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# Unsteady Couette Flow of A Generalized Second Grade Fluid through a Porous Medium with Uniform Injection And Suction

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***Abstract :*** *A study has been made on Couette flow of a generalized second grade fluid through a porous medium with suction. The approximate solution of the governing equation has been obtained by Variational Iteration Method. The effects of fractional calculus parameter $\beta$, porosity parameter $\sigma$ and the parameter $\eta$ on primary and secondary velocities have been illustrated graphically. It is found that both the primary and the secondary velocities decrease against the distance from the lower plate with the increase in the value of the fractional calculus parameter $\beta$.*
***Keywords:*** *Couette flow, porous medium, variational iteration method, Caputo operator.*

## I. Introduction

The interest of non-Newtonian fluid flow between two infinite parallel plates has been considerably grown because of its wide range of application in engineering fields e.g. petroleum industry, polymer technology etc. During the last few decades fractional calculus approach has been successfully applied in describing the viscoelastic behavior of the non-Newtonian fluids. In the constitutive equation of the stress field of the non-Newtonian fluid, the integer order time derivative has been replaced by Riemann-Liouville fractional calculus operator $D_t^{\beta}$. Many researchers have investigated the flows of non-Newtonian fluids between two parallel plates through porous medium. An investigation on unsteady incompressible flow of generalized Oldroyd- B fluid between two parallel plates have made by Bose and Basu [1]. In their work they have used the constitutive equation with fractional order time derivative representing the stress field related to the motion. He in [2-4] established variational Iteration Method (VIM) in (1999) that has been used by many researchers to handle linear and non-linear model conveniently. A study on incompressible flow of generalized Oldroyd-B fluid in a rotating system has been made by Bose and Basu [5]. Safari [6] in his paper has explained the Variational Iteration Method in finding the analytical solution of space fractional diffusion equation. Tan, Xian and Wei [7] investigated unsteady Couette flow of generalized second grade fluid. Fetecau, Hayat, Fetecau and Ali [8] have studied unsteady flow of a second grade fluid between two side walls perpendicular to a plate.

In present paper an investigation on unsteady Couette flow of generalized second grade fluid through a porous medium with suction has been made. The expression for the velocity field for the aforesaid flow has been obtained by Variational Iteration Method. The effects of fractional calculus parameter $\beta$, porosity parameter $\sigma$ and parameter $\eta$ have been illustrated graphically.

## II. Mathematical Analysis Of The Problem

We consider unsteady incompressible flow of generalized second grade fluid between two infinite parallel porous plates embedded in a porous medium. The upper and lower plates are located at $y = 0$ and $y = h$ respectively and extended from $x = -\infty$ to $x = +\infty$ and $z = -\infty$ to $z = +\infty$. The plates together with the fluid were initially at rest. At $t > 0$ the lower plate begins to move with uniform velocity U but the upper plate remains fixed. We choose a Cartesian co-ordinate system with x-axis along the lower plate in the direction of the flow, y-axis normal to the plates and the z-axis perpendicular to xy-plane. The flow is generated due to the motion of the lower plate moving with uniform velocity U parallel to itself in the direction of the flow. Since the plates are infinite in length in the x-and z-directions, all the physical quantities are functions of y and t only. The velocity vector of the fluid is $V(y,t) = u(y,t)\hat{\imath} + v(y,t)\hat{\jmath}$
The continuity equation gives

$$\frac{\partial v}{\partial y} = 0 \qquad (2.1)$$

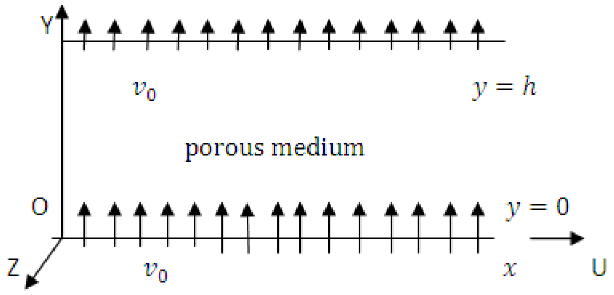


**Figure1. Sketch of the problem**

which on integration yields $v = \text{constant} = v_0$ where $v_0 > 0$ for suction and $v_0 < 0$ for the blowing at the plate.
Using the form of the velocity field as assumed earlier the governing equation can be written as

$$\frac{\partial u}{\partial t} + \frac{v_0}{\rho}\frac{\partial u}{\partial y} = \left(\upsilon + \alpha^\beta D_t^\beta\right)\frac{\partial^2 u(y,t)}{\partial y^2} - \frac{\upsilon u}{K} \tag{2.2}$$

