IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 11, Issue 3 Ver. I (May - Jun. 2015), PP 73-77
www.iosrjournals.org

Systems of exponential ratio-based and exponential product-based estimators with their efficiency

K.B.Panda[1]& N.Sahoo[2]
[1]*Reader, Department of Statistics, Utkal University, Bhubaneswar, Odisha, India*
[2]*Ph.D. Scholar,Department of Statistics, Utkal University, Bhubaneswar, Odisha, India*

Abstract: *Following Agrawal and Sthapit [1], we have developed systems of exponential ratio-based and exponential product-based estimators. The proposed exponential ratio-based estimator of order k and the proposed exponential product-based estimator of order k are, under practical conditions, found to be more efficient than the customary exponential ratio and the mean per unit estimator and the customary exponential product estimator and the mean per unit estimator, respectively, when k is optimally chosen. Under the optimal value of k, the k^{th}order exponential ratio-based and exponential product-based estimators are found to be as efficient as the linear regression estimator. With a view to establishing the supremacy of the proposed estimators over the existing estimators, numerical illustrations in respect of real populations have been considered.*

Keywords: *Simple random sampling, exponential ratio estimator, exponential product estimator, predictive estimation*

I. Introduction

In survey sampling, when the auxiliary variable x is positively correlated with the study variable y and complete information on x is available, the ratio method of estimation is followed to estimate the population mean ($\overline{Y}$) or the population total (Y). The conventional ratio estimator

$$\bar{y}_R = \frac{\bar{y}}{\bar{x}}\bar{X}$$

is preferred to the mean per unit estimator($\bar{y}$) if

$$\rho > \frac{1}{2}\frac{C_x}{C_y},(1.1)$$

where c_x and c_y are the coefficients of variation of x and y, respectively, and ρ, the correlation coefficient between x and y, is positive. Similarly, whenx and y are negatively correlated, Murthy [2]proposed the usual productestimator given by

$$\bar{y}_P = \bar{y}\frac{\bar{x}}{\bar{X}},$$

which is more efficient than the simple mean if the condition

$$\rho^* < -\frac{1}{2}\frac{C_x}{C_y}, \qquad (1.2)$$

where ρ^*, the correlation coefficient between x and y, is negative, holds true. It is worthwhile to note that the conditions (1.1) and (1.2) hold good in practice very often. Bahland Tuteja[3] have proposed the exponential ratio estimator andthe exponential product estimator which are listed, respectively,as

$$\bar{y}_{Re} = \bar{y}\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right), \qquad (1.3)$$

$$\text{and } \bar{y}_{Pe} = \bar{y}\exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right). \qquad (1.4)$$

While $\bar{y}_{Re}$ is more efficient than the classical ratio estimator ($\bar{y}_R$) and the mean per unit estimator($\bar{y}$) if

$$\frac{1}{4} < \rho\frac{C_y}{C_x} < \frac{3}{4},(1.5)$$

$\bar{y}_{Pe}$ fares better than the usual product estimator ($\bar{y}_P$) and the mean per unit estimator($\bar{y}$) if

$$-\frac{3}{4} < \rho\frac{C_y}{C_x} < -\frac{1}{4}. \qquad (1.6)$$

It may be noted here that the conditions (1.5) and (1.6) too are satisfied in several practical situations.

Following the predictive approach due to Basu [4] and Smith [5] for a fixed population set-up and then, with recursive use of this intuitive predictive format, Agrawal and Sthapit [1] have generated sequences of ratio-based and product-based estimators which, under certain conditions, perform better than the customary well-known estimators such as $\bar{y}, \bar{y}_R$ and$\bar{y}_P$. We have, in this paper, developed systems of exponential ratio-based and exponential product-based estimators as per the lines of Agrawal and Sthapit [1]. The estimators thus proposed are, under real-life conditions, found to fare better than the customary exponential ratio estimator$\bar{y}_{Re}$

and exponential product estimator $\bar{\mathbf{y}}_{\mathbf{Pe}}$ and the mean per unit estimator $\bar{\mathbf{y}}$. The work has been supported by some practical numerical illustrations.

