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Inequalities of a Generalized Class of K-Uniformly Harmonic Univalent Functions

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Abstract: *In this paper we define the inequalities for the classes k-USH(α) and k-HCV(α) are considered and obtain inequality for G(z). A class k-USH(α) is the class of k uniformely harmonic starlike function of order (α).and the class k-HCV(α) is the class of k uniformely convex function of order(α).These two classes are obtained by the generalization of class k-USH(μ,ν,α) [8].*
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I. Introduction

1.1 Let SH denotes the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the unit disk $\Delta = \{z : |z| < 1\}$ for which the function is normalized by the condition $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in SH$ the analytic functions h and g may be expressed as

- (1.1) $h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \; g(z) = \sum_{n=1}^{\infty} b_n z^n, \; |b_1| < 1.$
- For analytic function $h(z) \in S$ Salagean [1] introduced an operator defined as follows:

$D^0 h(z) = h(z)$, $D^1 h(z) = D(h(z)) = zh'(z)$ and

$$D^{\nu} h(z) = D(D^{\nu-1} h(z)) = z(D^{\nu-1} h(z))' = z + \sum_{n=2}^{\infty} n^{\nu} a_n z^n, \quad \nu \in N = \{1,2,...\}.$$

This operator D is called the Salagean operator.
Whereas, Jahangiri et al. [4] defined the modified Salagean operator of harmonic univalent functions f as

$$D^{\nu} f(z) = D^{\nu} h(z) + (-1)^{\nu} \overline{D^{\nu} g(z)}, \; \nu \in N_0 = \{0,1,2,\ldots\}$$

where $D^{\nu} h(z) = z + \sum_{n=2}^{\infty} n^{\nu} a_n z^n$ and $D^{\nu} g(z) = \sum_{n=1}^{\infty} n^{\nu} b_n z^n$.

Class Condition For The Class k-USH(μ,ν,α) [8]

For $0 \le \alpha < 1, \; 0 \le k < \infty, \; \mu > \nu$, k-USH$(\mu,\nu,\alpha)$ denotes a class of functions $f = h + \overline{g}$ satisfying

(1.2) $$\mathrm{Re}\left\{(1 + ke^{i\phi}) \frac{D^{\mu} f(z)}{D^{\nu} f(z)} - ke^{i\phi}\right\} \ge \alpha, \; \phi \in R.$$

Also, k-UTH$(\mu,\nu,\alpha) \subseteq$ k-USH(μ,ν,α) consists of harmonic functions $f_\mu = h + \overline{g}_\mu$ so that

(1.3) $$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad g_\mu(z) = (-1)^{\mu-1} \sum_{n=1}^{\infty} |b_n| z^n, \; |b_1| < 1.$$

It is noted that $k - USH(1,0,\alpha) \equiv k - USH(\alpha)$,
$k - USH(2,1,\alpha) \equiv k - HCV(\alpha)$ and classes $k - UTH(\alpha) \subseteq k - USH(\alpha)$ and $k - THCV(\alpha) \subseteq k - HCV(\alpha)$ consist of functions $f_\mu = h + \overline{g_\mu}$ of the form (1.3).

Also $1 - USH(0) \equiv G_H$, $1 - HCV(0) \equiv HCV$.

Generalization Of Class k-USH(μ,ν,α) [8]

The class k-UTH(μ,ν,α) generalizes several classes of harmonic univalent functions defined earlier. For k=0, $\mu=1,\nu=0$, this class reduces to $SH(\alpha)$ the class of univalent harmonic starlike functions of order α which was studied by Jahangiri [2] and for $k=0,\ \mu=2,\ \upsilon=1$, it reduces to the class $KH(\alpha)$, the class of univalent harmonic convex function of order α which is studied by Jahangiri [3]. For k=1, $\mu=1,\nu=0$, this class reduces to $G_H(\alpha)$ which was studied by Rosy et al. [6]. For k=1, $\mu=\nu+1$, this class reduces to $RS_H(\nu,\alpha)$ which was studied by Yalcin et al. [5].

(1.1) Results For The Class k-USH(μ,ν,α) **[8]**

In this section necessary and sufficient coefficient inequality for the class k-USH(μ,ν,α),extreme points,distortion bounds,neighbourhood property are defined in the form of corollaries.