$\upsilon = \frac{\mu}{\rho}$ is the kinematic viscosity, $\rho$ is the fluid density, $\alpha^\beta = \frac{\alpha_1^\beta}{\rho}$, $K$ is the permeability of porous medium.
The initial and boundary conditions can be written as

$$u = 0 \text{ for } 0 \le y \le h, t < 0 \tag{2.3}$$

$$u = U, v = v_0 \text{ at } y = 0, t > 0 \tag{2.4}$$

$$u = 0, v = v_0 \text{ at y = h for t > 0} \tag{2.5}$$

Introducing non-dimensional variables $u^* = \frac{u}{U}, v^* = \frac{v}{v_0}, y^* = \frac{y}{h}, t^* = \frac{\upsilon t}{h^2}$ and dropping the mark * for convenience the governing equation can be rewritten as

$$\frac{\partial u}{\partial t} = -R_e\frac{\partial u}{\partial y} + \left(1 + \eta^\beta D_t^\beta\right)\frac{\partial^2 u}{\partial y^2} - \frac{u}{\sigma^2} \tag{2.6}$$

Where $R_e = \frac{v_0}{\rho\upsilon}$ is Renolds number and $\eta^\beta = \frac{\alpha^\beta}{\upsilon}, \sigma^2 = \frac{K}{h^2}$
In non-dimensional variables the initial and boundary conditions are of the following forms

$$u = 0 \text{ for } 0 \le y \le 1, t \le 0 \tag{2.7}$$

$$u = 1, v = 1 \text{ at } y = 0, t > 0 \tag{2.8a}$$

$$u = 0, v = 1 \text{ at y = 1 for t > 0} \tag{2.8b}$$

## III. Fundamental Idea Of He's Variational Iteration Method

We consider the differential equation $Lu(y,t) + Nu(y,t) = f(y,t)$ (3.1)
We consider the differential equation Where $L$ is the linear operator, $N$ is the non-linear operator, $f(y,t)$ is the source in-homogeneous term.

According to VIM we can write down a correctional functional as follows

$$u_{n+1}(y,t) = u_n(y,t) + \int_0^t \lambda(Lu_n(\xi) + N\widetilde{u_n}(\xi) - f(\xi))d\xi, n \ge 0 \tag{3.2}$$

Where $\lambda$ is the general Lagrangian Multiplier which can identified optimally via variational theory. The second term on the right hand side is called the correction and $\widetilde{u_n}$ is considered as a restricted variation i.e. $\delta\widetilde{u_n} = 0$.
So, we first determine the Lagrangian multiplier that will be identified optimally via integration by parts. The successive approximations of the solution will be obtained using the obtained Lagrangian multiplier and using initial solution $u_0$. Consequently the solution is

$$u(y,t) = \lim_{n\to\infty} u_n(y,t) \tag{3.3}$$

## IV. Finding The Solution Of The Problem By Variational Iteration Method

Now we try to solve the governing equation (2.6) by VIM subject to the initial and boundary conditions given by equations (2.7) and (2.8) respectively.

We first construct the correctional functional in y and t for $u(y,t)$ as

$$u_{n+1}(y,t) = u_n(y,t) - \int_0^t \left(\frac{\partial u(y,\tau)}{\partial \tau} + R_e\frac{\partial u}{\partial y} - \frac{\partial^2 u(y,\tau)}{\partial y^2} - \eta^\beta\frac{\partial^{2+\beta} u(y,\tau)}{\partial \tau^\beta \partial y^2} + \frac{u(y,\tau)}{\sigma^2}\right)d\tau \tag{4.1}$$

Here we have taken the Lagrangian multiplier $\lambda = -1$

We start with the initial approximation $u(x,0) = u_0 = (1-y)^2$ (4.2)

