magnetic diffusivity of the fluid and ν is the kinematic viscosity. As pointed out by Moffat[19] the analogy has limitations as $\boldsymbol{\vartheta}$ and $\mathbf{B}$ are not related. This analogy has been the basis of many studies on vorticity especially vortex knots.

III. Conclusion

While computing the force on a surface, the surface over which the stress tensor is integrated need not correspond to a physical surface. This leads to the question of how the electromagnetic force is transmitted to the physical matter inside the surface. The answer given is, via the electromagnetic field that enter into the stress tensor. In the days of Maxwell, more explanations that are physical were considered necessary which led to the concept of ether, whose velocity acts as the vector potential for magnetic induction. This explanation perhaps applies more to the vorticity stress tensor, since the origin of vorticity is velocity.

It is difficult to take into account the stretching of vortex lines in three dimensions. Most of these studies make use of local induction approximation (LIA) or perturbation methods. But the stress tensor associated with vorticity can be made use of in such studies.

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