II. A system of exponential ratio-based estimators with performance

Under the predictive set-up, we can write the population total Y as

$$\mathbf{Y} = \sum_{i\epsilon s}\mathbf{y_i}+\sum_{i\epsilon \bar{s}}\mathbf{y_i}, \quad (2.1)$$

where s denotes the sample of selected units and $\bar{\mathbf{s}}$ its complement.To estimate $\mathbf{Y}$, we have to predict $\mathbf{y_i(i\epsilon\bar{s})}$.In otherwords, the predictive format for the estimation of $\mathbf{Y}$ is

$$\hat{\mathbf{Y}} = \sum_{i\epsilon s}\mathbf{y_i}+\sum_{i\epsilon \bar{s}}\hat{\mathbf{y}}_i, \quad (2.2)$$

where $\hat{\mathbf{y}}_i$ is the implied predictor of $\mathbf{y_i}$ $\mathbf{(i\epsilon\bar{s})}$.

Now, if we use $\bar{\mathbf{y}}_{\mathbf{Re}}$ as an intuitive predictor of $\mathbf{y_i(i\epsilon\bar{s})}$ in (2.2), we obtain

$$\hat{\mathbf{Y}} = \sum_{i\in s}\mathbf{y_i} + (\mathbf{N}-\mathbf{n})\bar{\mathbf{y}}_{\mathbf{Re}}$$

or equivalentely,

$$\hat{\bar{\mathbf{Y}}}=\bar{\mathbf{y}}_{\mathbf{Re}}^{(1)},$$

where $\bar{\mathbf{y}}_{\mathbf{Re}}^{(1)}=\emptyset_1\bar{\mathbf{z}}_{\mathbf{Re}}+\bar{\mathbf{y}}_{\mathbf{Re}}$, with $\emptyset_1 = \mathbf{1}+\boldsymbol{\lambda}\emptyset_0$, $\emptyset_0 = \mathbf{0}$, $\boldsymbol{\lambda} = \mathbf{1}-\frac{\mathbf{n}}{\mathbf{N}}$ (2.3)

and

$$\bar{\mathbf{z}}_{\mathbf{Re}} = \frac{\mathbf{n}}{\mathbf{N}}\bar{\mathbf{y}}\left[\mathbf{1}-\mathbf{exp}\left(\frac{\bar{\mathbf{X}}-\bar{\mathbf{x}}}{\bar{\mathbf{X}}+\bar{\mathbf{x}}}\right)\right]$$

A second iteration with $\bar{\mathbf{y}}_{\mathbf{Re}}^{(1)}$ as an intuitive predictor of $\mathbf{y_i(i\epsilon\bar{s})}$ in (2.2) would result in $\bar{\mathbf{y}}_{\mathbf{Re}}^{(2)}$ given by

$$\bar{\mathbf{y}}_{\mathbf{Re}}^{(2)}=\emptyset_2\bar{\mathbf{z}}_{\mathbf{Re}}+\bar{\mathbf{y}}_{\mathbf{Re}},$$

where $\emptyset_2 = \mathbf{1}+\boldsymbol{\lambda}\emptyset_1$.Continuing in this way, we would ,at the k[th]iteration, arrive at

$$\bar{\mathbf{y}}_{\mathbf{Re}}^{(\mathbf{k})}=\emptyset_{\mathbf{k}}\bar{\mathbf{z}}_{\mathbf{Re}}+\bar{\mathbf{y}}_{\mathbf{Re}},$$

where $\emptyset_{\mathbf{k}} = \mathbf{1}+\boldsymbol{\lambda}\emptyset_{\mathbf{k-1}}=\frac{\mathbf{1}-\boldsymbol{\lambda}^{\mathbf{k}}}{\mathbf{1}-\boldsymbol{\lambda}}$.

