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Solutions of some system of non-linear PDEs using Reduced Differential Transform Method

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Abstract : *In this paper, we present analytical solutions of some system of non-linear partial differential equations using Reduced Differential Transform Method (RDTM). The solutions are considered as infinite series expansions which converge rapidly to their exact solutions. Solutions obtained by RDTM are in close conformity with the exact solutions. The results obtained show that the RDTM technique is accurate and efficient and require less effort in comparison to the other analytical and numerical methods.*
Keywords ***– A****nalytical solutions , Non-linear System, PDEs, RDTM.*

I. Introduction

Differential equations are the mathematical expressions of some real life problems arising out of the real world around us such as physical, biological, engineering, financial or sociological fields. Solutions of differential equations involving non-linear terms are extremely difficult and in most of the situations and in some situations they are even not possible to solve completely. In such situations, Mathematicians usually apply some methods which involve lots of compromise and approximations. Other than exact solutions obtained by analytical methods, all these methods are computationally intensive because they are trial and error in nature and need complicated computations. Some of the numerical methods involving approximate solutions in solving systems of differential equations are - the Euler method, the Taylor method, the Runge–Kutta method, etc. The Differential Transform method is one of the numerical methods for solving ordinary or partial differential equations. This method uses polynomials based on Taylor's series expansion as the approximation to the exact solutions. Solutions of differential equations means to establish the relationship among the variables or the physical quantities involved in the process. Only the simplest differential equations are solvable by explicit formulas available in the literature; however, some properties of solutions of a given differential equation may be determined without finding their exact form. If a self-contained formula for the solution is not available, the solution may be numerically approximated using computers.

The differential transform method is one of the numerical methods in solving ordinary and partial differential equations. The concept of the Differential transform was first introduced by Zhou [1] and applied to solve initial value problems for electric circuit analysis. The method is based on Taylor's series expansion and can be applied to solve both linear and non linear ordinary differential equations (ODEs) as well as partial differential equations (PDEs). Already many authors have contributed their works using DTM. Some of them are Chen and Ho [2], Chen et. al. [3], Chen and Liu [4], Malik and Dang [5], Chen and Ho [6], Chen and Ho [7], Jang et. al. [8], [9], [10], Halim Hassan [11], [12], Chen and Chen [13], [14] Ayaz [15] Arikoglu and Ozkol [16], Haziqah et. al. [17], Marwan [18], Moon et. al. [19] and Umavathi and Shekar [20]. The method has been receiving much attention day-by-day and used by many authors to obtain solutions of their complicated problems.

Reduced Differential Transform method (RDTM) has been introduced by Keskin and Oturanc [21], defining a set of transformation rules to overcome the complicated complex calculations of traditional Differential Transform Method (DTM). Since then a number of authors like Keskin and Oturanc [22], [23], [24], [25], Cenesiz, et. al. [26], Taha [27], Taha and Wahab [28] solved many equations using this method. This paper is presented to solve few non-linear systems of differential equations using RDTM introduced by Keskin and Oturanc. The operational properties of DTM and RDTM are mentioned here for easy understanding.

II. Analysis Of The Method

Definition 1

A Taylor series of a polynomial of degree n is defined as follows:

$$F_n(x) = \sum_{k=0}^{n} \frac{1}{k!}\left(f^{(k)}(c)\right)(x-c)^k \tag{1}$$

Theorem 1

If the function $f(x)$ has $(n+1)$ derivatives on an interval $(c-r, c+r)$ for some *r> 0*, and $\lim_{n\to\infty} R_n(x) = 0,$ for all $x \in (c-r, c+r), where\ R_n(x)$ is the error between $F_n(x)$ and the polynomial function $f(x)$ then the Taylor series expanded about $x = c$ converges to $f(x)$. Thus,

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(f^k(c)\right)(x-c)^k \text{ , for all } x \in (c-r, c+r) \tag{2}$$

The Differential Transform Method (DTM):

Definition 2:

The Differential Transform *F(k)* of the function $f(x)$ for the k^{th} order derivative is defined as follows:

$$F(k) = \frac{1}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x\,=\,x_0} \tag{3}$$

Definition 3:

The inverse differential transform of *F(k)* is defined as follows:

$$f(x) = \sum_{k=0}^{\infty} (x-x_0)^k F(k) \tag{4}$$

This equation (4) is the Taylor series expansion of *f(x)* at $x=x_0$. From the above equations (3) and (4), the following basic operations of differential transform can be deduced.

