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Orthogonal Generalized (σ,τ) Derivations in Semiprime Semiring

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Abstract: *Motivated by some results on orthogonal (σ,τ) derivations in semiprime gamma rings, in [6], the authors defined the notion of (σ,τ) derivations and generalized (σ,τ) derivations in semiprime gamma rings. In this paper, we also introduce the notion of orthogonal generalized (σ,τ) derivations in semiprime semiring and derived some interesting results.*
keywords: *Semirings, (σ,τ) derivation, generalized(σ,τ) derivation, orthogonal generalized (σ,τ) derivation*

I. Introduction

This paper has been inspired by the work of Shakir Ali and Mohammad Salahuddin Khan [6]. Ashraf and Jamal, in [2], introduced the notion of orthogonality for two derivations on gamma rings, and established several necessary and sufficient conditions for derivations d and g to be orthogonal. Further in [3], they introduced orthogonal generalized derivation in gamma rings and obtained some results concerning orthogonal generalized derivations. In this paper, we introduce the notion of orthogonality of two generalized (σ,τ) derivations on semiprime semiring and we presented some interesting results..

II. Preliminaries

Definition: 2.1

A **semiring** (S,+,•) is a non-empty set S together with two binary operations, + and•such that(1). (S,+) is a commutative monoid with identity element 0

(2). (S,•) is a monoid with identity element 1

(3). For all a, b, c ϵ S, a . (b + c) = a.b + a . c and (b + c) . a = b. a + c . a

Definition: 2.2

A semiring S is said to be **2- torsionfree** if 2x = 0 ⇒ x = 0, ∀x ∈ S.

Definition: 2.3

A semiring S is **prime** if xSy = 0 ⇒ x = 0 or y = 0, ∀x, y∈S and S is **semiprime**if xSx = 0⇒ x = 0,∀ x∈ S.

Definition: 2.4

An additive map d : S → S is called a **derivation** if d(xy) = d(x) y + x d(y),∀ x, y ∈ S

Definition: 2.5

Let d, g be two additive maps from S to S. They are said to be **orthogonal** if
d(x)Sg(y) = 0 = g(y)Sd(x), ∀ x, y ∈ S.

Lemma: 2.6

Let S be a 2-torsion free semiprime semiring and a,b ∈ S. Then the following conditions are equivalent

(i) axb = 0, ∀ x ∈ S
(ii) bxa = 0, ∀ x ∈ S
(iii) axb + bxa = 0, ∀ x ∈ S. If one of the conditions is fulfilled, then ab = ba = 0.

Lemma: 2.7

Let S be a semiprime semiring and suppose that additive mappings d and g of S into itself satisfy d(x) S g(x) = 0, ∀ x ∈ S. Then d(x) S g(y) = 0, ∀ x ∈ S.

Theorem: 2.8

Let S be a 2-torsion free semiprime semiring. Let d, g be (σ,τ)derivations of S. Then d(x) g(y) + g(x) d(y) = 0, ∀ x, y ∈ S iff d and g are orthogonal.

Theorem: 2.9

Let S be a 2-torsion free semiprime semiring. Suppose d, g be (σ, τ) derivations of S such that dσ = $\sigma\, d$, dτ = $\tau\, d$, gσ = $\sigma\, g$, gτ = $\tau\, g$. Then the following conditions are equivalent.

(i) d and g are orthogonal
(ii) dg = 0
(iii) gd = 0
(iv) dg + gd = 0
(v) dg is a (σ^2, τ^2)derivation of S

III. Orthogonal Generalized $(\boldsymbol{\sigma}, \boldsymbol{\tau})$ Derivation

Definition: 3.1

Let σ and τ be automorphisms of S. An additive mapping d : S → S is called a $(\boldsymbol{\sigma}, \boldsymbol{\tau})$ **derivation** if d(xy) = d(x) $\sigma(y)$ + $\tau(x)$ d(y) , $\forall\, x, y \in S$

Definition: 3.2

An additive mapping D : S → S is called **generalized** $(\boldsymbol{\sigma}, \boldsymbol{\tau})$ **derivation** if there exists a (σ, τ) derivation d of S such that D(xy) = D(x) $\sigma(y)$ + $\tau(x)$ d(y) , $\forall\, x, y \in S$

Note:

Every generalized derivation is a generalized (σ, τ) derivation with $\sigma = \tau = I_S$, the identity map on S, but the converse need not be true in general.

