IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 11, Issue 5 Ver. I (Sep. - Oct. 2015), PP 63-73
www.iosrjournals.org

Optimality and Efficiency of Circular Neighbor Balanced Design for Second Order Circular Auto Regressive Process

Jeevitha.M[1] and Dr.C.Santharam[2]
[1]*Student, Department of Statistics, Loyola College,*
[2]*Faculty, Department of Statistics, Loyola College,*

Abstract: *This research paper deals with the optimality of Circular Neighbor Balanced Designs for total effects when the observation errors are correlated according to second order circular stationary autoregressive process. Few results pertaining to the optimality conditions under some specified conditions are provided and the efficiencies of circular neighbor balanced designs relative to the optimal continuous block designs are also investigated. The efficiency of the Circular Neighbor Balanced Designs is illustrated corresponding to the optimal continuous block designs.*

Key Words: *Auto regressive process, second order Block Design, Circular, Correlated observation, Total effect, Universal optimality.*

I. Introduction:

In many practical problems it is inevitable that a particular plot is being affected by neighboring effects. Even though it is harm in many cases, the plot is being gained by the neighboring effects in few cases. So it was necessary for the researchers to study the neighboring effects. Under the linear models with the neighbor effects, many optimality results of block designs are established for treatment and neighbor effects separately. Hedayat and Afsarinejad (1978), Cheng and Wu (1980), Kunert (1984b) and Kushner (1997) for cross-over designs, Kunert (1984a) and AzaÃ³s, Bailey and Monod(1993), Druilhet (1999). After studying the characters of neighboring effects, it was reasonable to make the assumptions on the dependency of the observations, because practically speaking, in many of the experiments,the observations are dependent on each other, if not overall within block at least. Hence the researches have invoked their thoughts to the models where the observations are dependent.

The optimal designs or highly efficient experiments (when the observations are dependent) have been studied by many authors. H.B.Kushner (1997) derived the necessary and sufficient condition for the universal optimality in the case repeated measurement designs. Kunert and Martin (2000b) have generalized the Kushner's condition by demonstrating the method of deriving the optimal designs in the case of two dimensional neighboring models. Filipiak and Markiewicz (2005) were dealt with circular neighbor- balanced designs.

It is very important to determine which treatment combination in a block will be optimal for the better result. Hence many authors have stepped into the next level of finding out the optimal continuous sequences. For example see Kunert and Martin (2000), Filipiak and Markiewicz (2005) and Ai, He and Yu (2009). Ai, Yu and He (2009) have discussed the optimality and efficiency of one dimensional and two dimensional neighboring designs when the errors are correlated according to first order circular auto regressive process.

In this paper we study the universal optimality of circular neighbor-balanced designs for total effects, but when the observation errors are correlated according to a second-order circular autoregressive process.

In this paper, Section 2 deals with some definitions and preliminaries. Section 3 presents the main results that circular neighbor- balanced designs are universally optimal under some conditions for the total effects in linear models when the observation errors are correlated according to a second-order circular autoregressive process. In order to discuss the efficiency of circular neighbor-balanced designs among all possible block designs with the same parameters, the optimal continuous block designs are characterized in Section 4. Section 5 presents the efficiency of circular neighbor-balanced designs with blocks of small size based on the previous structure of optimal equivalence classes of sequences.

II. Model And Definition

Consider a set of circular block designs $\Omega_{(t,b,k)}$. For a design $d \in \Omega_{(t,b,k)}$, the linear effect additive model with the left and two sided neighbor effects can be written in the vector form as

$(M_1) \Rightarrow Y = 1_{bk}\mu + T_d\tau + L_d\lambda + (I_b \otimes I_k)\beta + \varepsilon$ --- (1)

$(M_2) \Rightarrow Y = 1_{bk}\mu + T_d\tau + L_d\lambda + R_d\rho + (I_b \otimes I_k)\beta + \varepsilon$ --- (2)

Where Y =(Y_{11},...,Y_{1k},...,Y_{b1},...,Y_{bk})’,$Y_{ij}$is the observation response on plot j of block i, μ is the general mean, τ, λ and ρ are, respectively, the t-dimensional vectors of the direct effects, left-Neighbor effects and right-Neighbor effects of the t treatments, T_d, L_d and R_d are the corresponding incidence matrices, β is the b-dimensional vector of the block effects, and ε is the vector of random errors and 1_ndenotean n-dimensional vector of ones and the symbol⊗denote the Kro- Necker product.

Information Matrix, as an inverse of variance - co variance matrix:

As like many cases of design of experiments, the amount of information obtained from the experiment is measured in terms of information matrix. Also we know that, the information matrix can also be viewed as the inverse of variance co- variance matrix. Hence in this research work, we consider the inverse of Variance Co-Variance matrix.

For any m x n matrix A, we define $Q_A = I_m - A(A'A)^{-}A'$, where $(A'A)^{-}$ denotes the generalized inverse of $(A'A)$. Then from Kunert and Martin (2000a) the information matrix of d for estimating τ in the model (1) under normality is,

$$C_d = T_d^{'}(I_b \otimes S)^{-1/2} Q_{(I_b \otimes S)^{-1/2}(L_d : (I_b \otimes 1_k)}(I_b \otimes S)^{-1/2} T_d$$

Where $(I_b \otimes S)^{-1/2}$ is an bkxbk matrix with the property $(I_b \otimes S)^{-1/2}(I_b \otimes S)^{-1/2} = (I_b \otimes S)^{-1}$

Definition:

A block design is said to be a circular block design neighbor-balancedat distance i≤k-1 if it is a circular binary block design in $\Omega_{(t,b,k)}$, and is a BIBDsuch that for each ordered pair of distinct treatments, there exist exactly m plots suchthat each of these plots receives the first chosen treatment and the right-neighbor of it at distance i receives the second treatment. A circular block design is said to beneighbor-balanced at distances up to γ , abbreviated by CNBD(γ), if it is neighborbalanced at distance i for all 1≤i≤γ .