Sl.No.	Original function	Transformed function
1	$f(x)=g(x)+h(x)$	$F(k)=G(k)+H(k)$
2	$f(x)=\lambda g(x)$,	$F(k)=\lambda G(k)$, *where* λ *is a constant.*
3	$f(x) = \frac{dg(x)}{dx}$	$F(k)=(k+1)G(k+1)$
4	$f(x) = \frac{d^2g(x)}{dx^2}$	$F(k)=(k+1)(k+2)G(k+2)$
5	$f(x) = \frac{d^ng(x)}{dx^n}$	$F(k)=\frac{(k+1)!}{k!}G(k+n)$
6	$f(x)=g(x)h(x)$	$F(x) = \sum_{i=0}^{k} G(i)H(k-i)$
7	$f(x)=g^2(x)$	$F(x) = \sum_{i=0}^{k} G(i)G(k-i)$
8	$f(x)=g(x)h(x)q(x)$	$F(x) = \sum_{j=0}^{k} \sum_{i=0}^{j} G(i)H(j-i)Q(k-j)$
9	$f(x)=g^3(x)$	$F(x) = \sum_{j=0}^{k} \sum_{i=0}^{j} G(i)G(j-i)G(k-j)$
10	$f(x)=x^n$	$F(k) = \delta(k-n)$, where $\delta(k-n) = \begin{cases} 1 & k=n \\ 0 & k\neq n \end{cases}$
11	$f(x)=(1+x)^n$	$F(k)=\frac{n(n-1)\ldots\ldots\ldots(n-k+1)}{k!}$
12	$f(x)=e^{\lambda x}$	$F(k)=\frac{\lambda^k}{k!}$, *where* λ *is a constant.*
13	$f(x)=\sin(\omega x+\alpha)$	$F(k)=\frac{\omega^k}{k!}\sin\left(\frac{k\pi}{2}+\alpha\right)$, *where* ω *and* α *are constants.*
14	$f(x)=\cos(\omega x+\alpha)$	$F(k)=\frac{\omega^k}{k!}\cos\left(\frac{k\pi}{2}+\alpha\right)$, *where* ω *and* α *are constants.*

III. Reduced Differential Transform Method (Rdtm):

The reduced differential transform method was first proposed by the Turkish mathematician Keskin and Oturanc in the year 2009. This method has received much attention since then and applied to solve a wide variety of problems. This method as introduced by Keskin and Oturanc is a very reliable and efficient quite accurate method to solve linear and non-linear ordinary as well as partial differential equations. This reduced differential transform method is introduced mainly to overcome the demerits of complex calculation of the usual differential transform method. The main advantage is that it provides its users with an analytical approximation, in many cases, an exact solution, in a rapidly convergent sequence with elegantly computed terms as mentioned by Keskin and Oturanc. The traditional differential transform is already illustrated in the previous section with few examples.

The basic definition of the method is as follows:

Definition No-4: If the function $u(x,t)$ is analytic and differentiable continuously with respect to time t and space x in the domain of interest, then let

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial x^k} u(x,t)\right] \; at\ t = 0 \tag{5}$$

where the t-dimensional spectrum function $U_k(x)$ is the transformed function of $u(x,t)$.Here the lower case function $u(x,t)$ represents the original function while the upper case function $U_k(x)$ stands for the transformed function.

Definition No-5: The inverse differential transform of $U_k(x)$ is defined as follows:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^k \tag{6}$$

Thus combining (5) and (6), we can express the solution as follows :

$$u(x,t) = \sum_{k=0}^{\infty} \left(\frac{1}{k!}\left[\frac{\partial^k}{\partial x^k} u(x,t)\right] \; at\ t = 0\right) t^k \tag{7}$$

The basic concept of reduced differential transform method mainly comes from the power series expansion. Few fundamental mathematical operations performed by this reduced differential method are listed below:

Original Function	Transformed function
$u(x,t)$	$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial x^k} u(x,t)\right] \; at\ t = 0$
$w(x,t) = u(x,t) \pm v(x,t)$	$W_k(x) = U_k(x) \pm V_k(x)$
$w(x,t) = cu(x,t)$	$W_k(x) = cU_k(x)$, where *c* a is constant
$w(x,t) = x^m t^n$	$W_k(x) = x^m \delta(k-n)$, where $\delta(k-n) = \begin{cases} 1, k = n \\ 0, k \neq n \end{cases}$
$w(x,t) = x^m t^n u(x,t)$	$W_k(x) = x^m U_{k-n}(x)$
$w(x,t) = u(x,t)v(x,t)$	$W_k(x) = \sum_{r=0}^{k} U_r(x)\ V_{k-r}(x) = \sum_{r=0}^{k} V_r(x)\ U_{k-r}(x)$
$w(x,t) = \frac{\partial^r}{\partial t^r} u(x,t)$	$W_k(x) = \frac{(k+r)!}{k!} U_{k+r}(x)$