Corollary 1.1.1 (Sufficient coefficient condition for k-USH(μ,ν,α)) [8]

Let $f=h+\overline{g}$ be given by (1.1). Furthermore, let

(1.1.1) $$\sum_{n=1}^{\infty}\{\psi(\mu,\nu,\alpha)\,|\,a_n\,|+\theta(\mu,\nu,\alpha)\,|\,b_n\,|\}\le 2$$

where, $$\psi(\mu,\nu,\alpha)=n^{\nu}+\frac{(n^{\mu}-n^{\nu})(1+k)}{1-\alpha}$$

$$\theta(\mu,\nu,\alpha)=(-1)^{\nu-\mu}n^{\nu}+\frac{(n^{\mu}-(-1)^{\nu-\mu}n^{\nu})(1+k)}{1-\alpha}$$

with $a_1=1, 0\le\alpha<1, 0\le k<\infty,\ \mu\in N=\{1,2,\ldots\},\ \ \nu\in N_o=\{0,1,2,\ldots\}$ and $\mu>\nu$, then f is harmonic univalent, sense-preserving in Δ and $f\in$k-USH(μ,ν,α). The result is sharp also.

Corollary 1.1.2 (Coefficient inequality for k-UTH(μ,ν,α))

Let $f_{\mu}=h+\overline{g}_{\mu}$ where h and $\overline{g}_{\mu}$ be given by (1.3). Then $f_{\mu}\in$k-UTH(μ,ν,α) if and only if

(1.1.2) $$\sum_{n=1}^{\infty}\{\psi(\mu,\nu,\alpha)\,|\,a_n\,|+\theta(\mu,\nu,\alpha)\,|\,b_n\,|\}\le 2$$

where $a_1=1,\ 0\le\alpha<1,\ \mu\in N,\ \nu\in N_0\ \text{and}\ \mu>\nu$.

Corrolary1.1.3 (Extreme Points)

Let f_{μ} be of the form (1.3) then $f_{\mu}\in$k-UTH(μ,ν,α) if and only if

(1.1.3) $$f_{\mu}(z)=\sum_{n=1}^{\infty}\left[x_nH_n(z)+y_nG_n(z)\right]$$

where,

$$H_1(z)=z,\ H_n(z)=z-\frac{1}{\psi(\mu,\nu,\alpha)}z^n,\ (n=2,3\ldots)$$

and $$G_n(z)=z+(-1)^{\mu-1}\frac{1}{\theta(\mu,\nu,\alpha)}\overline{z}^n,\quad (n=1,2,3\ldots.)$$

$x_n\ge 0,\ y_n\ge 0,\ x_1=1-\sum_{n=2}^{\infty}x_n-\sum_{n=1}^{\infty}y_n$. In particular, the extreme points of k-UTH(μ,ν,α) are $\{H_n\}$ and $\{G_n\}$.

Corollary 1.1.4 (Distortion Bounds)

Let $f_{\mu}(z)\in$k-UTH(μ,ν,α). then for $|\,z\,|=r<1$

$$|\,f_{\mu}(z)\,|\le(1+|\,b_1\,|)r+\{\Omega(\mu,\nu,\alpha)-\delta(\mu,\nu,\alpha)\,|\,b_1\,|\}r^2$$

and

$$|\,f_{\mu}(z)\,|\ge(1-|\,b_1\,|)r-\{\Omega(\mu,\nu,\alpha-\delta(\mu,\nu,\alpha)\,|\,b_1\,|\}r^2$$

where,

$$\Omega(\mu,\nu,\alpha)=\frac{1-\alpha}{2^{\mu}(1+k)-2^{\nu}(k+\alpha)}$$

$$\delta(\mu,\nu,\alpha)=\frac{(1+k)-(-1)^{\nu-\mu}(k+\alpha)}{2^{\mu}(1+k)-2^{\nu}(k+\alpha)}.$$