Here assume that the errors in each block are correlated according to a second-order circular autoregressive process, denoted by AR(2,C),as in the case of Kunert and Martin (1987), Richard Cutler (1993) for first order and Martin.O.Grodona (1989) for second order. The AR (2, C) process can be represented in the recursive form $\varepsilon_i = \rho_1\varepsilon_{i-1} + \rho_2\varepsilon_{i-2} + \eta_i$ with $|\rho_i|<1$, i=1,2. where the η_i’s are uncorrelated noises with E(η_i)=0 and Var (η_i)=σ^2, and E(ε_0)=0.Then E(ε) = 0 Cov (ε) = $\sigma^2 I_b$ ⊗Sand The covariance function of a second order autoregressive process satisfies the difference equation(Fuller, 1976 p.53)

$$\sigma_\omega(h) - \rho_1\sigma_\omega(h-1) - \rho_2\sigma_\omega(h-2) = 0; h > 0$$

$$\sigma_\omega(h) - \rho_1\sigma_\omega(h-1) - \rho_2\sigma_\omega(h-2) = \sigma^2; h = 0$$

Where, $\sigma_\omega(h) \equiv \text{cov}(\omega_{ij}, \omega_{i+h,j})$, for all i=1,2,..

Let $S \equiv \text{var}(\underline{\omega}_j)$ where $\underline{\omega}_j$ is the error vector from the j –th block. Then S^{-1}, the inverse of S, is given by (Wise, 1955;Siddiqui 1958, **Martin D.Gronda 1985**)

$$\sigma^2 S^{-1} = (1+\rho_1^2+\rho_2^2)I_k + (\rho_1^2+\rho_2^2)H_3 + (1+\rho_1^2)H_4 - \rho_1(H+H^{'}) - \rho_2(H+H^{'}) + (\rho_1+\rho_2)H_5 + \rho_2 H_6 + \frac{1}{1-\rho_2}H_7 \quad \text{----- (3)}$$

$$\sigma^2 S^{-1} = \begin{bmatrix} 1 & a_4 & a_3 & 0 & 0 & ... & 0 & 0 \\ a_4 & a_5 & a_2 & a_3 & 0 & ... & 0 & 0 \\ a_3 & a_2 & a_1 & a_2 & a_3 & ... & 0 & 0 \\ 0 & a_3 & a_2 & a_1 & a_2 & ... & 0 & 0 \\ 0 & 0 & a_3 & a_2 & a_1 & ... & 0 & 0 \\ 0 & 0 & 0 & a_3 & a_2 & ... & 0 & 0 \\ . & . & . & . & . & ... & . & . \\ . & . & . & . & . & ... & . & . \\ 0 & 0 & 0 & 0 & 0 & ... & a_5 & a_4 \\ 0 & 0 & 0 & 0 & 0 & ... & a_4 & 1 \end{bmatrix}$$

Where, $a_1 = 1+\rho_1^2+\rho_2^2$
$a_2 = -\rho_1(1-\rho_2)$
$a_3 = -\rho_2$
$a_4 = -\rho_1$
$a_5 = 1+\rho_1^2$

Where H denotes the $k\times k$matrix with $h_{1k}= 1$ and the $(i,j)^{th}$element $h_{ij}= 1$ if $i-j=1$ and 0 otherwise, and

$$H_3=\begin{bmatrix}1&0&0&0&.&0&0\\0&0&0&0&.&0&0\\0&0&0&0&.&0&0\\0&0&0&0&.&0&0\\.&.&.&.&.&.&.\\0&0&0&0&.&0&0\\0&0&0&0&.&0&1\end{bmatrix} H_4=\begin{bmatrix}0&0&0&0&.&0&0\\0&1&0&0&.&0&0\\0&0&0&0&.&0&0\\0&0&0&0&.&0&0\\.&.&.&.&.&.&.\\0&0&0&0&.&1&0\\0&0&0&0&.&0&0\end{bmatrix} H_5=\begin{bmatrix}0&0&0&0&.&0&1\\0&0&0&0&.&0&0\\0&0&0&0&.&0&0\\0&0&0&0&.&0&0\\.&.&.&.&.&.&.\\0&0&0&0&.&0&0\\1&0&0&0&.&0&0\end{bmatrix}$$

$$H_6=\begin{bmatrix}0&0&0&0&.&0&0\\0&0&0&0&.&1&0\\0&0&0&0&.&0&0\\0&0&0&0&.&0&0\\.&.&.&.&.&.&.\\0&1&0&0&.&0&0\\0&0&0&0&.&0&0\end{bmatrix} H_7=\begin{bmatrix}0&1&0&0&.&0&0\\1&0&0&0&.&0&0\\0&0&0&0&.&0&0\\0&0&0&0&.&0&0\\.&.&.&.&.&.&.\\0&1&0&0&.&0&1\\0&0&0&0&.&1&0\end{bmatrix}$$

Note that when $\rho_1,\rho_2= 0$, the structure of errors is reduced to the popular i.i.d. case.

III. Universal Optimality Of CNBD (2):

In this paper we follow the universal optimality criterion defined byKiefer (1975).

LEMMA 1:

Let $C_d[\alpha]$ be the information matrix for some effect α based on a designd. Assume that a design $d \in \Omega_{(t,b,k)}$ has its information matrix completely symmetric,Then, dis universally optimal for the effect α over a class $\Omega_{(t,b,k)}$ of designs if and onlyiftr($C_{d^*}[\alpha]$) = $\max_{d\in D} tr(C_d[\alpha])$

Let φ and ψ denote the total effects of the t treatments in the models (M_1) and (M_2), respectively, that is $\varphi=\tau+\lambda$ and $\psi=\tau+\lambda+\rho$. Thus, we can obtainthe following universal optimality results of CNBD's for the total effects.

THEOREM 1

For $3 \le k \le t$, a CNBD (2) in $\Omega_{(t,b,k)}$ is Universally Optimal for the total effects in the model (M_1) among all the designs with no treatment Neighbor of itself when $0 \le \rho < 1$, and among all the designs with no treatment Neighbor ofitself at distance 1 or 2 when $-1 < \rho < 0$.