$w(x,t) = \frac{\partial}{\partial x} u(x,t)$	$W_k(x) = \frac{\partial}{\partial x} U_k(x)$
$w(x,t) = \frac{\partial^2}{\partial x^2} u(x,t)$	$W_k(x) = \frac{\partial^2}{\partial x^2} U_k(x)$

For three dimensional function:

Original Function	Transformed function
$u(x,y,t)$	$U_k(x,y) = \frac{1}{k!}\left[\frac{\partial^k}{\partial x^k} u(x,y,t)\right] \ at\ t = 0$
$w(x,y,t) = u(x,y,t) \pm v(x,y,t)$	$W_k(x,y) = U_k(x,y) \pm V_k(x,y)$
$w(x,y,t) = cu(x,y,t)$	$W_k(x,y) = cU_k(x,y)$, where c a is constant
$w(x,y,t) = x^m y^n t^p$	$W_k(x) = x^m y^n \delta(k-p)$, where $\delta(k-p) = \begin{cases} 1, k = p \\ 0, k \neq p \end{cases}$
$w(x,y,t) = x^m y^n t^p u(x,y,t)$	$W_k(x,y) = x^m y^n U_{k-p}(x,y)$
$w(x,y,t) = u(x,y,t)v(x,y,t)$	$W_k(x,y) = \sum_{r=0}^{k} U_r(x,y)\ V_{k-r}(x,y) = \sum_{r=0}^{k} V_r(x,y)\ U_{k-r}(x,y)$
$w(x,y,t) = \frac{\partial^r}{\partial t^r} u(x,y,t)$	$W_k(x,y) = \frac{(k+r)!}{k!} U_{k+r}(x,y)$
$w(x,y,t) = \frac{\partial}{\partial x} u(x,y,t)$	$W_k(x,y) = \frac{\partial}{\partial x} U_k(x,y)$
$w(x,y,t) = \frac{\partial^2}{\partial x^2} u(x,y,t)$	$W_k(x,y) = \frac{\partial^2}{\partial x^2} U_k(x,y)$
$w(x,y,t) = \frac{\partial^2}{\partial y^2} u(x,y,t)$	$W_k(x,y) = \frac{\partial^2}{\partial y^2} U_k(x,y)$

IV. Examples

Here, we use the reduced differential transform method to solve few non-linear partial differential equations which behave like heat equations. Since the equations are non-linear, so we consider them in some simple environment.

Example 1. To illustrate the method, we first consider the one dimensional initial value problem describing heat-like equations as follows:

$$\frac{\partial u(x,t)}{\partial t} = (x^2+5)\frac{\partial^2 U(x,t)}{\partial x^2} \tag{8}$$

with the initial condition $u(x,0) = x^2 + x$ (9)

Applying the Reduced Differential Transform Method, we obtain the recurrence relation as follows:

$$(k+1)U_{k+1}(x,t) = (x^2+5)\frac{\partial^2}{\partial x^2}U_k(x) \tag{10}$$

where $U_k(x)$ is the transform function. From the initial condition, we have

$$U_0(x) = x^2 + x \tag{11}$$

Substituting $U_0(x)$ in the recurrence relation, we obtain the following $U_k(x)$ values successively:

$$U_1(x) = 2(x^2+5) \tag{12}$$

$$U_2(x) = \frac{2^2}{2!}(x^2+5) \tag{13}$$

$$U_3(x) = \frac{2^3}{3!}(x^2+5) \tag{14}$$

$$U_4(x) = \frac{2^4}{4!}(x^2+5) \tag{15}$$

and so on.

Finally the inverse differential transform of $U_k(x)$ is obtained from the relation

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k \tag{16}$$

as follows:

$$U(x,t) = (x^2+5)e^{2t} + x - 5 \tag{17}$$

which is the exact solution.