II. Neighbourhoods

The modified δ-neighbourhood of f_μ which is of the form (1.3) is defined as the set

$$N_\delta(f_\mu)=\Big[F_\mu=z-\sum_{n=2}^{\infty}|A_n|z^n+(-1)^{\mu-1}\sum_{n=1}^{\infty}\overline{|B_n|z^n}:\sum_{n=2}^{\infty}\Big[\big\{(n^\nu+(n^\mu-n^\nu)(1+k)\big\}|a_n-A_n|$$
$$+\big\{(-1)^{\nu-\mu}n^\nu+(n^\mu-(-1)^{\nu-\mu}n^\nu)(1+k)\big\}|b_n-B_n|\Big]$$
$$+\big\{k\big\{1-(-1)^{\nu-\mu}\big\}+1\big\}\quad|b_1-B_1|\le\delta,\delta>0\Big].$$

Theorem 2.22 (Neighbourhood property)

Let f_μ satisfies the condition

$$\sum_{n=2}^{\infty}n^{\nu+1}\Big[\big\{1+(n^{\mu-\nu}-1)(1+k)\big\}|a_n|\Big]$$
$$+\sum_{n=1}^{\infty}n^{\nu+1}\Big[\big\{(-1)^{\nu-\mu}+(n^{\mu-\nu}-(-1)^{\nu-\mu})(1+k)\big\}|b_n|\Big]\le 1$$

and

$\delta=\frac{1}{2}\Big[1-3|b_1|\{k(1-(-1)^{\nu-\mu})+1\}\Big]$ with $|b_1|<\frac{1}{3\{k(1-(-1)^{\mu-\nu})+1\}}$

then, $N_\delta(f_\mu)\subset$ k-UTH$(\mu,\nu,0)$.

1.2 In this section two classes k-USH(α) and k-HCV(α) are considered and obtain inequality for G(z) to be sense-preserving and to be in the class k-USH(α). For the class k-HCV(α) a sufficient condition is also derived.

It is also shown that these sufficient conditions are also necessary for the function for $G_1(z)$ to be in classes k-UTH(α) and k-THCV(α) respectively. Further a necessary and sufficient condition for convolution function of f and G to be in k-UTH(α) class is derived.

Consider

$$G(z)=\phi_1(z)+\overline{\phi_2(z)}$$

where ,

(1.2.1) $$\phi_1(z)=\phi_1(a_1,b_1;c_1;z)=zF(a_1,b_1;c_1;z)$$
$$=z+\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}z^n,$$

(1.2.2) $$\phi_2(z)=\phi_2(a_2,b_2;c_2;z)=F(a_2,b_2;c_2;z)-1$$
$$=\sum_{n=1}^{\infty}\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}z^n,a_2b_2<c_2.$$

Also, consider

$$G_1(z)=z\left(2-\frac{\phi_1(z)}{z}\right)+\overline{\phi_2(z)}$$

where $\phi_1(z)$ and $\phi_2(z)$ are given by (1.2.1) and (1.2.2) respectively.
The following corollaries are used in the theorems-

Corollary 1 [7]
Let $f = h + \overline{g} \in SH$ be given by (2.1) and if

$$\sum_{n=1}^{\infty} n\{(n+nk-k-\alpha)|a_n| + (n+nk+k+\alpha)|b_n|\} \le 2(1-\alpha)$$

where $|a_1| = 1$ and $0 \le \alpha < 1$ then f is sense-preserving univalent in Δ and $f \in$ k-HCV(α).
Further on taking $\mu = 1, \nu = 0$ the following corollary is obtained.

Corollary 2
Let $f = h + \overline{g} \in SH$ be given by (2.1) and if

$$\sum_{n=1}^{\infty} \{(n+nk-k-\alpha)|a_n| + (n+nk+k+\alpha)|b_n|\} \le 2(1-\alpha)$$

where $|a_1| = 1$ and $0 \le \alpha < 1$ then f is sense-preserving, univalent in Δ and $f \in$ k-USH(α).

Corollary 3 [7]
Let $f = h + \overline{g} \in SH$ be given by (1.3). Then $f \in$ k-THCV(α) if and only if

$$\sum_{n=1}^{\infty} n\{(n+nk-k-\alpha)|a_n| + (n+nk+k+\alpha)|b_n|\} \le 2(1-\alpha)$$

where $|a_1| = 1$ and $0 \le \alpha < 1$.
For $\mu = 1, \nu = 0$ the following corollary is obtained.

Corollary 4
Let $f = h + \overline{g} \in SH$ be given by (1.3) then $f \in$ k-UTH(α) if and only if

$$\sum_{n=1}^{\infty} \{(n+nk-k-\alpha)|a_n| + (n+nk+k+\alpha)|b_n|\} \le 2(1-\alpha)$$

where $|a_1| = 1$ and $0 \le \alpha < 1$.