Thus, $\bar{\mathbf{y}}_{\mathbf{Re}}^{(\mathbf{k})}$ can be expressed as

$$\bar{\mathbf{y}}_{\mathbf{Re}}^{(\mathbf{k})} = (\mathbf{1}-\boldsymbol{\lambda}^{\mathbf{k}})\bar{\mathbf{y}}+\boldsymbol{\lambda}^{\mathbf{k}}\bar{\mathbf{y}}_{\mathbf{Re}}. \quad (2.4)$$

We will address $\bar{\mathbf{y}}_{\mathbf{Re}}^{(\mathbf{k})}$ the exponential ratio-based estimator of the kth order. It may be noted here that , for $\mathbf{k}=\mathbf{0}$, $\bar{\mathbf{y}}_{\mathbf{Re}}^{(\mathbf{k})}$ (i.e. $\bar{\mathbf{y}}_{\mathbf{Re}}^{(\mathbf{0})}$) is the usual exponential ratio estimator given in (1.1) and that , as $\mathbf{k}\rightarrow$,we have $\boldsymbol{\lambda}^{\mathbf{k}}\rightarrow\mathbf{0}$ and $\bar{\mathbf{y}}_{\mathbf{Re}}^{(\mathbf{k})}\rightarrow\bar{\mathbf{y}}$.Here, we assume N to be finite because, if we draw samples of fixed sizes from an infinite population, then $\bar{\mathbf{y}}_{\mathbf{Re}}^{(\mathbf{k})}$ will be no different from $\bar{\mathbf{y}}_{\mathbf{Re}}$ as λ becomes 1. The bias of $\bar{\mathbf{y}}_{\mathbf{Re}}^{(\mathbf{k})}$ to $\mathbf{O(n^{-1})}$ can be obtained as

$$\mathbf{B}(\bar{\mathbf{y}}_{\mathbf{Re}}^{(\mathbf{k})})= \boldsymbol{\lambda}^{\mathbf{k}}\left(\frac{1}{n}-\frac{1}{N}\right)\bar{\mathbf{Y}}\left(\frac{3}{8}\mathbf{C}_{\mathbf{x}}^2-\frac{1}{2}\boldsymbol{\rho}\mathbf{C}_{\mathbf{y}}\mathbf{C}_{\mathbf{x}}\right) \quad (2.5)$$

For $\mathbf{k}\geq\mathbf{1}$, this is invariably smaller than the bias of customary exponential ratio estimator $\bar{\mathbf{y}}_{\mathbf{Re}}$.The mean square error or variance of $\bar{\mathbf{y}}_{\mathbf{Re}}^{(\mathbf{k})}$ to $\mathbf{O(n^{-1})}$ can be found out as

$$\mathbf{MSE}(\bar{\mathbf{y}}_{\mathbf{Re}}^{(\mathbf{k})})=\frac{\lambda}{n}\bar{\mathbf{Y}}^2\mathbf{C}_{\mathbf{y}}^2\left[\mathbf{1}+\frac{\lambda^{2k}}{4}\left(\frac{C_x}{C_y}\right)^2-\boldsymbol{\lambda}^{\mathbf{k}}\boldsymbol{\rho}\left(\frac{C_x}{C_y}\right)\right]. \quad (2.6)$$

Then, when **k i**s determined optimally with a view to minimizing (2.6), we find

$$\boldsymbol{\lambda}^{\mathbf{k}} = \mathbf{2}\,\boldsymbol{\rho}\frac{C_y}{C_x}. \quad (2.7)$$

It follows that $\mathbf{MSE}(\bar{\mathbf{y}}_{\mathbf{Re}}^{(\mathbf{k})})$, under the condition (2.7), will overlap with the approximate variance of the usual linear regression estimator, say $\bar{\mathbf{y}}_{\mathbf{lr}}$, given by

$$\mathbf{MSE}(\bar{\mathbf{y}}_{\mathbf{lr}})=\frac{\lambda}{n}\bar{\mathbf{Y}}^2\mathbf{C}_{\mathbf{y}}^2(\mathbf{1}-\boldsymbol{\rho}^2). \quad (2.8)$$