Taking $\beta = 0.2$ and using the iteration formula (3.2) we get the successive approximations as

$$u_0(y,t) = (1-y)^2 \tag{4.3}$$

$$u_1(y,t) = \left(\frac{1}{\exp\left(\frac{t}{\sigma^2}\right)} - 1\right)(y-1)^2 + (y-1)^2 - 2\sigma^2\left(\frac{1}{\exp\left(\frac{t}{\sigma^2}\right)} - 1\right) - 2\eta^{0.2}\sigma^2\left(\frac{1}{\exp\left(\frac{t}{\sigma^2}\right)} - 1\right) + 2R_e\sigma^2\left(\frac{1}{\exp\left(\frac{t}{\sigma^2}\right)} - 1\right)(y-1) \tag{4.4}$$

$$\begin{aligned} u_2(y,t) = {} & \left(\frac{1}{\exp\left(\frac{t}{\sigma^2}\right)} - 1\right)(y-1)^2 + (y-1)^2 - 2\sigma^2\left(\frac{1}{\exp\left(\frac{t}{\sigma^2}\right)} - 1\right) - 2\eta^{0.2}\sigma^2\left(\frac{1}{\exp\left(\frac{t}{\sigma^2}\right)} - 1\right) \\ & - \left(\left(\exp\left(\frac{t}{\sigma^2}\right) - 1\right)\left(2\sigma^2\exp\left(\frac{t}{\sigma^2}\right) + 2\eta^{0.2}\sigma^2\exp\left(\frac{t}{\sigma^2}\right) + 2R_e\sigma^2\exp\left(\frac{t}{\sigma^2}\right) - 2R_e\sigma^2 y\exp\left(\frac{t}{\sigma^2}\right)\right)\right)\frac{1}{exp\left(\frac{2t}{\sigma^2}\right)} \\ & + \left(2\sigma^2\left(exp\left(\frac{t}{\sigma^2}\right) - 1\right)\right)\frac{1}{exp\left(\frac{2t}{\sigma^2}\right)} + \left(2\eta^{0.2}\sigma^2\left(exp\left(\frac{t}{\sigma^2}\right) - 1\right)\right)\frac{1}{exp\left(\frac{2t}{\sigma^2}\right)} + 2R_e\sigma^2\left(\frac{1}{\exp\left(\frac{t}{\sigma^2}\right)} - 1\right)(y-1) \\ & - \left(2R_e\sigma^2\left(\exp\left(\frac{t}{\sigma^2}\right) - 1\right)\left(y + R_e\sigma^2 - R_e\sigma^2 exp\left(\frac{t}{\sigma^2}\right) - 1\right)\right)\frac{1}{exp\left(\frac{2t}{\sigma^2}\right)} \end{aligned} \tag{4.5}$$

Substituting the expression of $u_2(y,t)$ for $u$ from the Equation (4.5) into the stress equation

$$\tau_{xy} = \left(\mu + \alpha_1^{\beta} D_t^{\beta}\right)\frac{\partial u}{\partial y} \tag{4.6}$$

and putting $y = 0$ we obtain the stress field at the moving plate as follows

$$\tau_{xy} = -2\mu\exp\left(-\frac{t}{\sigma^2}\right) - 2R_e\sigma^2\alpha^{\beta}\frac{t^{-\beta}}{\Gamma(1-\beta)} - 2\alpha^{\beta}\left(-\frac{1}{\sigma^2}\right)^{\beta}\exp\left(-\frac{t}{\sigma^2}\right) \tag{4.7}$$

## V. Conclusion And Numerical Results

The unsteady Couette flow of a generalized second grade fluid through a porous medium has been investigated. The approximate solution for the velocity field is obtained by Variational Iteration Method (VIM). The effect of the fractional calculus parameter $\beta$, porosity parameter $\sigma$ and $\eta$ on the velocity field has been discussed graphically.The unsteady Couette flow of a generalized second grade fluid through a porous medium with uniform suction and injection has been investigated. **Fig 2** shows that the velocity field $u$ of fluid decreases with time t for higher values of fractional calculus parameter $\beta$ near the moving plate. In **Fig 3** it can be seen that as $\beta$ increases the velocity field decreases against y, the distance from the lower plate. The velocity field $u$ increases for increasing values of the porosity parameter $\sigma$ for small values of time t in **Fig 4** near the moving plate. **Fig 5** shows that the velocity field $u$ increases with time t for higher values of the parameter $\eta$. In **Fig 6** the velocity $u$ increases for higher values of parameter $\eta$ with the distance y from the lower moving plate. From **Fig7** it is evident that the velocity field decreases with the increase in the Reynolds number $R_e$. From **Fig 8** it is evident that as Reynolds number $R_e$ takes higher values the velocity field decreases against the time t. **Fig 9** shows the stress fields against time t for different values of the parameter β.