PROOF:

We already have,

$$\tilde{S} = S^{-1} - (1_k' S^{-1} 1_k)^{-1} S^{-1} 1_k 1_k' S^{-1}$$

$$\tilde{S} = S^{-1} - \frac{(1-\rho_1-\rho_2)^4(1+\rho_1^2-2\rho_1-\rho_2+\rho_1\rho_2)^4(1+\rho_1^2+\rho_2^2-2\rho_1-2\rho_2-2\rho_1\rho_2}{2(k-4)(1+\rho_1^2+\rho_2^2)-2(k-3)\rho_1(1-\rho_2)-2(k-2)\rho_2-4\rho_1+2(1+\rho_1^2)} 1_k 1_k'$$

But,

$$S^{-1} = (1+\rho_1^2+\rho_2^2)I_k + (\rho_1^2+\rho_2^2)H_3 + (1+\rho_1^2)H_4 - \rho_1(H+H') - \rho_2(H+H') + (\rho_1+\rho_2)H_5 + \rho_2 H_6 + \frac{1}{1-\rho_2}H_7$$

Hence,

$$\tilde{S} = (1+\rho_1^2+\rho_2^2)I_k + (\rho_1^2+\rho_2^2)H_3 + (1+\rho_1^2)H_4 - \rho_1(H+H') - \rho_2(H+H') + (\rho_1+\rho_2)H_5$$
$$+ \rho_2 H_6 + \frac{1}{1-\rho_2}H_7 - A1_k1_k'$$

Where,

$$A = \frac{(1-\rho_1-\rho_2)^4(1+\rho_1^2-2\rho_1-\rho_2+\rho_1\rho_2)^4(1+\rho_1^2+\rho_2^2-2\rho_1-2\rho_2-2\rho_1\rho_2}{2(k-4)(1+\rho_1^2+\rho_2^2)-2(k-3)\rho_1(1-\rho_2)-2(k-2)\rho_2-4\rho_1+2(1+\rho_1^2)}$$

For a design, $d \in \Omega_{(t,b,k)}$, the information matrix $C_d[\alpha]$ for the effect α=[τ',λ']' in the model (1) can be expressed as,

$$C_d[\alpha] = (T_d, L_d)'(I_b \otimes S^{-1/2})' pr^{\perp}_{(I_b \otimes S^{-1/2} I_k)}(I_b \otimes S^{-1/2})(T_d, L_d)$$
$$= (C_{d_{ij}})1 \le i, j \le 2,$$

Where the submatrices $(C_{d_{ij}})1 \le i, j \le 2,$ have the forms

$$C_{d_{11}} = T_d'(I_b \otimes \tilde{S})T_d$$
$$C_{d_{12}} = T_d'(I_b \otimes \tilde{S})L_d$$
$$C_{d_{22}} = L_d'(I_b \otimes \tilde{S})L_d$$

Since S is a cyclic matrix, so $HSH' = H'SH = S.$ For a circular design d, $L_{du} = HT_{du}, 1 \le u \le b.$ It implies that $C_{d_{11}} = C_{d_{22}}$.

For a CNBD (2) d*, we have

$$T_{d^*}'(I_b \otimes H)T_{d^*} = T_{d^*}'(I_b \otimes H')T_{d^*} = T_{d^*}'(I_b \otimes H'H')T_{d^*} = \frac{bk}{t(t-1)}(1_t1_t' - I_t)$$

Then,

$$C_{d_{11}^*} = T_{d*}'(I_b \otimes \tilde{S})T_{d^*}$$

$$= T_{d^*}'\{I_b \otimes (1+\rho_1^2+\rho_2^2)I_k + (\rho_1^2+\rho_2^2)H_3 + (1+\rho_1^2)H_4 - \rho_1(H+H') - \rho_2(H+H') + (\rho_1+\rho_2)H_5$$
$$+ \rho_2 H_6 + \frac{1}{1-\rho_2}H_7 - A1_k1_k')\}T_{d^*}$$
$$= (1+\rho_1^2+\rho_2^2)T_{d^*}'(I_b \otimes I_k)T_{d^*} + (\rho_1^2+\rho_2^2)T_{d^*}'(I_b \otimes H_3)T_{d^*} + (1+\rho_1^2)T_{d^*}'(I_b \otimes H_4)T_{d^*}$$
$$- \rho_1 T_{d^*}'(I_b \otimes H)T_{d^*} - \rho_1 T_{d^*}'(I_b \otimes H')T_{d^*} - \rho_2 T_{d^*}'(I_b \otimes H)T_{d^*} - \rho_2 T_{d^*}'(I_b \otimes H')T_{d^*}$$
$$+ (\rho_1+\rho_2)T_{d^*}'(I_b \otimes H_5)T_{d^*} + \rho_2 T_{d^*}'(I_b \otimes H_6)T_{d^*} + \frac{1}{1-\rho_2}T_{d^*}'(I_b \otimes H_7)T_{d^*}$$
$$- AT_{d^*}'(I_b \otimes 1_k1_k')T_{d^*}$$

Similarly,

$$C_{d_{12}^*} = T_{d*}'(I_b \otimes \tilde{S})L_{d^*} = T_{d*}'(I_b \otimes \tilde{S})HT_{d^*}$$

$$= (1+\rho_1^2+\rho_2^2)T_{d^*}'(I_b \otimes I_k)HT_{d^*} + (\rho_1^2+\rho_2^2)T_{d^*}'(I_b \otimes H_3)HT_{d^*} + (1+\rho_1^2)T_{d^*}'(I_b \otimes H_4)HT_{d^*}$$
$$- \rho_1 T_{d^*}'(I_b \otimes H)HT_{d^*} - \rho_1 T_{d^*}'(I_b \otimes H')HT_{d^*} - \rho_2 T_{d^*}'(I_b \otimes H)HT_{d^*} - \rho_2 T_{d^*}'(I_b \otimes H')HT_{d^*}$$
$$+ (\rho_1+\rho_2)T_{d^*}'(I_b \otimes H_5)HT_{d^*} + \rho_2 T_{d^*}'(I_b \otimes H_6)HT_{d^*} + \frac{1}{1-\rho_2}T_{d^*}'(I_b \otimes H_7)HT_{d^*} - AT_{d^*}'(I_b \otimes 1_k1_k')HT_{d^*}$$