There will be a natural problem in case the optimal value of $\mathbf{k}$ is found to be a non-integer and,as such, it may have to be replaced by the nearest integer. Also, if $\boldsymbol{\rho}\frac{C_y}{C_x}$ exceeds $1/2$,then it is apparent that a suitable value of $\mathbf{k}$ cannot be found from (2.7).In general, $\bar{\mathbf{y}}_{\mathbf{Re}}^{(\mathbf{k})}$ will be more efficient than $\bar{\mathbf{y}}_{\mathbf{Re}}$ if

$$\boldsymbol{\rho}\frac{C_y}{C_x}<\frac{1}{4}(\mathbf{1}+\boldsymbol{\lambda}^{\mathbf{k}}) \quad (2.9)$$

And, furthermore, $\bar{\mathbf{y}}_{\mathbf{Re}}^{(\mathbf{k})}$ fares better than $\bar{\mathbf{y}}$ if

$$\boldsymbol{\rho}\frac{C_y}{C_x}>\frac{1}{4}\boldsymbol{\lambda}^{\mathbf{k}} \quad (2.10)$$

Thus, combining conditions (2.9) and (2.10), we find that $\bar{\mathbf{y}}_{\mathbf{Re}}^{(\mathbf{k})}$ performs better than both $\bar{\mathbf{y}}_{\mathbf{Re}}$ and $\bar{\mathbf{y}}$ if

$$\frac{1}{4}\boldsymbol{\lambda}^{\mathbf{k}}<\rho\frac{C_y}{C_x}<\frac{1}{4}(\mathbf{1}+\boldsymbol{\lambda}^{\mathbf{k}}). \quad (2.11)$$

The bounds on $\boldsymbol{\rho}\, \mathbf{C_y/C_x}$ given in (2.11) will be called the efficiency bounds. Because of its role, $\boldsymbol{\rho}\, \mathbf{C_y/C_x}$ may be looked upon as some sort of a pivotal quantity.By choosing values of the sampling fraction $\mathbf{f(=\frac{n}{N})}$ and, hence, $\boldsymbol{\lambda} = (\mathbf{1} - \mathbf{f})$,we have prepared Table-1, which presents the bounds on $\boldsymbol{\rho}(\mathbf{C_y/C_x})$ for which $\bar{\mathbf{y}}_{\mathbf{Re}}^{(\mathbf{k})}$ (for various values of k) will be more efficient than $\bar{\mathbf{y}}_{\mathbf{Re}}$ and $\bar{\mathbf{y}}$.

Table-1: Efficiency bounds of $\boldsymbol{\rho}\, \mathbf{C_y/C_x}$ for various values of f and k

	k					
f	**1**	**2**	**5**	**8**	**10**	**50**
0.05	**(0.237,0.487)**	**(0.226,0.476)**	**(0.193,0.443)**	**(0.166,0.416)**	**(0.150,0.340)**	**(0.019,0.269)**
0.10	**(0.225,0.475)**	**(0.202,0.452)**	**(0.148,0.398)**	**(0.108,0.358)**	**(0.087,0.337)**	**(0.001,0.251)**
0.20	**(0.200,0.450)**	**(0.160,0.410)**	**(0.082,0.332)**	**(0.042,0.292)**	**(0.027,0.277)**	**(0,0.250)**
0.50	**(0.125,0.375)**	**(0.062,0.312)**	**(0.008,0.258)**	**(0,0.250)**	**(0,0.250)**	**(0,0.250)**
0.80	**(0.050,0.262)**	**(0.010,0.260)**	**(0,0.250)**	**(0,0.250)**	**(0,0.250)**	**(0,0.250)**
0.90	**(0.025,0.275)**	**(0.002,0.253)**	**(0,0.250)**	**(0,0.250)**	**(0,0.250)**	**(0,0.250)**