Definition: 3.3

Two generalized derivations (D,d) and (G,g) of S are called **orthogonal** if D(x) S G(y) = 0 = G(y) S D(x), $\forall\, x, y \in S$

Lemma: 3.4

Suppose that two generalized (σ, τ) derivations (D,d) and (G,g) of S are orthogonal. Then following relations hold

(i) D(x) G(y) = G(x) D(y) = 0, and hence D(x) G(y) + G(x) D(y) = 0 , $\forall\, x, y \in S$
(ii) d and G are orthogonal and d(x) G(y) = G(y) d(x) = 0, $\forall\, x, y \in S$
(iii) g and D are orthogonal and g(x) D(y) = D(y) g(x) = 0, $\forall\, x, y \in S$
(iv) d and g are orthogonal
(v) If Dσ = σ D, Dτ = τ D, Gσ = $\sigma\, G$, Gτ = $\tau\, G$ and dσ = $\sigma\, d$, dτ = $\tau\, d$, gσ = $\sigma\, g$, gτ = $\tau\, g$, then dG = Gd = 0, gD = Dg = 0 and DG = GD = 0

Proof:

(i) By the hypothesis, D(x) s G(y) = 0, $\forall\, x, y \in S$
By lemma 2.6, D(x) G(y) = 0 = G(y) D(x)
∴ D(x) G(y) + G(y) D(x) = 0, $\forall\, x, y \in S$

(ii) By (i), we have D(x) G(y) = 0, $\forall\, x, y \in S$
Replace x by zx, we get
0 = D(zx) G(y)
= [D(z) $\sigma(x)$ + $\tau(z)$ d(x)] G(y)
= D(z) $\sigma(x)$ G(y) + $\tau(z)$ d(x) G(y)
= $\tau(z)$ d(x) G(y) [∵ D and G are orthogonal

Since τ is an automorphism of S, d(x) G(y) S d(x) G(y) = 0, $\forall\, x, y \in S$
∴ d(x) G(y) = 0,$\forall\, x, y \in S$ [∵S is semiprime (1)
Replacing x by xz, d(xz) G(y) = 0 ⇒ d(x) $\sigma(z)$ G(y) + $\tau(x)$ d(z) G(y) = 0

∴d(x) S G(y) = 0, $\forall\, x, y \in S$ [∵by (1)
By lemma 2.6, G(y) S d(x) = 0, $\forall\, x, y \in S$
∴d and G are orthogonal and hence d(x) G(y) = G(y) d(x) = 0, $\forall\, x, y \in S$

(iii) Similar proof in (ii)

(iv) By the assumption, D(x) S G(y) = 0, $\forall\, x, y \in S$
⇒ D(x) s G(y) = 0, $\forall\, x, y \in S$. Replacing x by xz and y by yw,
D(xz) s G(yw) = 0 ⇒[D(x) $\sigma(z)$ + $\tau(x)$ d(z)] s [G(y) $\sigma(w)$ + $\tau(y)$ g(w)] = 0

⇒D(x) $\sigma(z)$ s G(y) $\sigma(w)$ + D(x) $\sigma(z)$ s $\tau(y)$ g(w) + $\tau(x)$ d(z) s G(y) $\sigma(w)$ + $\tau(x)$ d(z) s $\tau(y)$ g(w) = 0
Using (ii) and (iii), we get $\tau(x)$ d(z) s $\tau(y)$ g(w) = 0 , $\forall x,y,z,w,s \in S$
Since τ is an automorphism of S, d(z) s g(w) S d(z) s g(w) = 0, $\forall z,w,s \in S$
The semiprimeness of S, d(z) s g(w) = 0. Using lemma 2.6, g(w) s d(z) = 0
∴d and g are orthogonal

