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A New Complex Continuous Wavelet Family

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Abstract : *A family of functions is identified and all of its properties to be a complex continuous wavelet are verified and are applied in analyzing time series by using popular software Matlab. We discuss the cross wavelet spectrum and wavelet coherence using this new wavelet for examining relationships in time frequency space between two time series. We demonstrate how phase angle statistics can be used to gain confidence in causal relationships and test mechanistic models of physical relationships between the time series. The different graphical representation provided by the wcoher.m function in Matlab is used.*
Keywords: *One dimensional complex continuous wavelets, Wavelet cross spectrum and Wavelet coherence for Complex-valued Wavelets.*

I. Introduction

Many scientists have made use of the complex continuous wavelet method in analyzing time series, often using popular software like Matlab. We found a list of functions which are successive derivatives of single function and are satisfying the entire requirement to be a complex continuous wavelet. By using these wavelets, we discuss the cross wavelet transform and wavelet coherence for examining relationships in time frequency space between two time series. We demonstrated this by two examples. The first one is Two Sine Waves in Gaussian Noise and the second one is Sine and Doppler Signal [8]. In section-2 we discuss the requirement for a function to be Mother wavelet [1], [2], [3] and [5], in section-3 we verified all the conditions for new family of function to be a wavelet [7] and in the last section-4 we demonstrate the use of the new wavelets in examining relationships in time frequency space between two time series. [6], [4], [8] and [9].

II. Requirement for Complex Continuous Wavelet

In order to be classified as a wavelet, a function must satisfy certain mathematical criteria. These are:

1.1 The wavelet must have finite energy i.e.

$$\Psi(t) \in L^2(R) \text{ Or } E = \int_{-\infty}^{\infty} |\Psi(t)|^2 \, dt < \infty \qquad (1)$$

1.2 If $\hat{\Psi}(f)$ is the Fourier transform of $\Psi(t)$ i.e.

$$\hat{\Psi}(f) = \int_{-\infty}^{\infty} \Psi(t) e^{-i(2\pi f)t} dt \qquad (2)$$

Then the following condition must hold

$$C_\psi = \int_0^{\infty} \frac{|\hat{\Psi}(f)|^2}{f} df < \infty \qquad (3)$$

(The above equation is known as admissibility condition and C_ψ is called the admissibility constant)

This implies that the wavelet has no zero frequency component i.e. $\hat{\Psi}(0) = 0$ or to put this in another way, the wavelet $\Psi(t)$ must have zero mean i.e.

$$\int_{-\infty}^{\infty} \Psi(t) \, dt = 0 \qquad (4)$$

III. The New Wavelet Family

The present family is built with different order by starting from the complex function $\psi(x) = e^{-ix} \cdot \frac{1}{1+x^2}$ and by taking the k^{th} derivative of ψ like the Complex Gaussian Family which start with the complex Gaussian function $f(t)=e^{-ix}e^{-x^2}$. The integer k is the parameter of this family and represents the order of the wavelet of the family i.e. $\psi_k(x)$ is a Complex Continuous Wavelet for each k and

$$\psi_k(x)=C_k\frac{d^k}{dx^k}(\psi(x)) \quad (5)$$

In the previous formula, C_k is such that $\|\psi_k(x)\|^2=1$ where $\psi_k(x)$ is the k^{th} derivative of ψ Where

$$C_k=\left(\int_{-\infty}^{\infty}\left|\frac{d^k}{dx^k}[\psi(x)]\right|^2 dx\right)^{-1/2} \quad (6)$$

Such that $\int_{-\infty}^{\infty}|\psi_k(x)|^2 dx=1$, $\forall \quad k=1,2,3.........,7$. [Refer Table-I]

It has been verified that for the above functions $\psi_k(x)$,

1.3 $\int_{-\infty}^{\infty}|\psi_k(x)|^2 dx=1<\infty$, $\forall \quad k=1,2,3.........,7$.

1.4 $C_{\psi_k}=\int_{-\infty}^{\infty}\frac{\left|\hat{\Psi}_k(f)\right|^2}{f}df<\infty$ and, $\int_{-\infty}^{\infty}\psi_k(x)\,dx=0 \quad \forall \quad k=1,2,3.........,7$. [Refer Table-I]

Moreover $\int_{-\infty}^{\infty}x\psi_k(x)\,dx=0$, $\forall \quad k=2,3.........,7$.