$$= (1+\rho_1^2+\rho_2^2)T_{d^*}^{'}(I_b \otimes H)T_{d^*} + (\rho_1^2+\rho_2^2)T_{d^*}^{'}(I_b \otimes H_3H)T_{d^*} + (1+\rho_1^2)T_{d^*}^{'}(I_b \otimes H_4H)T_{d^*}$$
$$-\rho_1 T_{d^*}^{'}(I_b \otimes HH)T_{d^*} - \rho_1 T_{d^*}^{'}(I_b \otimes H^{'}H)T_{d^*} - \rho_2 T_{d^*}^{'}(I_b \otimes HH)T_{d^*} - \rho_2 T_{d^*}^{'}(I_b \otimes H^{'}H)T_{d^*}$$
$$+(\rho_1+\rho_2)T_{d^*}^{'}(I_b \otimes H_5H)T_{d^*} + \rho_2 T_{d^*}^{'}(I_b \otimes H_6H)T_{d^*} + \frac{1}{1-\rho_2}T_{d^*}^{'}(I_b \otimes H_7H)T_{d^*} - AT_{d^*}^{'}(I_b \otimes 1_k1_k^{'}H)T_{d^*}$$
$$= (1+\rho_1^2+\rho_2^2)T_{d^*}^{'}(I_b \otimes H)T_{d^*} - \rho_1 T_{d^*}^{'}(I_b \otimes HH)T_{d^*} - \rho_1 T_{d^*}^{'}(I_b \otimes I_k)T_{d^*} - \rho_2 T_{d^*}^{'}(I_b \otimes HH)T_{d^*}$$
$$-\rho_2 T_{d^*}^{'}(I_b \otimes I_k)T_{d^*} - AT_{d^*}^{'}(I_b \otimes 1_k1_k^{'})T_{d^*}$$

Hence all $C_{d_{ij}^*}$ (1≤ i ,j≤2) are completely symmetric.

Rewrite $\phi = K^{'}\alpha$ with $K = 1_2 \otimes I_t$. It is obvious that $K^{'}K = 2I_t$.

Hence,

$$C_d[\phi] \le \frac{1}{4}K^{'}C_d[\alpha]K$$
$$= \frac{1}{4}(C_{d_{11}} + C_{d_{12}} + C_{d_{21}} + C_{d_{22}})$$
$$= \frac{1}{4}(T_d^{'}(I_b \otimes \tilde{S})T_d + T_d^{'}(I_b \otimes \tilde{S})L_d + L_d^{'}(I_b \otimes \tilde{S})T_d + L_d^{'}(I_b \otimes \tilde{S})L_d$$
$$= \frac{1}{4}\{(1+\rho_1^2+\rho_2^2)T_{d^*}^{'}(I_b \otimes I_k)T_{d^*} - \rho_1 T_{d^*}^{'}(I_b \otimes H)T_{d^*} - \rho_1 T_{d^*}^{'}(I_b \otimes H^{'})T_{d^*} - \rho_2 T_{d^*}^{'}(I_b \otimes H)T_{d^*}$$
$$-\rho_2 T_{d^*}^{'}(I_b \otimes H^{'})T_{d^*} - AT_{d^*}^{'}(I_b \otimes 1_k1_k^{'})T_{d^*} + (1+\rho_1^2+\rho_2^2)T_{d^*}^{'}(I_b \otimes I_k)T_{d^*} - \rho_1 T_{d^*}^{'}(I_b \otimes H)T_{d^*}$$
$$-\rho_1 T_{d^*}^{'}(I_b \otimes H^{'})T_{d^*} - \rho_2 T_{d^*}^{'}(I_b \otimes H)T_{d^*} - \rho_2 T_{d^*}^{'}(I_b \otimes H^{'})T_{d^*} - AT_{d^*}^{'}(I_b \otimes 1_k1_k^{'})T_{d^*}$$
$$+(1+\rho_1^2+\rho_2^2)T_{d^*}^{'}(I_b \otimes H)T_{d^*} - \rho_1 T_{d^*}^{'}(I_b \otimes HH)T_{d^*} - \rho_1 T_{d^*}^{'}(I_b \otimes I_k)T_{d^*} - \rho_2 T_{d^*}^{'}(I_b \otimes HH)T_{d^*}$$
$$-\rho_2 T_{d^*}^{'}(I_b \otimes I_k)T_{d^*} - AT_{d^*}^{'}(I_b \otimes 1_k1_k^{'})T_{d^*} + (1+\rho_1^2+\rho_2^2)T_{d^*}^{'}(I_b \otimes H)T_{d^*} - \rho_1 T_{d^*}^{'}(I_b \otimes HH)T_{d^*}$$
$$-\rho_1 T_{d^*}^{'}(I_b \otimes I_k)T_{d^*} - \rho_2 T_{d^*}^{'}(I_b \otimes HH)T_{d^*} - \rho_2 T_{d^*}^{'}(I_b \otimes I_k)T_{d^*} - AT_{d^*}^{'}(I_b \otimes 1_k1_k^{'})T_{d^*}\}$$
$$= \frac{1}{4}\{2(1+\rho_1^2+\rho_2^2-2\rho_1-2\rho_2)T_{d^*}^{'}(I_b \otimes I_k)T_{d^*} - 4AT_{d^*}^{'}(I_b \otimes 1_k1_k^{'})T_{d^*}$$
$$+(1+\rho_1^2+\rho_2^2-2\rho_1-2\rho_2)T_{d^*}^{'}(I_b \otimes H)T_{d^*} - 2(\rho_1+\rho_2)T_{d^*}^{'}(I_b \otimes HH)T_{d^*}\} \quad \text{---- (4)}$$