Table-1, in general, is of great help in locating a suitable value of $\mathbf{k}$ for given values of$\boldsymbol{\rho}(\mathbf{C_y/C_x})$ and $\mathbf{f}$. As regards knowledge of $\boldsymbol{\rho}(\mathbf{C_y/C_x})$, it can besaid that the quantities $\boldsymbol{\rho}$, $\mathbf{C_x}$ and $\mathbf{C_y}$ can be based on pilot survey, or can be available from a past survey, if any, because they remain stable over a considerable period of time.For a specified value of $\boldsymbol{\rho}(\mathbf{C_y/C_x})$, Table-1 offers more than one value of $\mathbf{k}$ which ensures better performance of $\bar{\mathbf{y}}_{\mathbf{Re}}^{(\mathbf{k})}$ relative to $\bar{\mathbf{y}}_{\mathbf{Re}}$ and$\bar{\mathbf{y}}$. However, the optimal value of$\mathbf{k}$ is obtainable from equation (2.7), provided that $\boldsymbol{\rho}(\mathbf{C_y/C_x}) < \mathbf{1/2}$. Even if the exact optimal value of $\mathbf{k}$ is not available, a suitable value of k that renders $\bar{\mathbf{y}}_{\mathbf{Re}}^{(\mathbf{k})}$ superior to $\bar{\mathbf{y}}_{\mathbf{Re}}$ and $\bar{\mathbf{y}}$ might still be found as evidenced by Table-1.Furthermore, with a view to finding the gainsin efficiency of $\bar{\mathbf{y}}_{\mathbf{Re}}$and $\bar{\mathbf{y}}_{\mathbf{Re}}^{(\mathbf{k})}$with respect to $\bar{\mathbf{y}}$, when$\mathbf{k}$is optimally determined, Table-2 presents for various values of $\mathbf{f}$, $\boldsymbol{\rho}$ and $\mathbf{C_y/C_x}$ the percentage gain $\mathbf{G_1}$ and $\mathbf{G_2}$ defined as

$$\mathbf{G_1} = \left[\frac{V(\bar{y})}{V(\bar{y}_{Re})} - \mathbf{1}\right] \times \mathbf{100}$$
$$\mathbf{G_2} = \left[\frac{V(\bar{y})}{V(\bar{y}_{Re}^{(k)})} - \mathbf{1}\right] \times \mathbf{100}.$$

Table-2.Optimum values of k, along with ($\mathbf{G_1}$and $\mathbf{G_2}$), for various combinations of f, ρ and $\mathbf{C_y/C_x}$

f	$\mathbf{C_y/C_x}$	**ρ**			
		0.25	**0.50**	**0.80**	**0.95**
0.10	**0.10**	**28 (-95.7, 6.4)**	**22 (-95.2, 33.3)**	**17 (-94.4, 177.8)**	**16 (-94.1, 900.0)**
	0.25	**20 (-75.0, 6.4)**	**13 (-66.7, 33.3)**	**9 (-44.4, 177.8)**	**7 (-16.7, 1011.1)**
	0.50	**13 (-33.3, 6.4)**	**6 (0, 33.3)**	**2 (150.0, 177.8)**	**0 (900, 900)**
	0.80	**9 (-7.4, 6.4)**	**2 (29.8, 33.3)**		
	1.00	**6 (0, 6.4)**	**0 (33.3, 33.3)**		
	2.00	**0 (6.4, 6.4)**			
0.25	**0.10**	**10 (-95.7, 6.4)**	**8 (-95.2, 33.3)**	**6 (-94.4, 170.3)**	**6 (-94.1, 900.0)**
	0.25	**7 (-75.0, 6.4)**	**5 (-66.7, 33.3)**	**3 (-44.4, 177.8)**	**2 (-16.7, 733.3)**
	0.50	**5 (-33.3, 6.4)**	**2 (0, 31.6)**	**1 (150.0, 177.8)**	**0 (900, 900)**
	0.80	**3 (-7.4, 6.4)**	**1 (29.8, 33.3)**	**0 (156.4, 156.4)**	
	1.00	**2 (0, 6.4)**	**0 (33.3, 33.3)**		
	2.00	**0 (6.4, 6.4)**			