Hence the functions $\psi_k(x)$, $\forall \quad k=1,2,3.........,7$ forms a family of One dimensional complex continuous wavelets.
Here after this family is named as complex raees wavelet family, shortly crsw associated with numbers 1, 2……7, according to their order. The graphical representation of the family is given from Fig-1.1 to Fig-1.7.

IV. Relationships in time frequency space between two time series using new Wavelets

By using complex wavelets we can separate the phase and amplitude components within the signal.
In two of the following examples, complex-valued wavelets-3 is used. When a complex wavelet is used, the CWT, $C_x(a,b)$ of a real-valued time series, x is a complex-valued function of the scale parameter a and the location parameter b.
Example 1: Two Sine Waves in Gaussian Noise:
The first example introduces the different graphical representations provided by the wcoher.m function in Matlab. The example also highlights the usefulness of the phase information obtained from using complex-valued wavelets.
Consider two sine functions
x = sin(16πt)+0.25×randn(size(t));
y = sin(16πt+π/4)+0.25×randn(size(t));
on the interval [0, 1] using 2048 points. Both sine functions have a frequency of 8 Hz. One of the functions has an initial phase offset of π/4 radians. Both sine functions are corrupted by additive zero-mean Gaussian noise with a variance of 0.5.

Consider the CWT of the two signals (denoted by x and y) using the crsw3 complex wavelet for integer scales from 1 to 512. The common period of the signals at scale 128 is clearly detected in the moduli of the individual wavelet spectra in figure 2.1.

note that this corresponds to a frequency of 8 Hz, which is equivalent to (1/8)×2048 = 256 samples per period with the given sampling frequency. The wavelet spectrum, defined for each signal, is characterized by the modulus and the phase of the CWT obtained using the complex-valued wavelet. Denote the individual wavelet spectra as $C_x(a,b)$ and $C_y(a,b)$. The two decompositions are exactly the same, up to a translation, since the CWT is translation-invariant. To examine the relationship between the two signals in the time-scale plane, consider the wavelet cross spectrum $C_{xy}(a,b)$, which is defined as

$$C_{xy}(a,b)=\overline{C_x(a,b)}\,C_y(a,b)$$

Where $\bar{z}$ denotes the complex conjugate of z .

A smoothed version of this function is depicted in the figure 2.2. The magnitude of the wavelet cross spectrum can be interpreted as the absolute value of the local covariance between the two time series in the time-scale plane. In this example, this non-normalized quantity highlights the fact that both signals have a significant contribution around scale 128 at all positions. The next figure 2.3 displays the wavelet coherence and is the most important. The empirical wavelet coherence for x and y is defined as the ratio:

$$\frac{S\left(C_{xy}\left(a,b\right)\right)}{\sqrt{S\left(\left|C_{x}\left(a,b\right)\right|^{2}\right)}\sqrt{S\left(\left|C_{y}\left(a,b\right)\right|^{2}\right)}}$$

Where S stands for a smoothing operator in time and scale. The wavelet coherence can be interpreted as the local squared correlation coefficient in the time-scale plane. The arrows in the figure represent the relative phase between the two signals as a function of scale and position. The relative phase information is obtained from the smoothed estimate of the wavelet cross spectrum, $S\left(C_{xy}\left(a,b\right)\right)$. The plot of the relative phases is superimposed on the wavelet coherence. The relative phase information produces a local measure of the delay between the two time series. Note that for scales around 128, the direction of the arrows captures the relative phase difference between the two signals of π/4 radians.

Example 2: Sine and Doppler Signal:

A 4-Hz sine wave with additive Gaussian noise is sampled on a grid of 1024 points over the interval [0, 1]. The second time series is a Doppler signal with decreasing frequency over time.

x = -sin(8πt) + 0.4×randn(1,1024);

x = x/max(abs(x));

y = wnoise('doppler',10); (ref Matlab)