Since $C_{d_{ij}^*}$ (1≤ i, j ≤2) are completely symmetric $C_{d_{12}^*} = C_{d_{21}^*}$. So $C_{d^*}[\alpha]$ commutes with $pr_{(K)} = \frac{1}{2}(1_21_2^{'} \otimes I_t)$. Then

$$C_{d^*}[\phi] = \frac{1}{4}K^{'}C_{d^*}[\alpha]K = \frac{1}{4}(C_{d_{11}^*} + C_{d_{12}^*} + C_{d_{21}^*} + C_{d_{11}^*})$$

and consequently $C_{d^*}[\phi]$ is also completely symmetric. Consider now (4). When -1<ρ<0, for a design d in $\Omega_{t,b,k}$ with no treatment neighbor of itself at distance 1 or 2, the traces of $T_d^{'}(I_b \otimes H)T_d$, $T_d^{'}(I_b \otimes H^{'})T_d$, $T_d^{'}(I_b \otimes HH)T_d$, $T_d^{'}(I_b \otimes H^{'}H^{'})T_d$ are all zero, and tr($T_d^{'}(I_b \otimes I_k)T_d$) is a constant. So tr$\{C_d[\phi]\}$ depends only on tr $T_d^{'}(I_b \otimes 1_k1_k^{'})T_d$. Moreover, a CNBD(2) is a balanced block design, so it also minimizestr $T_d^{'}(I_b \otimes 1_k1_k^{'})T_d$ among all possible designs of the same size. Therefore tr$\{C_d[\phi]\}$ attains the maximum. When $0 \le \rho < 1$, the traces of both $T_d^{'}(I_b \otimes HH)T_d$, $T_d^{'}(I_b \otimes H^{'}H^{'})T_d$ must be non-negative. However, for a CNBD(2) d*, they are all zero. So for a design with no treatment neighbor of Therefore tr$\{C_d[\phi]\}$ attains the maximum. When $0 \le \rho < 1$, the traces of both $T_d^{'}(I_b \otimes HH)T_d$ $T_d^{'}(I_b \otimes H^{'}H^{'})T_d$, must be non-negative. However, for a CNBD(2) d*, they are all zero. So for a design with no treatment neighbor of itself at

distance 1, it still holds that $tr\{C_{d^*}[\phi]\} \geq tr\{C_d[\phi]\}$.
Hence the theorem follows from Lemma 1.

THEOREM

For $4 \leq k \leq t$, a CNBD (3) in $\Omega_{(t,b,k)}$is Universally Optimal for the total effects in the model (M_2) among all the designs with no treatment Neighbor of itself when $0 \leq \rho < 1$, and among all the designs with no treatment Neighbor ofitself at distance 1 or 2 when $-1 < \rho < 0$.

PROOF:

For a design, $d \in \Omega_{(t,b,k)}$, the information matrix $C_d[\alpha]$ for the effect α=[τ',λ']' in the model (1) can be expressed as,

$$C_d[\alpha] = (T_d, L_d)^{'}(I_b \otimes S^{-1/2})^{'} pr^{\perp}_{(I_b \otimes S^{-1/2} I_k)} (I_b \otimes S^{-1/2})(T_d, L_d)$$

$$= (C_{d_{ij}})1 \leq i, j \leq 2,$$

Where the submatrices $(C_{d_{ij}})1 \leq i, j \leq 2,$ have the forms

$$C_{d_{11}} = T_d^{'}(I_b \otimes \tilde{S})T_d$$

$$C_{d_{12}} = T_d^{'}(I_b \otimes \tilde{S})L_d$$

$$C_{d_{13}} = T_d^{'}(I_b \otimes \tilde{S})R_d$$

$$C_{d_{22}} = L_d^{'}(I_b \otimes \tilde{S})L_d$$

$$C_{d_{23}} = L_d^{'}(I_b \otimes \tilde{S})R_d$$

$$C_{d_{23}} = R_d^{'}(I_b \otimes \tilde{S})R_d$$

Since S is a cyclic matrix, so $HSH^{'} = H^{'}SH = S.$ For a circular design d, $L_{du} = HT_{du}, 1 \leq u \leq b.$ It implies that $C_{d_{11}} = C_{d_{22}}$.
For a CNBD (3) d*, we have

$$T_{d^*}^{'}(I_b \otimes HHH)T_{d^*} = T_{d^*}^{'}(I_b \otimes H^{'}H'H^{''})T_{d^*} = \frac{bk}{t(t-1)}(1_t 1_t^{'} - I_t)$$

Then,

$$C_{d^*_{13}} = C_{d^*_{31}} = T_{d*}^{'}(I_b \otimes \tilde{S})R_{d^*}$$

$$= T_{d^*}^{'}\{I_b \otimes (1+\rho_1^2+\rho_2^2)I_k + (\rho_1^2+\rho_2^2)H_3 + (1+\rho_1^2)H_4 - \rho_1(H+H^{'}) - \rho_2(H+H^{'}) + (\rho_1+\rho_2)H_5$$

$$+ \rho_2 H_6 + \frac{1}{1-\rho_2}H_7 - A1_k1_k^{'})\}R_{d^*}$$

$$= (1+\rho_1^2+\rho_2^2)T_{d^*}^{'}(I_b \otimes I_k)R_{d^*} + (\rho_1^2+\rho_2^2)T_{d^*}^{'}(I_b \otimes H_3)R_{d^*} + (1+\rho_1^2)T_{d^*}^{'}(I_b \otimes H_4)R_{d^*}$$
$$- \rho_1 T_{d^*}^{'}(I_b \otimes H)R_{d^*} - \rho_1 T_{d^*}^{'}(I_b \otimes H^{'})R_{d^*} - \rho_2 T_{d^*}^{'}(I_b \otimes H)R_{d^*} - \rho_2 T_{d^*}^{'}(I_b \otimes H^{'})R_{d^*}$$
$$+ (\rho_1+\rho_2)T_{d^*}^{'}(I_b \otimes H_5)R_{d^*} + \rho_2 T_{d^*}^{'}(I_b \otimes H_6)R_{d^*} + \frac{1}{1-\rho_2}T_{d^*}^{'}(I_b \otimes H_7)R_{d^*} - AT_{d^*}^{'}(I_b \otimes 1_k1_k^{'})R_{d^*}$$
$$= (1+\rho_1^2+\rho_2^2)T_{d^*}^{'}(I_b \otimes H^{'})T_{d^*} - \rho_1 T_{d^*}^{'}(I_b \otimes HH^{'})T_{d^*} - \rho_1 T_{d^*}^{'}(I_b \otimes H^{'}H^{'})T_{d^*}$$
$$- \rho_2 T_{d^*}^{'}(I_b \otimes HH')T_{d^*} - \rho_2 T_{d^*}^{'}(I_b \otimes H^{'}H^{'})T_{d^*} - AT_{d^*}^{'}(I_b \otimes 1_k1_k^{'})T_{d^*}$$