Table-2 reflects that , for various configurations of$\mathbf{f}$, $\boldsymbol{\rho}$ and,$\mathbf{C_y/C_x}$ the estimator$\bar{\mathbf{y}}_{\mathbf{Re}}^{(\mathbf{k})}$ fares better than $\bar{\mathbf{y}}$, and the usual exponential ratio estimator $\bar{\mathbf{y}}_{\mathbf{Re}}$ could be considerably less efficient than $\bar{\mathbf{y}}$when the quantity $\boldsymbol{\rho}(\mathbf{C_y/C_x})$ is small, e.g., less than 0.125. It can also be noted from table-2 that, even for configurations of $\boldsymbol{\rho}$ and $\mathbf{C_y/C_x}$ for which $\bar{\mathbf{y}}_{\mathbf{Re}}$ is more efficient than $\bar{\mathbf{y}}$, i.e. when $\boldsymbol{\rho}(\mathbf{C_y/C_x}) > \frac{\mathbf{1}}{\mathbf{4}}$, $\bar{\mathbf{y}}_{\mathbf{Re}}^{(\mathbf{k})}$ is more efficient than both $\bar{\mathbf{y}}_{\mathbf{Re}}$ and $\bar{\mathbf{y}}$, provided that $\boldsymbol{\rho(C_y/C_x)\leq 1/2}$. It is also observed from Table 2 that if $\boldsymbol{\rho(C_y/C_x)}< 1/2$, then $\mathbf{G_2}$ increases with $\boldsymbol{\rho}$ for a given value of $\mathbf{C_y/C_x}$, and remains invariant to $\mathbf{C_y/C_x}$ for a given value of $\boldsymbol{\rho}$.

III. A system of exponential product-based estimators with performance

As stated in Section 1, when the auxiliary variable **x** is negatively correlated with the study variable **y**, the exponential product estimator $\bar{\mathbf{y}}_{\mathbf{Pe}}$ as defined in (1.2) is used to estimate the population mean $\bar{\mathbf{Y}}$.

Like $\bar{\mathbf{y}}_{\mathbf{Re}}$ in Section2, we proceed also for $\bar{\mathbf{y}}_{\mathbf{Pe}}$ so as to find

$$\bar{\mathbf{y}}_{\mathbf{Pe}}^{(\mathbf{k})} = (\mathbf{1}-\boldsymbol{\lambda}^{\mathbf{k}})\bar{\mathbf{y}} + \boldsymbol{\lambda}^{\mathbf{k}}\bar{\mathbf{y}}_{\mathbf{Pe}}. \quad (3.1)$$

Now,

$$\mathbf{B}(\bar{\mathbf{y}}_{\mathbf{Pe}}^{(\mathbf{k})}) = \boldsymbol{\lambda}^{\mathbf{k}}\left(\frac{1}{n}-\frac{1}{N}\right)\bar{\mathbf{Y}}\left(\frac{1}{2}\boldsymbol{\rho}^{*}\mathbf{C_y C_x}-\frac{1}{8}\mathbf{C_x^2}\right) \quad (3.2)$$

and

$$\mathbf{MSE}(\bar{\mathbf{y}}_{\mathbf{Pe}}^{(\mathbf{k})}) = \frac{\lambda}{n}\bar{\mathbf{Y}}^2\mathbf{C_y^2}\left[\mathbf{1}+\frac{\lambda^{2k}}{4}\left(\frac{C_x}{C_y}\right)^2+\boldsymbol{\lambda}^{\mathbf{k}}\boldsymbol{\rho}^{*}\left(\frac{C_x}{C_y}\right)\right], \quad (3.3)$$

where $\boldsymbol{\rho}^{*}$ is the correlation coefficient between the negatively correlated variables **x** and y and an optimum value of **k** that minimizes the MSE or variance given in(3.3) is obtained from

$$\boldsymbol{\lambda}^{\mathbf{k}} = -\mathbf{2}\boldsymbol{\rho}^{*}\frac{C_y}{C_x}, \quad (3.4)$$

which, when substituted in condition (3.3) will render it equal to the approximate variance of the linear regression estimator. It can easily be seen that $\bar{\mathbf{y}}_{\mathbf{Pe}}^{(\mathbf{k})}$ performs better than the customary exponential product estimator $\bar{\mathbf{y}}_{\mathbf{Pe}}$ if