Consider the CWT of the two signals (denoted by x and y) using the crsw3 complex wavelet for integer scales from 1 to 512. In figure 2.4 the analysis of the sine function on the left exhibits the scale associated with the period (which is equal to 1024/8 = 128). The analysis of the Doppler signal highlights a typical time-scale picture illustrating the decreasing frequency (increasing scale) as a function of time. The wavelet cross spectrum is shown in the figure 2.5. The magnitude is the more instructive and shows the similarity of the local frequency behavior of the two time series in the time-scale plane. Both signals have a similar contribution around scale 128 over the interval [300, 700]. This is consistent with the behavior observed by visual inspection of the time-domain plot. Additional interesting information is discernible in the wavelet coherence in figure 2.6. The phase information can be interpreted by locating different regions of the time-scale plane and highlighting some coherent behaviors. Some transient minor contributions to the variability of the time series occur at small scales at the beginning of the Doppler signal, which exhibits rapid oscillations. The behavior is not coherent and the phase changes very quickly. However, at positions greater than 150 and scales greater than 130, numerous coherent regions can be easily detected, delimited by the stability of the phase information. Because phase information is so useful in determining coherent behavior, another representation tool is available for focusing on the phase. The phase information is coded both by the arrow, or vector, orientation and by the background color. The background color is associated with a mapping onto the interval [-π, π] in figure 2.7.