$$= (1+\rho_1^2+\rho_2^2)T_{d^*}^{'}(I_b \otimes H^{'})T_{d^*} - (\rho_1+\rho_2)T_{d^*}^{'}(I_b \otimes I_k)T_{d^*} - (\rho_1+\rho_2)T_{d^*}^{'}(I_b \otimes H^{'}H^{'})T_{d^*}$$
$$- AT_{d^*}^{'}(I_b \otimes 1_k 1_k^{'})T_{d^*}$$
$$C_{d_{23}^*} = C_{d_{32}^*} = L_{d*}^{'}(I_b \otimes \tilde{S})R_{d^*}$$

$$= L_{d^*}^{'}\{I_b \otimes (1+\rho_1^2+\rho_2^2)I_k + (\rho_1^2+\rho_2^2)H_3 + (1+\rho_1^2)H_4 - \rho_1(H+H^{'}) - \rho_2(H+H^{'}) + (\rho_1+\rho_2)H_5$$
$$+ \rho_2 H_6 + \frac{1}{1-\rho_2}H_7 - A1_k 1_k^{'})\}R_{d^*}$$
$$= (1+\rho_1^2+\rho_2^2)L_{d^*}^{'}(I_b \otimes I_k)R_{d^*} + (\rho_1^2+\rho_2^2)L_{d^*}^{'}(I_b \otimes H_3)R_{d^*} + (1+\rho_1^2)L_{d^*}^{'}(I_b \otimes H_4)R_{d^*}$$
$$- \rho_1 L_{d^*}^{'}(I_b \otimes H)R_{d^*} - \rho_1 L_{d^*}^{'}(I_b \otimes H^{'})R_{d^*} - \rho_2 L_{d^*}^{'}(I_b \otimes H)R_{d^*} - \rho_2 L_{d^*}^{'}(I_b \otimes H^{'})R_{d^*}$$
$$+ (\rho_1+\rho_2)L_{d^*}^{'}(I_b \otimes H_5)R_{d^*} + \rho_2 L_{d^*}^{'}(I_b \otimes H_6)R_{d^*} + \frac{1}{1-\rho_2}L_{d^*}^{'}(I_b \otimes H_7)R_{d^*} - AL_{d^*}^{'}(I_b \otimes 1_k 1_k^{'})R_{d^*}$$
$$= (1+\rho_1^2+\rho_2^2)T_{d^*}^{'}(I_b \otimes I_k)T_{d^*} - (\rho_1+\rho_2)T_{d^*}^{'}(I_b \otimes H)T_{d^*} - (\rho_1+\rho_2)T_{d^*}^{'}(I_b \otimes H^{'})T_{d^*}$$
$$- AT_{d^*}^{'}(I_b \otimes 1_k 1_k^{'})T_{d^*}$$

Hence all $C_{d_{ij}^*}$ (1≤ i, j≤2) are completely symmetric.

Rewrite $\psi = K^{'}\alpha$ with $K = 1_3 \otimes I_t$. It is obvious that K'K=2I$_t$. By Lemma and equation, for any design $d \in \Omega_{(t,b,k)}$,

$$C_d[\psi] \le \frac{1}{9}K^{'}C_d[\alpha]K$$
$$= \frac{1}{4}(C_{d_{11}} + C_{d_{12}} + C_{d_{13}} + C_{d_{21}} + C_{d_{22}} + C_{d_{23}} + C_{d_{31}} + C_{d_{32}} + C_{d_{33}})$$
$$= \frac{1}{9}[3T_{d*}^{'}(I_b \otimes \tilde{S})T_{d^*} + 2T_{d*}^{'}(I_b \otimes \tilde{S})L_{d^*} + 2L_{d*}^{'}(I_b \otimes \tilde{S})T_{d^*} + L_{d*}^{'}(I_b \otimes \tilde{S})R_{d^*} + L_{d*}^{'}(I_b \otimes \tilde{S})L_{d^*}$$
$$= \frac{1}{3}\{(3+3\rho_1^2+3\rho_2^2-4\rho_1-4\rho_2)T_d^{'}(I_b \otimes I_k)T_d + (2+2\rho_1^2+2\rho_2^2-3\rho_1-4\rho_2)T_d^{'}(I_b \otimes H)T_d +$$
$$(2+2\rho_1^2+2\rho_2^2-7\rho_1-7\rho_2)T_d^{'}(I_b \otimes H^{'})T_d - 2(\rho_1+\rho_2)T_d^{'}(I_b \otimes HH)T_d +$$
$$(2+2\rho_1^2+2\rho_2^2-2\rho_1-2\rho_2)T_d^{'}(I_b \otimes H^{'}H^{'})T_d - 2(\rho_1+\rho_2)T_d^{'}(I_b \otimes H^{'}H^{'}H^{'})T_d - 9AT_d^{'}(I_b \otimes 1_k 1_k^{'})T_d$$

For any design in $d \in \Omega_{(t,b,k)}$ with no treatment neighbor of itself at distance upto 3, $tr(T_d^{'}(I_b \otimes H)T_d), tr(T_d^{'}(I_b \otimes H^{'})T_d), tr(T_d^{'}(I_b \otimes HH)T_d), tr(T_d^{'}(I_b \otimes H^{'}H^{'})T_d),$ $tr(T_d^{'}(I_b \otimes HHH)T_d), tr(T_d^{'}(I_b \otimes H^{'}H^{'}H^{'})T_d)$ areall zero. Moreover, both $tr(T_d^{'}(I_b \otimes HHH)T_d), tr(T_d^{'}(I_b \otimes H^{'}H^{'}H^{'})T_d)$ are non-negative. The remainder of the proof follows from the proof of the previous theorem.