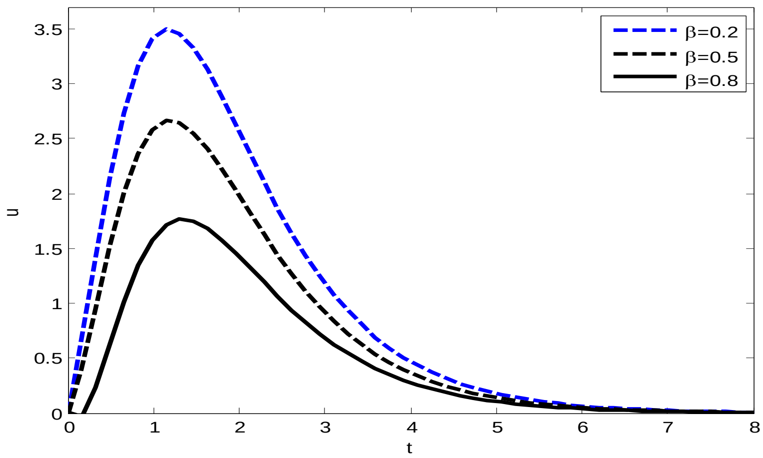


**Fig 2. The velocity field u is depicted against t for different values of β.** $\boldsymbol{\eta = 1.0,\ \sigma = 1.0,\ y = 0.3, Re = 2.0}$

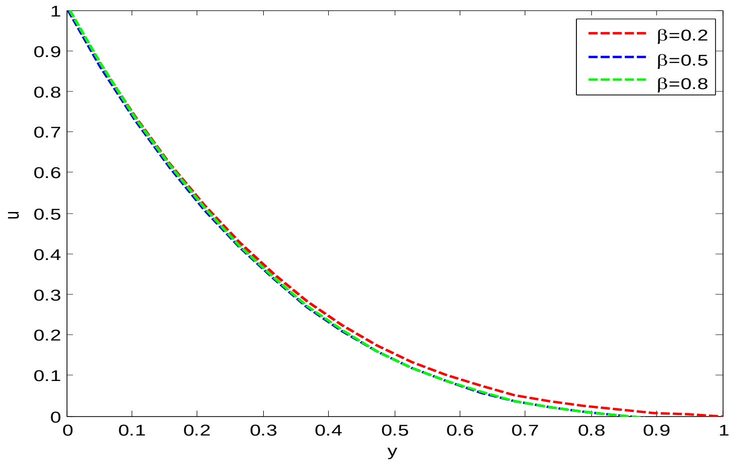


**Fig 3. The velocity field u is depicted against the distance y from the lower plate.** $\boldsymbol{\eta = 1.0,\ \sigma = 1.0,\ t = 0.1,\ Re = 2.}$

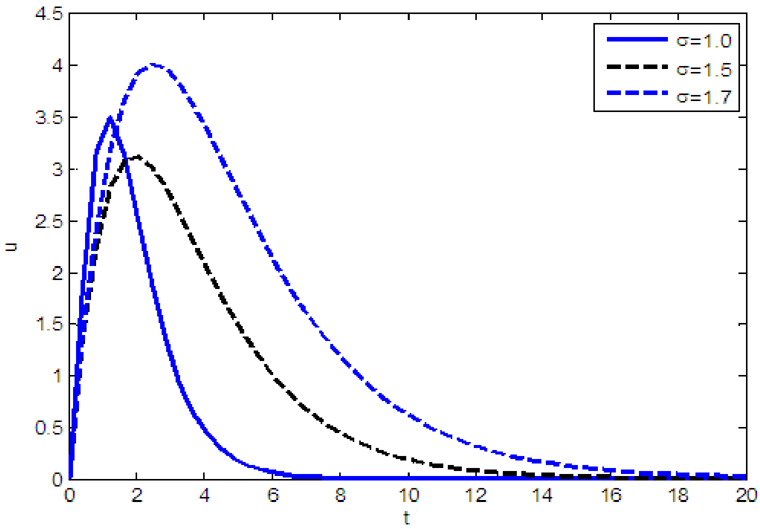


**Fig 4.** The velocity field u is depicted against time for different value of $\sigma$.
$\boldsymbol{\eta = 1,\ \beta = 0.2,\ y = 0.3,\ Re = 2.0}$