Theorem 1.2.1
If $a_j, b_j > 0$, $c_j > a_j + b_j + 1$ for j=1,2 then a sufficient condition for $G = \phi_1 + \overline{\phi_2}$ where ϕ_1 and ϕ_2 are given in (1.2.1) and (1.2.2) respectively to be sense-preserving harmonic univalent in Δ and $G \in$ k-USH(α) is that

$$\text{(1.2.3)} \quad F(a_1,b_1;c_1;1)\left[\frac{a_1 b_1}{c_1-a_1-b_1-1}(k+1)+1-\alpha\right] + F(a_2,b_2;c_2;1)\left[\frac{a_2 b_2}{c_2-a_2-b_2-1}(k+1)+k+\alpha\right] \le 2-\alpha+k.$$

Proof
To prove that G is sense-preserving in Δ, it needs to show that

$$|\phi_1'(z)| > |\phi_2'(z)|, \quad z \in \Delta$$

$$|\phi_1'(z)| = \left|1 + \sum_{n=2}^{\infty} n \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} z^{n-1}\right|$$

$$= \left|1 + \sum_{n=2}^{\infty} (n-1) \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} z^{n-1} + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} z^{n-1}\right|$$

$$\ge \left[1 - \sum_{n=2}^{\infty} (n-1) \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} - \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}\right]$$

$$=\left[1-\frac{a_1b_1}{c_1}\sum_{n=1}^{\infty}\frac{(a_1+1)_{n-1}(b_1+1)_{n-1}}{(c_1+1)_{n-1}(1)_{n-1}}-\sum_{n=1}^{\infty}\frac{(a_1)_n(b_1)_n}{(c_1)_n(1)_n}\right]$$

$$=\left[2-\frac{a_1b_1}{c_1}\frac{\Gamma(c_1+1)\Gamma(c_1-a_1-b_1-1)}{\Gamma(c_1-a_1)\Gamma(c_1-b_1)}-\frac{\Gamma(c_1)\Gamma(c_1-a_1-b_1)}{\Gamma(c_1-a_1)\Gamma(c_1-b_1)}\right]$$

$$=\left[2-\left(\frac{a_1b_1}{c_1-a_1-b_1-1}+1\right)F(a_1,b_1;c_1;1)\right]$$

$$\geq\left[\left\{\frac{a_1b_1}{c_1-a_1-b_1-1}k-\alpha\right\}F(a_1,b_1;c_1;1)+\right.$$

$$\left.+\left\{\frac{a_2b_2}{c_2-a_2-b_2-1}(k+1)+k+\alpha\right\}\;F(a_2,b_2;c_2;1)+\alpha-k\right]\text{by (1.2.3)}$$

$$\geq\frac{a_2b_2}{c_2-a_2-b_2-1}F(a_2;b_2;c_2;1)$$

$$=\frac{a_2b_2}{c_2}\frac{\Gamma(c_2+1)\Gamma(c_2-a_2-b_2-1)}{\Gamma(c_2-a_2)\Gamma(c_2-b_2)}$$

$$=\frac{a_2b_2}{c_2}\sum_{n=1}^{\infty}\frac{(a_2+1)_{n-1}(b_2+1)_{n-1}}{(c_2+1)_{n-1}(1)_{n-1}}$$

$$=\sum_{n=1}^{\infty}n\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}$$

$$>\sum_{n=1}^{\infty}n\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}\,|z|^{n-1}$$

$$\geq\left|\sum_{n=1}^{\infty}n\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}z^{n-1}\right|=|\phi_2'(z)|.$$

So, G is sense-preserving in Δ. To show that G is univalent and $G\in k-USH(\alpha)$, applying Corollary (1) it only need to prove that

(1.2.4) $$\sum_{n=2}^{\infty}(n+nk-k-\alpha)\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}+$$

$$+\sum_{n=1}^{\infty}(n+nk-k+\alpha)\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}\leq 1-\alpha,\quad a_2b_2<c_2.$$