Example 2. We consider the following two dimensional initial value problem describing heat-like equations as follows:

$$\frac{\partial u(x,y,t)}{\partial t} = \frac{1}{2}\left[x^2\frac{\partial^2 u(x,y,t)}{\partial x^2} + y^2\frac{\partial^2 u(x,y,t)}{\partial y^2}\right] \tag{18}$$

with the initial condition $\quad u(x,y,0) = x^2 + y^2 \quad$ (19)

Applying the Reduced Differential Transform Method, we obtain the recurrence relation as follows:

$$(k+1)U_{k+1}(x,y) = \frac{1}{2}\left[x^2\frac{\partial^2}{\partial x^2}U_k(x,y) + y^2\frac{\partial^2}{\partial y^2}U_k(x,y)\right] \tag{20}$$

where $U_k(x,y)$ is the transform function. From the initial condition, we have

$$U_0(x,y) = x^2 + y^2 \tag{21}$$

Substituting $U_0(x,y)$ in the recurrence relation, we obtain the following $U_k(x,y)$ values successively:

$$U_1(x,y) = x^2 + y^2 \tag{22}$$

$$U_2(x,y) = \frac{1}{2!}(x^2+y^2) \tag{23}$$

$$U_3(x,y) = \frac{1}{3!}(x^2 + y^2) \tag{24}$$

$$U_4(x,y) = \frac{1}{4!}(x^2 + y^2) \tag{25}$$

and so on.

The solution for $u(x,t)$ is obtained as follows:

$$u(x,y,t) = \sum_{k=0}^{\infty} U_k(x,y)t^k \tag{26}$$

$$u(x,y,t) = (x^2 + y^2)\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots \infty\right) \tag{27}$$

$$= (x^2 + y^2)e^t$$

Example 3. We consider another two dimensional initial value problem describing heat-like equations as follows:

$$\frac{\partial u(x,y,t)}{\partial t} = x^2y^2 + \frac{1}{4}\left[x^2 \frac{\partial^2 u(x,y,t)}{\partial x^2} + y^2 \frac{\partial^2 u(x,y,t)}{\partial y^2}\right] \tag{28}$$

$$u(x,y,0) = 0 \tag{29}$$

Applying the Reduced Differential Transform Method, we obtain the recurrence relation as follows:

$$(k+1)U_{k+1}(x,y) = x^2y^2\delta(k) + \frac{1}{4}\left[x^2 \frac{\partial^2}{\partial x^2}U_k(x,y) + y^2 \frac{\partial^2}{\partial y^2}U_k(x,y)\right] \tag{30}$$

where $\delta(k) = 1, when\ k = 0$ and $\delta(k)$ =0, when k≠0

where $U_k(x,y)$ is the transformed function. From the initial condition, we have

$$U_0(x,y) = 0 \tag{31}$$

Substituting $U_0(x,y)$ in the recurrence relation, we obtain the following $U_k(x,y)$ values successively:

$$U_1(x,y) = x^2y^2 \tag{32}$$

$$U_2(x,y) = \frac{1}{2!}(x^2y^2) \tag{33}$$

$$U_3(x,y) = \frac{1}{3!}(x^2y^2) \tag{34}$$

$$U_4(x,y) = \frac{1}{4!}(x^2y^2) \text{ and so on.} \tag{35}$$

The solution for *u(x,t)* is obtained as follows:

$$u(x,y,t) = \sum_{k=0}^{\infty} U_k(x,y)t^k \tag{36}$$

$$u(x,y,t) = (x^2y^2)\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots \infty\right) \tag{37}$$

$$= (x^2y^2)e^t$$

Example 4.

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} - u^2(x,t) \tag{38}$$

with initial conditions $u(x,o)$=1 (39)

Applying the Reduced Differential Transform Method, we obtain the recurrence relation as follows:

$$(k+1)U_{k+1}(x) = \frac{\partial^2}{\partial x^2}\{U_k(x)\} - \sum_{i=0}^{k}[U_i(x)U_{k-i}(x)] \tag{40}$$

And the transformed initial condition becomes

$$U_0(x) = 1 \tag{41}$$

For different values of *k* we obtain the following results

$$U_1(x) = -1 \tag{42}$$

$$U_2(x) = 1 \tag{43}$$

$$U_3(x) = -1 \tag{44}$$

$U_4(x) = 1$, and so on. (45)

The solution for $u(x,t)$ is obtained as follows:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k = U_0(x) + U_1(x)t + U_2(x)t^2 + U_3(x)t^3 + \cdots \tag{46}$$

$$u(x,t) = 1 - t + t^2 - t^3 + \cdots \ldots = (1+t)^{-1} = \frac{1}{1+t} \tag{47}$$

V. Conclusion

In this paper, we have applied the reduced differential transform method to some non-lineat partial differential equations which have the resemblance of heat like situations. They may also describe diffusion like situations though their exact behavior could not be identified. The method is very effective and direct method. The method do not require any approximation or discretization. Rather it is a direct and effective method to approach the exact solution rapidly. The computational size is also small than the traditional differential transform method. Thus the method is very powerful and effective and can be utilized to tackle complex situations arising out of real world.

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