V. Figures and Tables

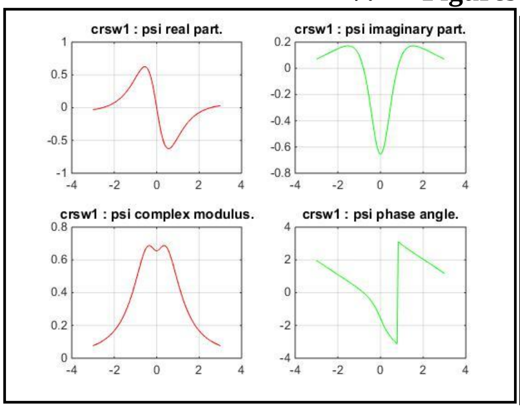

Fig-1.1

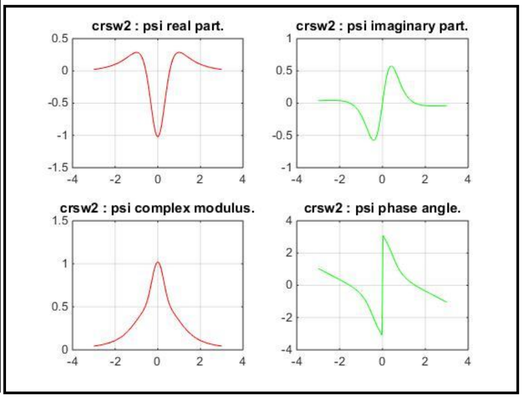

Fig-1.2

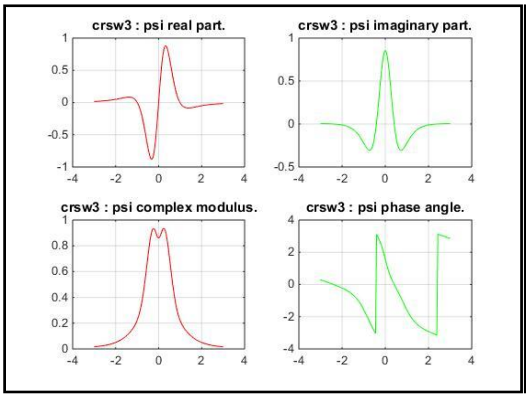

Fig-1.3

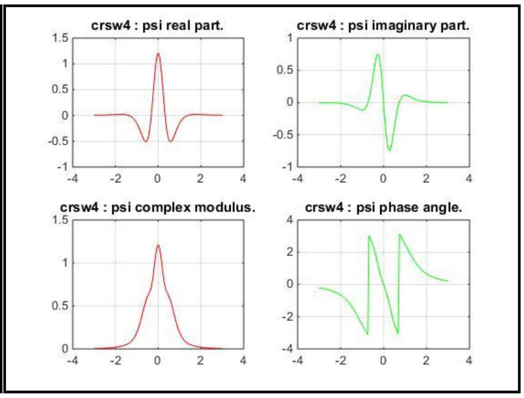

Fig-1.4

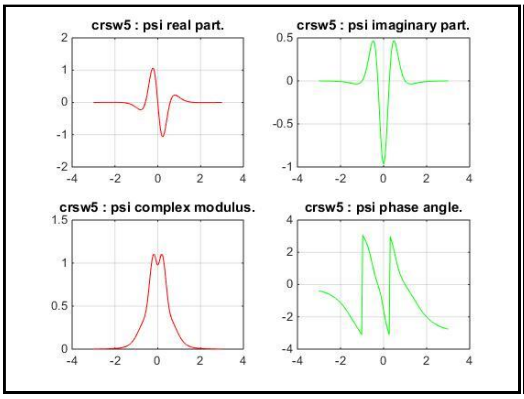

Fig-1.5

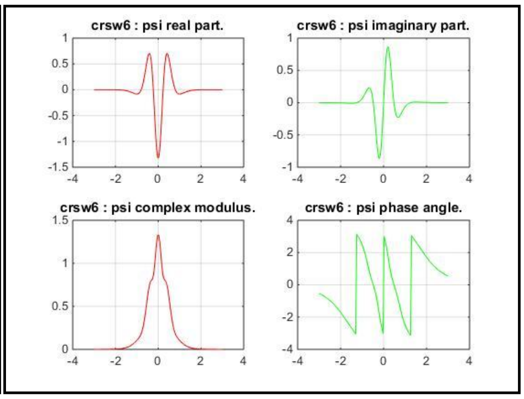

Fig-1.6

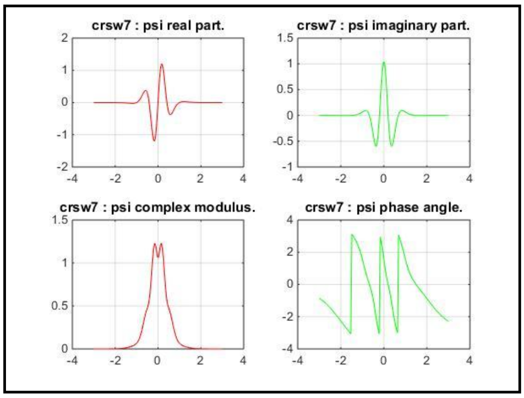

Fig-1.7

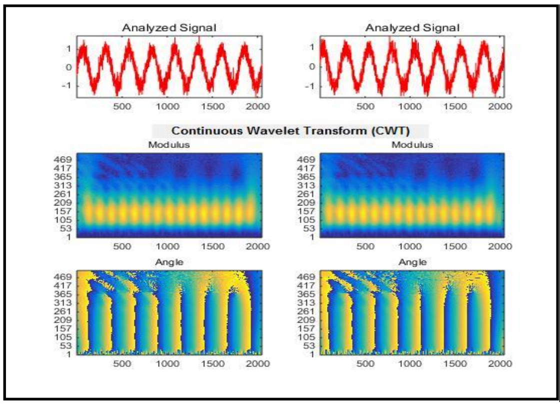

Fig – 2.1

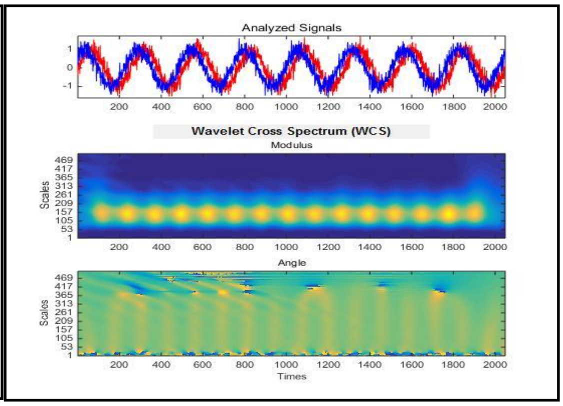

Fig-2.2

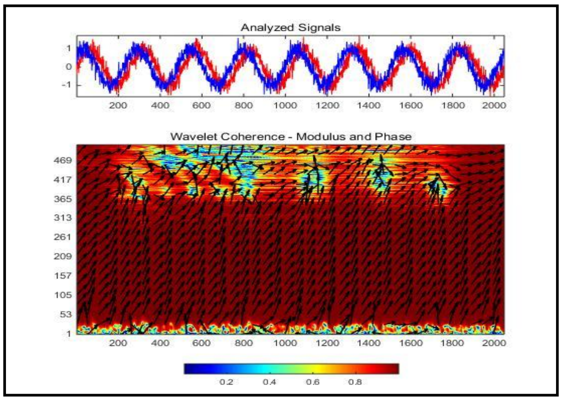

Fig-2.3

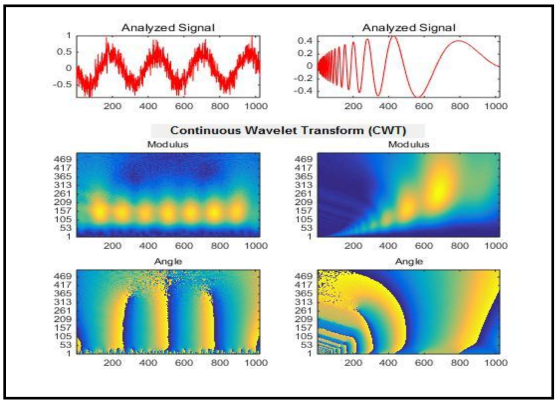

Fig-2.4

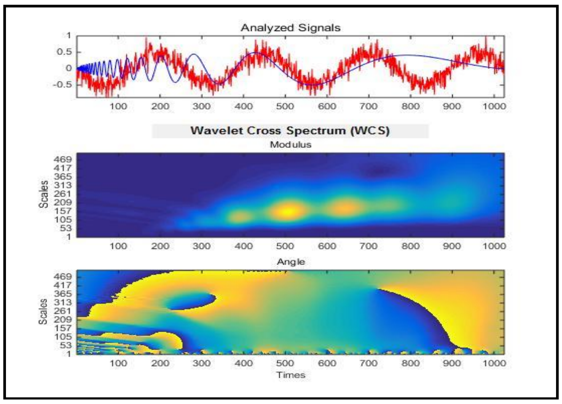

Fig-2.5

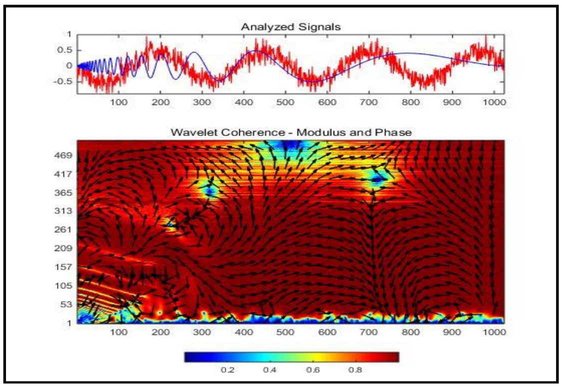