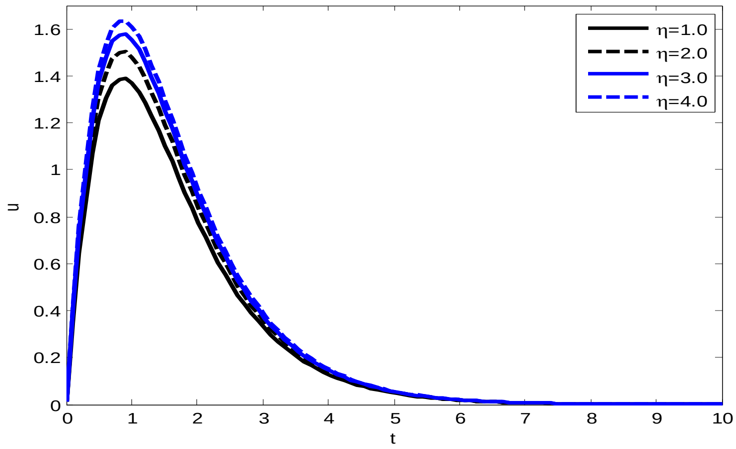


**Fig 5. The velocity field u is depicted against time t for different values of** $\boldsymbol{\eta}$**.**
$\boldsymbol{\sigma = 1.0,\ \beta = 0.2,\ y = 0.2, Re = 2.0}$

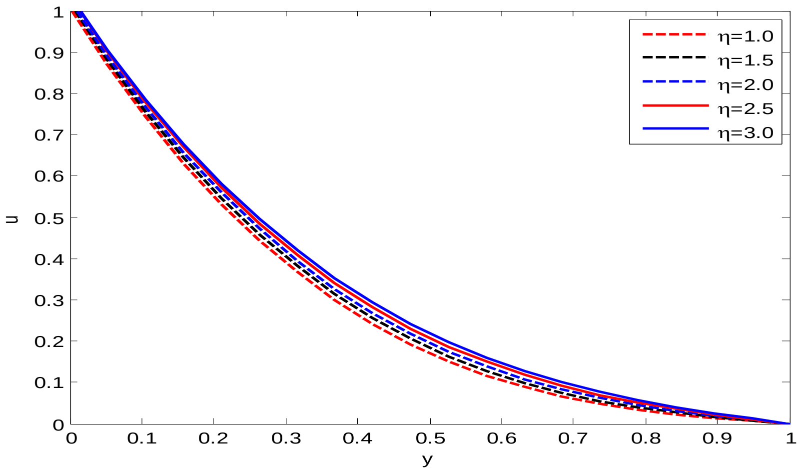


**Fig 6.** The velocity field u is depicted against the distance from the lower plate for different values of $\eta$.
$\boldsymbol{\beta = 0.2,\ \sigma = 1.0,\ t = 0.1, Re = 2.0}$

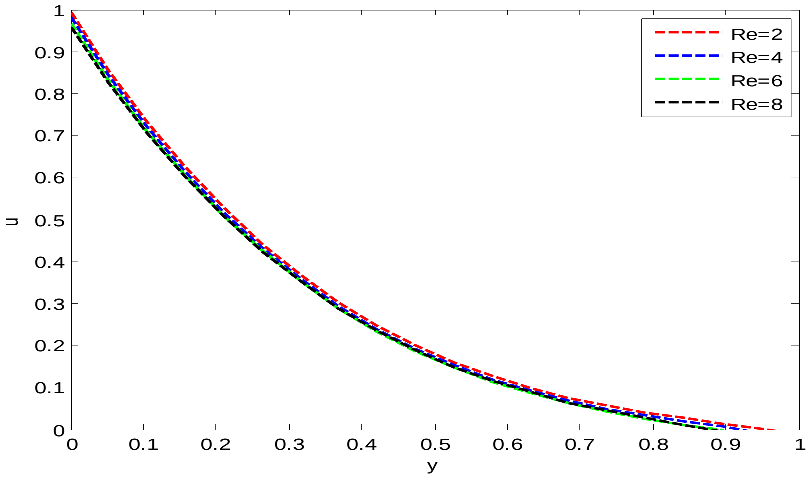


**Fig 7. The velocity field u is depicted against y for different values of the Reynolds number Re.**
$\boldsymbol{\eta = 1.0,\quad \sigma = 1.0,\ t = 0.1,\quad \beta = 0.2}$