(v) In view of (ii) d and G are orthogonal.
Hence, d(x) s G(y) = 0, $\forall x,y,s \in S$. Now G [d(x) s G(y)] = 0
⇒ G d(x) $\sigma(s)$ $\sigma(G(y))$ + $\tau(d(x))$ g(s) $\sigma\big(G(y)\big)$ + $\tau(d(x))\tau(s)$ g(G(y)) = 0
Since d$\tau = \tau\, d$,G$\sigma = \sigma\, G$ and d and g are orthogonal, G d(x) s_1 G(y_1) = 0 for all x, y_1, s_1 in S. Replacing y_1 by d(x) and using the semiprimeness of S, Gd = 0
Similarly, since each of the equalities d(G(x)) z d(y) = 0, D(g(x)) z D(y) = 0, g(D(x)) z g(y) = 0, D(G(x)) z D(y) = 0 and G(D(x)) z G(y) = 0 hold ∀x,y,z $\in S$
We conclude that dG = Dg = gD = DG = GD = 0 respectively.

Corollary: 3.5
Let (D,d) and (G,g) be orthogonal generalized (σ,τ) derivations of S such that D$\sigma = \sigma$ D, D$\tau = \tau$ D, G$\sigma = \sigma\, G$, G$\tau = \tau\, G$ and d $\sigma = \sigma\, d$, d$\tau = \tau\, d$, g$\sigma = \sigma\, g$, g$\tau = \tau\, g$. Then dg is a (σ^2,τ^2)derivation of S and (DG,dg) = (0,0) is a generalized(σ^2,τ^2)derivations of S.

Theorem: 3.6
Suppose (D,d) and (G,g) are generalized (σ,τ) derivations such thatD$\sigma = \sigma$ D, D$\tau = \tau$ D, G$\sigma = \sigma\, G$, G$\tau = \tau\, G$ and d $\sigma = \sigma\, d$, d$\tau = \tau\, d$, g$\sigma = \sigma\, g$, g$\tau = \tau\, g$. Then(D,d) and (G,g) are orthogonal iff one of the following holds
(i) a) D(x)G(y) + G(x)D(y) = 0, $\forall x,y \in S$
b) d(x) G(y) + g(x) D(y) = 0, $\forall x,y \in S$
(ii) D(x)G(y) = d(x) G(y) = 0, $\forall x,y \in S$
(iii) D(x)G(y) = 0, $\forall x,y \in S$ and dG = dg = 0
(iv) (DG,dg) is a generalized(σ^2,τ^2)derivation and D(x) G(y) = 0, $\forall x,y \in S$

Proof:
Using the above lemma 3.4 and corollary 3.5 we get ,
(D,d) and (G,g) are orthogonal⇒ (i), (ii), (iii) and (iv) holds.
Now we prove that the converse parts of each one.
(i)⇒(D,d) and (G,g)are orthogonal.
Assume that, D(x) G(y) + G(x) D(y) = 0, $\forall x,y \in S$. Replacing x by xz
0 = D(xz) G(y) + G(xz) D(y)
= D(x) $\sigma(z)$ G(y) + $\tau(x)$ d(z) G(y) + G(x) $\sigma(z)$ D(y) + $\tau(x)$ g(z) D(y)
Using (b) we get, D(x) $\sigma(z)$ G(y)+ G(x) $\sigma(z)$ D(y) = 0,$\forall x,y,z \in S$
Since σ is an automorphism of S, the above relation can be rewritten as
D(x) z_1 G(y)+ G(x) z_1 D(y) = 0,$\forall x,y,z_1 \in S$
By lemma 2.6, D(x) z_1 G(y) = 0 and G(x) z_1 D(y) = 0,$\forall x,y,z_1 \in S$
∴D and G are orthogonal.
(ii)⇒(D,d) and (G,g)are orthogonal.
Assume that D(x) G(y) = 0, $\forall x,y \in S$. Replacing x by xz
0 = D(xz) G(y) = D(x) $\sigma(z)$ G(y) + $\tau(x)$ d(z) G(y) = D(x) $\sigma(z)$ G(y)
Using lemma 2.6 and σ is an automorphism of S, we get (D,d) and (G,g) are orthogonal.
(iii)⇒(D,d) and (G,g)are orthogonal.
By the assumption, 0 = dG(xy) = d [G(x) $\sigma(y)$ + $\tau(x)$ g(y)]
= dG(x) $\sigma^2(y)$+$\tau(G(x))$d($\sigma(y)$+d($\tau(x)\sigma(g(y))$+$\tau^2(x)$ dg(y)
= $\tau(G(x))$ d($\sigma(y)$ + d($\tau(x)\sigma(g(y))$
Since G$\tau = \tau\, G$, g$\sigma = \sigma\, g$ and σ,τ is an automorphisms of S, we have
$G(x_1)$ d(y_1) + d(x_1)$g(y_1)$ = 0,$\forall x_1,y_1 \in S$. Using theorem 2.8 and lemma 2.6, $G(x_1)$ d(y_1)= 0, $\forall x_1,y_1 \in S$.
Replacing x_1 by xz,
0 = G(xz) d(y_1)
= [G(x) $\sigma(z)$ + $\tau(x)$ g(z)] d(y_1)
= G(x) $\sigma(z)$ d(y_1) + $\tau(x)$ g(z) d(y_1)