IV. Optimal Equivalence Classes Of Sequences

In the present section we discuss the optimality of continuous block designs, by applying the method derived by Kunert and Martin (2000b). For u = 1, 2, . . ., b, let T_{du} be the incidence matrix of the direct effects of the treatment in block u, $1 \le u \le b$.Then $T_d = (T_{d_1}, T_{d_2}, ..., T_{d_b})^{'}$ is just the incidence matrix of the direct effects. For each u, define $L_{du} = HT_{du}$ $R_{du} = H'T_{du}$. Thus, it is obvious that $L_d = (I_b \otimes H)T_d$ and $L_d = (I_b \otimes H')T_d$ are exactly the incidence matrices of the left-Neighbor effects and of the right-Neighbor effects.Now consider,

$$C_d[\phi] \le C_d[K^{'}\alpha K]$$

$$C_d[\phi] \le \sum_{u=1}^{b} C_{du}$$

Where $C_{du} = \frac{1}{4}\{2(1+\rho_1^2+\rho_2^2-2\rho_1-2\rho_2)T_{du}^{'}T_{du} - 4AT_{du}^{'}1_k1_k^{'}T_{du}$

$+2(1+\rho_1^2+\rho_2^2-2\rho_1-2\rho_2)T_{du}^{'}L_{du} - 2(\rho_1+\rho_2)T_{du}^{'}HHT_{du}\}$

Now deriving the trace of the matrix C_{du}, we get,

$$tr(C_{du}) = \frac{1}{2}tr\{(1+\rho_1^2+\rho_2^2-2\rho_1-2\rho_2)T_{du}^{'}T_{du} - 2AT_{du}^{'}1_k1_k^{'}T_{du}$$

$$+(1+\rho_1^2+\rho_2^2-2\rho_1-2\rho_2)T_{du}^{'}L_{du} - (\rho_1+\rho_2)T_{du}^{'}HHT_{du}\}$$

$$\Rightarrow \quad tr(C_{du}) = \frac{1}{2}\{(1+\rho_1^2+\rho_2^2-2\rho_1-2\rho_2)k - 2A\sum_{i=1}^{t} n_i^2$$

$$+(1+\rho_1^2+\rho_2^2-2\rho_1-2\rho_2)\sum_{i=1}^{t} m_i - (\rho_1+\rho_2)\sum_{i=1}^{t} p_i\}$$

Two sequencesoftreatmentsonablockareequivalentifonesequence can be obtained from the other by relabeling the treatments and denote by *s* the equivalence class of the sequence *l* on the block *u*. Because tr(C_{du})are in variant under permutations of treatment labels, so the value tr(C_{du}) remains the same for any sequence in the same equivalence class. Thus, we can define,

$$c(s) = tr(C_{du}) = \frac{1}{2}\left[(1+\rho_1^2+\rho_2^2-\rho_1-\rho_2)k + (1+\rho_1^2+\rho_2^2-2\rho_1-2\rho_2)\sum_{i=1}^{t} m_i - (\rho_1+\rho_2)\sum_{i=1}^{t} p_i - 2A\sum_{i=1}^{t} n_i^{\,2}\right] \quad \text{----- (5)}$$

Where,

$$A = \frac{(1-\rho_1-\rho_2)^4(1+\rho_1^2-2\rho_1-\rho_2+\rho_1\rho_2)^4(1+\rho_1^2+\rho_2^2-2\rho_1-2\rho_2-2\rho_1\rho_2)^{2(k-4)}}{2(k-4)(1+\rho_1^2+\rho_2^2)-2(k-3)\rho_1(1-\rho_2)-2(k-2)\rho_2-4\rho_1+2(1+\rho_1^2)}$$

$n_i$is the number of occurrences of treatment i in the sequence *l*,

$m_i$is thenumber of times treatment i is on the left-hand side of itself in the sequence *l*

$p_i$is the number of plots having treatment i both on the left-hand side and on theright-hand side.

In this section our ultimate aim would be in finding out the optimal equivalence classes of sequence. This optimal sequence is the sequence which maximizes the c(s) in (5) as explained by Kushner(1997).

PROPOSITION:

When $\rho_1, \rho_2 \in (0.2199,1)$ for any positive integer k ≥ 5, if k is odd, then the optimal sequence has the form of '$a_1a_2a_2a_3a_3\cdots a_{[k/2]}a_{[k/2]}$', while if k is even, then the optimal sequence has the form of '$a_1a_1a_2a_2\cdots a_{[k/2]}a_{[k/2]}$',where$a_1,...,a_{[k/2]}$ are distinct treatments.

PROOF:

If $\sum_{i=1}^{t} p_i$ decreases by one unit, then $\sum_{i=1}^{t} m_i$ decreases definitely by one unit, and correspondingly c(s) (5) will increase by $(\rho_1+\rho_2)-2(1+\rho_1^2+\rho_2^2-2\rho_1-2\rho_2)$. Also for the value ρ_1 and ρ_2 between 0.2199 and 1, the above increment takes the positive value. Thus from the Proposition 3 of Ai, Yu and He (2009), we have the proof of this theorem.

Now consider the blocks of size k=6. It contains the possible treatment sequences for k=6. Out of which we are going to consider the optimal treatment sequence.