The left hand side of (1.2.4) is equivalent to

$$\sum_{n=2}^{\infty}(n-1)k\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}+\sum_{n=2}^{\infty}(n-1)\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}+\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}$$

$$-\alpha\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}+\sum_{n=1}^{\infty}n(k+1)\frac{(a_2)_n(b_2)}{(c_1)_n(1)_n}+(k+\alpha)\sum_{n=1}^{\infty}\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}$$

$$=\frac{a_1b_1}{c_1-a_1-b_1-1}(k+1)F(a_1,b_1;c_1;1)+[F(a_1,b_1;c_1;1)-1]$$

$$-\alpha[F(a_1,b_1;c_1;1)-1]+(k+1)\frac{a_2b_2}{c_2-a_2-b_2-1}F(a_2,b_2;c_2;1)$$

$$+(k+\alpha)[F(a_2,b_2;c_21)-1]$$

$$= F(a_1,b_1;c_1;1)\left[\frac{a_1b_1}{c_1-a_1-b_1-1}(k+1)+1-\alpha\right]$$
$$+F(a_2,b_2;c_2;1)\left[\frac{a_2b_2}{c_2-a_2-b_2-1}(k+1)+k+\alpha\right]-1-k.$$

The last expression is bounded by $(1-\alpha)$ provided that (1.2.3) is satisfied. Therefore, $G \in$ k-USH(α). Consequently G is sense-preserving and univalent of order α in Δ.

On putting $\alpha = 0, k = 0$ the following result of Ahuja [4] is obtained.

Corollary 1.2.2 [4]

If $a_j, b_j > 0,\ c_j > a_j + b_j + 1$ for j=1,2, then a sufficient condition for $G = \phi_1 + \overline{\phi_2}$ to be harmonic univalent in Δ and $G \in S^*H$ is that

$$\left(1+\frac{a_1b_1}{c_1-a_1-b_1-1}\right)F(a_1,b_1;c_1;1)+\frac{a_2b_2}{c_2-a_2-b_2-1}F(a_2,b_2;c_2;1)\le 2$$

For $\alpha = 0, k = 1$ the following corollary is obtained.

Corollary 1.2.3

If $a_j, b_j > 0,\ c_j > a_j + b_j + 1$ for j=1,2 then a sufficient condition for $G = \phi_1 + \overline{\phi_2}$ to be harmonic univalent in Δ and $G \in G_H$ is that

$$\left(\frac{2a_1b_1}{c_1-a_1-b_1-1}+1\right)F(a_1,b_1;c_1;1)+\left(\frac{2a_2b_2}{c_2-a_2-b_2-1}+1\right)F(a_2,b_2;c_2;1)\le 3$$

Theorem 1.2.4

If $a_j, b_j > 0,\ c_j > a_j + b_j + 2$ for j=1,2 then a sufficient condition for $G = \phi_1 + \overline{\phi_2}$ to be harmonic univalent in Δ and $G \in$ k-HCV(α) is that

(1.2.5)
$$F(a_1,b_1;c_1;1)\left[\frac{(a_1)_2(b_1)_2}{(c_1-a_1-b_1-2)_2}+\frac{a_1b_1}{c_1-a_1-b_1-1}\{3-k-\alpha\}-\alpha+1+k\right]$$
$$+F(a_2,b_2;c_2;1)\left[\frac{(a_2)_2(b_2)_2}{(c_2-a_2-b_2-2)_2}+\frac{a_2b_2}{c_2-a_2-b_2-1}\{1+\alpha+k\}\right]\le 2(1-\alpha).$$

Proof

To prove the theorem applying Corollary (2) it needs to show that

(1.2.6)
$$\sum_{n=2}^{\infty}\left[n(n+nk-k-\alpha)\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}\right]+$$
$$+\sum_{n=1}^{\infty}\left[n(n+nk+k+\alpha)\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}\right]\le 1-\alpha$$

That is

$$\sum_{n=2}^{\infty}n^2\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}(k+1)-(k+\alpha)\sum_{n=2}^{\infty}n\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}+$$
$$+\sum_{n=1}^{\infty}n^2\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}+(k+\alpha)\sum_{n=1}^{\infty}n\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}\le 1-\alpha$$

or

$$\sum_{n=0}^{\infty}(n+2)^2\frac{(a_1)_{n+1}(b_1)_{n+1}}{(c_1)_{n+1}(1)_{n+1}}(k+1)-(k+\alpha)\sum_{n=2}^{\infty}(n-1)\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}$$

$$-(k+\alpha)\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}+\sum_{n=0}^{\infty}(n+1)^2\frac{(a_2)_{n+1}(b_2)_{n+1}}{(c_2)_{n+1}(1)_{n+1}}$$

$$+(k+\alpha)\sum_{n=1}^{\infty}n\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}\le 1-\alpha$$