$$\boldsymbol{\rho}^{*}\frac{C_y}{C_x} > -\frac{(1+\lambda^k)}{4}. \quad (3.5)$$

Also, $\bar{\mathbf{y}}_{\mathbf{Pe}}^{(\mathbf{k})}$ fares better than the usual simple mean $\bar{\mathbf{y}}$ if

$$\boldsymbol{\rho}^{*}\frac{C_y}{C_x} < -\frac{1}{4}\boldsymbol{\lambda}^{\mathbf{k}}. \quad (3.6)$$

Thus , $\bar{\mathbf{y}}_{\mathbf{Pe}}^{(\mathbf{k})}$ performs better than both $\bar{\mathbf{y}}_{\mathbf{Pe}}$ and $\bar{\mathbf{y}}$ if

$$-\frac{(1+\lambda^k)}{4} < \boldsymbol{\rho}^{*}\frac{C_y}{C_x} < -\frac{1}{4}\boldsymbol{\lambda}^{\mathbf{k}}. \quad (3.7)$$

IV. Numerical illustration:

The following examples reflect the potential gains from the use of the exponential ratio-based and exponential product-based estimators of order $\mathbf{k}$ in the place of the classical exponential ratio-based and exponential product-based estimators $\bar{\mathbf{y}}_{\mathbf{Re}}$ and $\bar{\mathbf{y}}_{\mathbf{Pe}}$ and the mean per unit estimator $\bar{\mathbf{y}}$.

IV (A). Exponential ratio-based estimator

Illustration 1. The data, which relate to the number of persons **(x)** and weekly expenditure on food **(y)** of 33 low-income families, have been taken from Cochran ([6], p.33). We have however, treated these data as the population data and computed the following quantities: $\bar{\mathbf{X}} = \mathbf{3.75}, \bar{\mathbf{Y}} = \mathbf{27.49}, \mathbf{S_x} = \mathbf{1.50}, \mathbf{S_y} = \mathbf{10.10}, \mathbf{C_x} = \mathbf{0.41}, \mathbf{C_y} = \mathbf{0.37}.$ Here, the correlation coefficient between **x** and **y**, $\boldsymbol{\rho}$, becomes $\mathbf{0.40}$. Based on these quantities gains in efficiencies ($\mathbf{G_1}$ and $\mathbf{G_2}$) for various sample sizes are computed below.

Case 1. Consider weekly expenditure as the main variate (**y**) and the number of persons as the auxiliary variate (**x**). Then, for $\mathbf{n = 5}$, the optimum value of $\mathbf{k}$ (rounded off to the nearest integer) is found to be $\mathbf{2}$; hence, $\mathbf{G_1 = 14.92}$ **and** $\mathbf{G_2 = 19.05}$.

Case 2. For $\mathbf{n = 10}$, the optimum value of $\mathbf{k}$ (rounded off to the nearest integer) is found to be $\mathbf{1}$; hence, $\mathbf{G_1 = 14.99}$ and $\mathbf{G_2 = 19.17}$.

Illustration 2 : We consider the following information given in Murthy ([7], p.228):

x: fixed capital, **y**: output, $\mathbf{N = 80, n = 10}, \bar{\mathbf{Y}} = \mathbf{5182.64}, \mathbf{C_x = 0.7507}, \mathbf{C_y = 0.3542}, \boldsymbol{\rho} = \mathbf{0.9413}.$

For $\mathbf{n = 10}$, the optimum value of $\mathbf{k}$ (rounded off to the nearest integer) is found to be $\mathbf{1}$; hence, $\mathbf{G_1 = 669.23}$ **and** $\mathbf{G_2 = 733.34}$.

It is clearly demonstrated through these illustrations that there are populations for which $\bar{\mathbf{y}}_{\mathbf{Re}}^{(\mathbf{k})}$ performs better than $\bar{\mathbf{y}}$ and $\bar{\mathbf{y}}_{\mathbf{Re}}$.