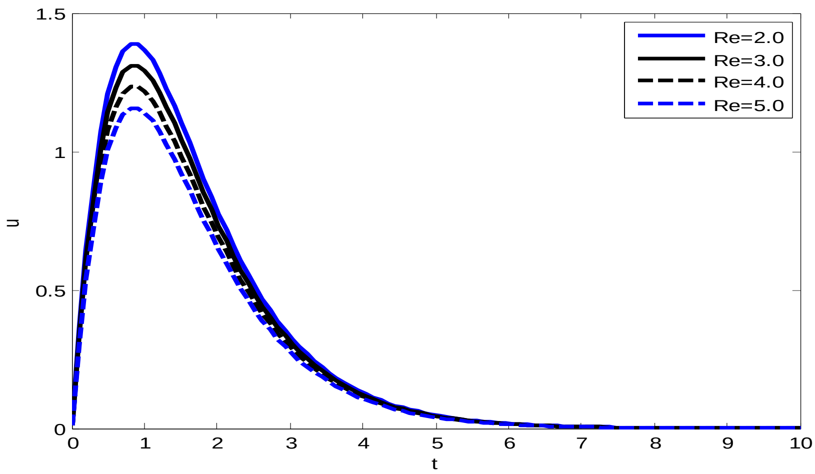


**Fig 8. The velocity field u is depicted against t for different values of the Reynolds number Re.**
$\boldsymbol{\eta = 1.0,\ \sigma = 1.0,\ t = 0.1, \beta = 0.2}$

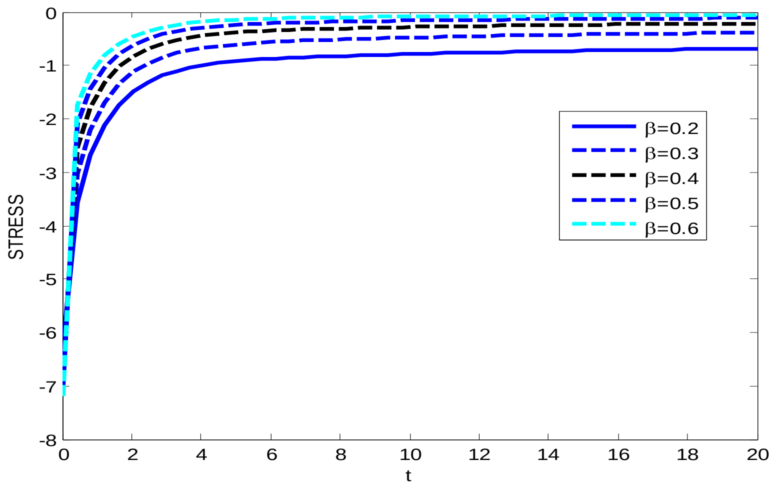


**Fig 9. The stress field is depicted against time t for different values of the parameter β.** $\eta = 1.0,\ \sigma = 1.0,\ Re = 2.0$

**Table 1. Shear stress $\tau_{xy}$ due to flow field when $\sigma = 1.0, \mu = 1.0, R_e = 1.0, \alpha = 0.2$**

| | General Solution | | | Solution for small times | | |
|---|---|---|---|---|---|---|
| **β\t** | **1** | **2** | **3** | **0.01** | **0.02** | **0.03** |
| **0.2** | -7.3935 | -6.2153 | -5.4616 | -15.6008 | -13.9737 | -13.1177 |
| **0.3** | -5.9945 | -4.7957 | -4.0644 | -17.9465 | -15.0971 | -13.6817 |
| **0.4** | -4.8495 | -3.7177 | -3.0482 | -20.4023 | -16.0844 | -14.0576 |
| **0.5** | -3.9244 | -2.9116 | -2.3214 | -22.6262 | -16.7080 | -14.0828 |

**Table 2. Shear stress $\tau_{xy}$ due to flow field when $\sigma = 1.0, \mu = 1.0, , \alpha = 0.2, \beta = 0.5$**

| | General Solution | | | Solution for small times | | |
|---|---|---|---|---|---|---|
| **$R_e$\t** | **1** | **2** | **3** | **0.01** | **0.02** | **0.03** |
| **2.0** | -5.9429 | -4.3389 | -3.4868 | -42.8112 | -30.9810 | -25.7366 |
| **3.0** | -7.9614 | -5.7662 | -4.6521 | -62.9963 | -45.2540 | -37.3905 |
| **4.0** | -9.9799 | -7.1935 | -5.8175 | -83.1813 | -59.5270 | -49.0443 |
| **5.0** | -11.9984 | -8.6208 | -6.9829 | -103.3664 | -73.8000 | -60.6982 |

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