= G(x) $\sigma(z)$ d(y_1) [sincetheorem 2.8
By lemma 2.6, d(y_1) G(x) = 0, $\forall\ x, y_1 \in S$, which satisfies (ii).
Therefore (iii) implies that (D,d) and (G,g) are orthogonal
(iv)⇒ (D,d) and (G,g) are orthogonal.
Since (DG,dg) is a generalized(σ^2,τ^2)derivation and dg is a(σ^2,τ^2)derivation
DG(xy) = DG(x) $\sigma^2(y)$+ $\tau^2(x)$ dg(y) , $\forall\ x, y \in S$ (1)

Also, DG(xy) = D [G(x) $\sigma(y)$ + $\tau(x)$ g(y)] = D (G(x) $\sigma(y)$) +D ($\tau(x)$ g(y))
= DG(x) $\sigma^2(y)$+$\tau(G(x))$d($\sigma(y)$+D ($(x)\sigma(g(y))$+ $\tau^2(x)$ dg(y) (2)

Comparing (1) and (2) we get,$\tau(G(x))$ d ($\sigma(y)$+D ($\tau(x)\sigma(g(y))$= 0, $\forall\ x, y \in S$
Since Gτ = $\tau\ G$, gσ = $\sigma\ g$ and σ,τ is an automorphisms of S, we have
$G(x_1)$ d(y_1) + D(x_1)$g(y_1)$ = 0,$\forall\ x_1, y_1 \in S$ (3)

Since 0 = D(x_1)$g(y_1)$ = D(x_1)$g(y_1\ z_1)$
= D(x_1)$G(y_1)\sigma(z_1)$ + D(x_1)$\tau(y_1)g(z_1)$=D(x_1)$\tau(y_1)g(z_1)$
By lemma 3.1, $g(z_1)$D(x_1) = 0, $\forall\ x_1, z_1 \in S$. Replacez_1 by$y_1\ z_1$
0 = g($y_1\ z_1$)D(x_1) = $g(y_1)\sigma(z_1)$ D(x_1) + $\tau(y_1)g(z_1)$ D(x_1) =$g(y_1)\sigma(z_1)$ D(x_1)

Since σ is an automorphisms of S and using lemma2.6, D(x_1) $g(y_1)$= 0, $\forall\ x_1, y_1 \in S$
(3) ⇒$G(x_1)$ d(y_1) = 0, $\forall\ x_1, y_1 \in S$. Replacing y_1 by $z_1 y_1$
0 = $G(x_1)$ d($z_1 y_1$) = G(x_1) d(z_1)$\sigma(y_1)$ + G(x_1)$\tau(z_1)$d(y_1) = G(x_1)$\tau(z_1)$d(y_1)
Since τ is an automorphisms of S, G(x_1)z_2d(y_1)= 0, $\forall\ x_1, y_1, z_2 \in S$
Using lemma 2.6, d(y_1) G(x_1) = 0,$\forall\ x_1, y_1 \in S$
By (ii), (D,d) and (G,g) are orthogonal

Theorem: 3.7

Let (D,d) and (G,g) be generalized (σ,τ) derivations of Ssuch thatd σ = $\sigma\ d$, dτ = $\tau\ d$, gσ = $\sigma\ g$, gτ = $\tau\ g$. Then the following conditions are equivalent.
(i) (DG,dg) is a generalized(σ^2,τ^2)derivation
(ii) (GD,gd) is a generalized (σ^2,τ^2)derivation
(iii) D and g are orthogonal also G and d are orthogonal