But

$$(k+1)\sum_{n=0}^{\infty}(n+2)^2\frac{(a_1)_{n+1}(b_1)_{n+1}}{(c_1)_{n+1}(1)_{n+1}}$$

$$=(k+1)\left\{\left[\frac{(a_1)_2(b_1)_2}{(c_1-a_1-b_1-2)_2}+\frac{3a_1b_1}{c_1-a_1-b_1-1}+1\right]F(a_1,b_1;c_1;1)-1\right\}$$

and

$$\sum_{n=0}^{\infty}(n+1)^2\frac{(a_2)_{n+1}(b_2)_{n+1}}{(c_2)_{n+1}(1)_{n+1}}$$

$$=\left[\frac{(a_2)_2(b_2)_2}{(c_2-a_2-b_2-2)_2}+\frac{a_2b_2}{c_2-a_2-b_2-1}\right]F(a_2,b_2;c_2;1)$$

Thus, the left hand side of (1.2.5) is equivalent to

$$=F(a_1,b_1;c_1;1)\left[(k+1)\left\{\frac{(a_1)_2(b_1)_2}{(c_1-a_1-b_1-2)_2}+\frac{3a_1b_1}{c_1-a_1-b_1-1}+1\right\}\right.$$

$$\left.-(k+\alpha)\frac{a_1b_1}{c_1-a_1-b_1-1}-k-\alpha\right]+F(a_2,b_2;c_2;1)\left[\frac{(a_2)_2(b_2)_2}{(c_2-a_2-b_2-2)_2}\right.$$

$$\left.+\frac{a_2b_2}{c_2-a_2-b_2-1}+(k+\alpha)\frac{a_2b_2}{c_2-a_2-b_2-1}\right]-k-1+k+\alpha$$

which is bounded above by $1-\alpha$ provided that (1.2.5) is satisfied. This completes the proof.

On taking $\alpha=0,\ k=0$ the following Corollary [4] is obtained.

Corollary 1.2.5 [4]

If $a_j,b_j>0,\ c_j>a_j+b_j+2$ for j=1,2 then a sufficient condition for $G=\phi_1+\overline{\phi_2}$ to be harmonic univalent in Δ and $G\in KH$ is that

$$\left(1+\frac{3a_1b_1}{c_1-a_1-b_1-1}+\frac{(a_1)_2(b_1)_2}{(c_1-a_1-b_1-2)_2}\right)F(a_1,b_1;c_1;1)$$

$$+\left(\frac{a_2b_2}{c_1-a_2-b_2-1}+\frac{(a_2)_2(b_2)_2}{(c_2-a_2-b_2-2)_2}\right)F(a_2,b_2;c_2;1)\le 2$$

For $\alpha=0,k=1$ the following Corollary is obtained.

Corollary 1.2.6

If $a_j,b_j>0,\ c_j>a_j+b_j+2$ for j=1,2 then a sufficient condition for $G=\phi_1+\overline{\phi_2}$ to be harmonic univalent in Δ and $G\in HCV$ is that

$$F(a_1,b_1;c_1;1)\left[\frac{(a_1)_2(b_1)_2}{(c_1-a_1-b_1-2)_2}+\frac{2a_1b_1}{c_1-a_1-b_1-1}+2\right]+$$

$$F(a_2,b_2;c_2;1)\left[\frac{(a_2)_2(b_2)_2}{(c_2-a_2-b_2-2)_2}+\frac{a_2b_2}{c_2-a_2-b_2-1}\right]\le 2.$$

Theorem 1.2.7

Let $a_j, b_j > 0,\ c_j > a_j + b_j + 1$, for j=1,2 and $a_2b_2 > c_2$. If $G_1(z) = z\left(2 - \frac{\phi_1(z)}{z}\right) + \overline{\phi_2(z)}$ then,