IV(B). Exponential product-based estimator

Illustration 1. The data relating to the automobile accidents rate in accidents per million vehicle miles (y) and length of the segment in miles (x) have been taken from Weisberg([8],p.179) . The data given here, however, are assumed to relate to the population and the following quantities are computed: $\mathbf{N = 39}, \bar{\mathbf{X}} = \mathbf{12.88}, \bar{\mathbf{Y}} = \mathbf{3.93}, \mathbf{S_x = 7.61}, \mathbf{S_y = 1.99}, \mathbf{C_x = 0.59}, \mathbf{C_y = 0.51}$ **and** $\boldsymbol{\rho}^{*} = -\mathbf{0.47}$. Invoking the formulae for gains in efficiency of $\bar{\mathbf{y}}_{\mathbf{Pe}}$ and $\bar{\mathbf{y}}_{\mathbf{Pe}}^{(\mathbf{k})}$ with respect to $\bar{\mathbf{y}}$ such as

$$\mathbf{G_1^*} = \left[\frac{V(\bar{y})}{V(\bar{y}_{Pe})} - \mathbf{1}\right] \times \mathbf{100}$$

$$\mathbf{G_2^*}=\left[\frac{\mathbf{V(\bar{y})}}{\mathbf{V(\bar{y}_{Pe}^{(k)})}}-\mathbf{1}\right]\times\mathbf{100}$$

and considering sample size $\mathbf{10}$, the optimum value of $\mathbf{k}$ (rounded off to the nearest integer) is found to be $\mathbf{1}$ and consequently$\mathbf{G_1^*=26.62}$ and $\mathbf{G_2^*=28.25}$. Thus, $\mathbf{\bar{y}_{Pe}^{(k)}}$ is more efficient than both $\mathbf{\bar{y}_{Pe}}$ and $\mathbf{\bar{y}}$.

Illustration 2. We refer to Steel and Torrie ([9], p.282), wherein the following information is considered:

$\mathbf{x}$: chlorine percentage,$\mathbf{y}$: log of leaf burn in sacs, $\mathbf{N=30, n=4, \bar{Y}=0.6860, C_x=0.7493, C_y=0.4803, \rho=-0.4996.}$

For $\mathbf{n=4}$,the optimum value of $\mathbf{k}$ (rounded off to the nearest integer) is found to be $\mathbf{3}$; hence,$\mathbf{G_1^*=20.51}$ and $\mathbf{G_2^*=35.05}$. The above illustrations clearly establish the fact that, $\mathbf{\bar{y}_{Pe}^{(k)}}$(with an optimum or near optimum value of $\mathbf{k}$) is more efficient than both $\mathbf{\bar{y}}$and$\mathbf{\bar{y}_{Pe}}$.

References

[1]. M.C. Agrawal and A.B. Sthapit: "Hierarchic predictive ratio-based and product-based estimators and their efficiency", Journal of Applied Statistics, Vol. 24, No. 1, 1997, 97-104.

[2]. M. N. Murthy. Product method of estimation, Sankhya, A26, 1964, pp.69 -74.

[3]. S. Bahl and R.K. Tuteja: Ratio and product type exponential estimator, Information and Optimization Science, 12, 1991, 159-163 D.Basu . An essay on the logical foundations of survey sampling. In: V. P. Godambe & D. A.Sprott (Eds), Foundations of Statistical Inference, 1971, pp. 203 -233 (Toronto, Holt, Rinehart and Winston).

[4]. T. M. F. Smith .The foundations of survey sampling: a review, Journal of the Royal Statistical Society, Series A , 139, 1976, pp. 183 -204.

[5]. W. G. Cochran. Sampling techniques (New York, John Wiley), 1977.

[6]. M. N. Murthy. Sampling theory and methods. Calcutta, India: Statistical Publishing Society.1967.

[7]. S. Weisberg . Applied linear regression (New York, John Wiley). 1980.

[8]. R.G.D. Steel, and J..H. Torrie. Principles and Procedures of Statistics. New York: Mc-Graw Hills, 1960.