Proof: (i)⇒(iii), Suppose (DG,dg) is a generalized (σ^2,τ^2) derivation.
By the simple simplification we get G(x) d(y) + D(x) g(y) = 0, $\forall\ x, y \in S$
Replacing y by yz, 0 = G(x) d(yz) + D(x) g(yz)
= G(x) d(y) $\sigma(z)$+G(x) $\tau(y)$d(z)+D(x) g(y) $\sigma(z)$+D(x) $\tau(y)$g(z)
= G(x) $\tau(y)$d(z)+D(x) $\tau(y)$g(z)
Since τ is an automorphisms of S, G(x) y_1d(z)+D(x) y_1 g(z) = 0, $\forall\ x, y_1, z \in S$ (1)

Since dg is a (σ^2,τ^2)derivation, so d and g are orthogonal.
Replacing y_1 by g(z) y and using the orthogonality of d and g, we get
G(x) g(z) y d(z) + D(x) g(z) y g(z) = 0 ⇒D(x) g(z) y g(z) = 0
Again replacing y by yD(x), D(x) g(z) y D(x) g(z) = 0 ⇒ D(x) g(z) = 0, $\forall\ x, z \in S$ (2)

Substituting yz for z in the above equation,
0 = D(x) g(yz) = D(x) g(y) $\sigma(z)$+D(x) $\tau(y)$g(z), $\forall\ x, y, z \in S$
Using (1) and τ is an automorphisms of S, we get D(x) y_1 g(z) = 0, $\forall\ x, y_1, z \in S$
Then by lemma 2.6, D and g are orthogonal. Hence (1) becomes, G(x) y_1d(z) = 0
Using lemma 2.6, G and d are orthogonal
(iii)⇒(i), By the orthogonality of D and g, D(x) s g(y) = 0, $\forall\ x, y, s \in S$ (3)

Replacing x by zx, 0 = D(zx) s g(y) =D(x) $\sigma(x)$ s g(y)+ $\tau(z)$ d(x) s g(y)
= $\tau(z)$ d(x) s g(y)
Since τ is an automorphisms of S and using semiprimeness of S, we get
d(x) s g(y) = 0, $\forall\ x, y, s \in S$. By lemma 2.6, d and g are orthogonal.
Then by result, dg is a(σ^2,τ^2)derivation . Now replacing s by g(y) s D(x) in (3),
D(x) g(y) s D(x) g(y) = 0, $\forall\ x, y, s \in S$. By the semiprimeness of S, D(x) g(y) = 0.

Similarly, by the orthogonality of G and d, we have G(x) d(z) = 0, $\forall\ x, z \in S$.
Thus DG(xy) = DG(x) $\sigma^2(y)$+ $\tau^2(x)$ dg(y) , $\forall\ x, y \in S$
Hence (DG,dg) is a generalized(σ^2, τ^2)derivation
(ii)⟺(iii), Using similar approach as we have used to prove (i)⟺(iii)

Corollary: 3.8
Let (D,d) and (G,g) be generalized derivations of S. Then the following conditions are equivalent
(i) (DG,dg) is a generalized derivation
(ii) (GD,gd) is a generalized derivation
(iii) D and g are orthogonal also G and d are orthogonal

Corollary: 3.9
Let (D,d) be generalized (σ, τ) derivations of S. If D(x) D(y) = 0, $\forall\ x, y \in S$,
then D = d = 0

Proof: GivenD(x) D(y) = 0, $\forall\ x, y \in S$ (1)

Replacing y by yz, we get 0 = D(x) D(yz)
= D(x) D(y) $\sigma(z)$+D(x) $\tau(y)$ d(z) = D(x) $\tau(y)$ d(z)
Since τ is an automorphisms of S and using lemma 2.6, we have d(z) D(x) = 0, $\forall\ x, z \in S$
Now replacing x by xz, we get, 0 = d(z) D(xz) = d(z) D(x) $\sigma(z)$+ d(z) $\tau(x)$ d(z)
= d(z) $\tau(x)$ d(z)
By the semiprimeness of S, d(z) = 0, $\forall\ z \in S$. Therefore d = 0.
Again in (1) we replace x by xz, and the same simplification we get D = 0.

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