$G_1 \in$ k-UTH(α) if and only if

$$F(a_1, b_1; c_1; 1)\left[\frac{a_1b_1}{(c_1 - a_1 - b_1 - 1)}(k+1) + 1 - \alpha\right] + F(a_2, b_2; c_2; 1)$$

$$\left[\frac{a_2b_2}{(c_2 - a_2 - b_2 - 1)}(k+1) + k + \alpha\right] \le 2 - \alpha + k.$$

Proof

It is observe that

$$G_1(z) = z - \sum_{n=2}^{\infty}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}z^n + \overline{\sum_{n=1}^{\infty}\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}z^n}$$

and k-UTH$(\alpha) \subset$ k-USH(α). In view of Theorem 1.2.1, it needs to show the necessary condition for G_1 to be in k-UTH(α). If $G_1 \in$ k-UTH(α), then G_1 satisfies the inequality in Corollary (3) and the result follows.

Theorem 1.2.8

Let $a_j, b_j > 0,\ c_j > a_j + b_j + 1$ for j=1,2 and $a_2b_2 < c_2$ if

$$G_1(z) = z\left(2 - \frac{\phi_1(z)}{2}\right) + \overline{\phi_2(z)}$$

then $G_1 \in$ k-THCV(α) if and only if

$$F(a_1, b_1; c_1; 1)\left[\frac{(a_1)_2(b_1)_2}{(c_1 - a_1 - b_1 - 2)_2} + \frac{a_1b_1}{c_1 - a_1 - b_1 - 1}\{3 - k - \alpha\} - \alpha + 1 + k\right]$$

$$+F(a_2, b_2; c_2; 1)\left[\frac{(a_2)_2(b_2)_2}{(c_2 - a_2 - b_2 - 2)_2} + \frac{a_2b_2}{c_2 - a_2 - b_2 - 1}\{1 + \alpha + k\}\right] \le 2(1 - \alpha).$$

Proof

It observe that

$$G_1(z) = z - \sum_{n=2}^{\infty}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}z^n + \overline{\sum_{n=1}^{\infty}\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}z^n}$$

and k-THCV$(\alpha) \subset$ k-HCV(α). In view of Theorem 1.2.4, it needs to show the necessary condition for G_1 to be in. k-THCV(α) If $G_1 \in$ k-THCV(α) then G_1 satisfies the inequality in Corollary(4) and the result follows.

Theorem 1.2.9

Let $a_j, b_j > 0,\ c_j > a_j + b_j + 1$, for j=1,2 and $a_2b_2 < c_2$. A necessary and sufficient condition such that

$f \,\overline{*}\, (\phi_1 + \overline{\phi_2}) \in$ k-UTH (α) for $f \in$ k-UTH (α) is that

(1.2.7) $F(a_1, b_1; c_1; 1) + F(a_2, b_2; c_2; 1) \le 3$

where ϕ_1 and ϕ_2 are defined respectively in (1.2.1) and (1.2.2).

Proof

Let $f \in$ k-UTH (α), then

$$(f \,\overline{*}\, (\phi_1 + \overline{\phi_2}))(z) = h(z) * \phi_1(z) + \overline{g(z) * \phi_2(z)}$$

$$= z - \sum_{n=2}^{\infty}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}|a_n|z^n + \sum_{n=1}^{\infty}\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}|b_n|\overline{z}^n$$

and

$$|a_n| \le \frac{1-\alpha}{n+nk-k-\alpha},\ |b_n| \le \frac{1-\alpha}{n+nk+k-\alpha}$$

In view of corollary 2.18, it needs to prove that $f \,\overline{*}\, (\phi_1 + \overline{\phi_2}) \in$ k-UTH(α), As an application of corollary 2.18

$$\sum_{n=2}^{\infty}(n+nk-k-\alpha)\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}|a_n|$$
$$+\sum_{n=1}^{\infty}(n+nk+k+\alpha)\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}|b_n|$$
$$\le (1-\alpha)\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + (1-\alpha)\sum_{n=1}^{\infty}\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}$$
$$= (1-\alpha)F(a_1,b_1;c_1;1) + (1-\alpha)F(a_2,b_2;c_2;1) - 2(1-\alpha).$$

The last expression is bounded above by $1-\alpha$ if and only if (2.3.7) is satisfied. This proves the result.

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