Fig-2.6

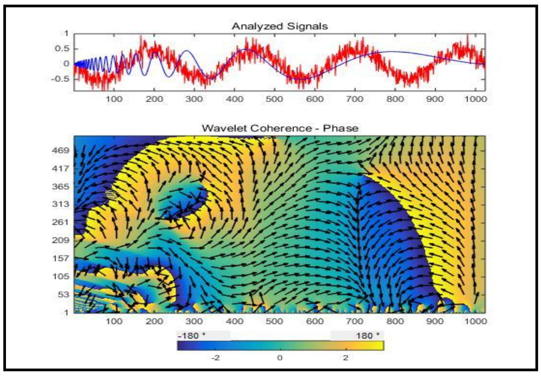

Fig- 2.7

Table-1

Order of the Wavelet (k)	C_{ψ_k}	C_k
1	$\pi^2 \exp(-2)/4$	$2/\sqrt{3\pi}$
2	$3\pi^2 \exp(-2)/8$	$2/\sqrt{11\pi}$
3	$15\pi^2 \exp(-2)/8$	$2\sqrt{2}/\sqrt{13\pi}$
4	$315\pi^2 \exp(-2)/16$	$2/\sqrt{1185\pi}$
5	$2835\pi^2 \exp(-2)/8$	$2\sqrt{2}/\sqrt{53329\pi}$
6	$155925\pi^2 \exp(-2)/16$	$2\sqrt{2}/\sqrt{1759861\pi}$
7	$6081075\pi^2 \exp(-2)/16$	$4/\sqrt{160147359\pi}$

VI. Conclusion

In this paper we introduce a new Complex Continuous Wavelet family (crsw) and are used in analyzing time frequency relationship between two time series.

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