Table 1.Optimal sequences for all possible pairs of *{v,v₁}* for k=6

S.No	OPTIMAL SEQUENCE	v	v_1	tr (C_{du})
1	aaabbb	2	0	$5\rho_1^2+5\rho_2^2-8\rho_1-8\rho_2-18A+5$
2	aabbbb	2	1	$\frac{1}{2}(10\rho_1^2+10\rho_2^2-17\rho_1-17\rho_2-52A+10)$
3	aabbcc	3	0	$\frac{3}{2}(3\rho_1^2+3\rho_2^2-4\rho_1-4\rho_2-8A+3)$
4	abbccc	3	0	$\frac{1}{2}(9\rho_1^2+9\rho_2^2-12\rho_1-12\rho_2-28A+9)$
5	abcdef	6	0	$3(\rho_1^2+\rho_2^2-\rho_1-\rho_2-2A+1)$

Here, $A=\dfrac{(1-\rho_1-\rho_2)^4(1+\rho_1^2-2\rho_1-\rho_2+\rho_1\rho_2)^4(1+\rho_1^2+\rho_2^2-2\rho_1-2\rho_2-2\rho_1\rho_2)^4}{6\rho_1^2+4\rho_2^2-10\rho_1-8\rho_2+6\rho_1\rho_2+6}$

From the above sequences, the sequence “aabbcc” is the optimal sequence by Proposition 3.

The below table represents all the optimal sequences for $6 \leq k \leq 11$. Also Note that the below table shows the optimal sequence and the last column lists the values tr(C_{du}) of a CNBD (2) d.

Block Size	Optimal sequence	c (s*)	tr(C_{du})
6	aabbcc	$1/3(\rho^2-\rho+1)$	$3/2\ (\rho^2-2\rho+1)$
7	aabbccc abbccdd	$1/7(27\rho^2-33\rho+27)$ $1/7(26\rho^2-31\rho+26)$	$2(\rho^2-2\rho+1)$
8	aabbccdd	$5\rho^2-6\rho+5$	$5/2(\rho^2-2\rho+1)$
9	aaabbbccc aabbccddd abbccddee	$6\rho^2-9\rho+6$ 1/9(15ρ2-11ρ+15) $1/9(51\rho^2-66\rho+51)$	$3(\rho^2-2\rho+1)$
10	aabbcccddd aabbccddee	$1/10(71\rho^2-102\rho+71)$ $7\rho^2-9\rho+7$	$7/2(\rho^2-2\rho+1)$
11	aabbbcccddd aabbccddeee abbccddeeff	$1/11(99\rho^2-144\rho+99)$ $1/11(89\rho^2-123\rho+89)$ $1/11(84\rho^2-113\rho+84)$	$4(\rho^2-2\rho+1)$

5. Efficiency of CNBD (2) with blocks of size 6≤ *k* ≤ 11

In this section we are going to discuss the Efficiency of CNBD (2) for various block size. In previous section we showed that for different block size k, the CNBD(2) is universally optimal over the class of all designs from $\Omega_{(t,b,k)}$for |ρ|< 1. Here the efficiency of CNBD (2) is demonstrated by having the optimal continuous block design as the base. Since the values tr(C_{du}) are invariant to any block *u* for aCNBD (2), so we can define the efficiency of aCNBD (2) *d* relative to the optimal continuous block design d^* as

$$Eff(d)=\frac{tr(C_d)}{tr(C_{d*})}=\frac{tr(C_{du})}{c(s^*)}$$

We will demonstrate the calculation of tr(C_{du}) by making use of the expression derived in the previous sections just by substituting different values for ρ_1, ρ_2 and we can find out the efficiency for various block size. We are going to assume the values for both ρ_1, ρ_2 to be -1 to +1 with 0.2 increments, avoiding the other combinations of ρ_1, ρ_2 since these combinations giving the negative values for the efficiency.The below tables show the calculations of tr(C_{du}) and c(s*) for k= 6, 7,..., 11

Table 2 Efficiency of CNBD (2) when the block size k=6

ρ_1	ρ_2	$tr(C_d)$	$tr(C_{d^*})$	Eff(d)
-1	-1	-62,934.60	-31,462.80	0.4999
-0.8	-0.8	-17,131.43	-8,562.29	0.4998
-0.6	-0.6	-3,796.02	-1,895.43	0.4993
-0.4	-0.4	-623.20	-309.62	0.4968
-0.2	-0.2	-57.20	-26.98	0.4716
0	0	5.00	4.00	0.8
0.2	0.2	4.87	4.05	0.8329
0.4	0.4	2.28	3.12	1.3684
0.6	0.6	1.08	3.12	2.8888
0.8	0.8	1.32	4.08	3.0915
1	1	3.00	6.00	2

Now we present the efficiency of CNBD(2) corresponding to the optimal continuous block design for different block size.

Table 3 Efficiency of CNBD (2)

ρ_1	ρ_2	Block Size						
		5	6	7	8	9	10	11
-1	-1	0.5555	0.4999	0.5384	0.4999	0.4283	0.4998	0.4397
-0.8	-0.8	0.5554	0.4998	0.5382	0.4997	0.4279	0.4996	0.4392
-0.6	-0.6	0.5551	0.4993	0.5376	0.4989	0.4265	0.4985	0.4374
-0.4	-0.4	0.5533	0.4968	0.5343	0.4949	0.4189	0.4929	0.4277
-0.2	-0.2	0.5372	0.4717	0.4987	0.4469	0.3352	0.4113	0.2990
0	0	1.0000	0.8000	0.7778	0.7273	0.7895	0.7059	0.7476
0.2	0.2	0.8646	0.8329	0.8522	0.8311	0.8974	0.8305	0.8838
0.4	0.4	0.5556	1.3684	1.3000	1.3684	1.6957	1.3684	1.6250
0.6	0.6	2.0968	2.8889	2.2750	2.8889	6.1579	2.8889	5.1071
0.8	0.8	2.1789	3.0916	2.3801	3.0911	7.2867	3.0910	5.8441
1	1	1.6667	2.0000	1.7500	2.0000	3.0000	2.0000	2.7500

V. Conclusion

In this research paper, the Optimality and efficiency of the circular neighbor balanced design have been investigated by having the assumption that the errors in each block are correlated according to second order circular auto regressive process. Few results pertaining to the universal optimal designs have been proved for second order model. The traces of optimal sequence of treatments for different block size have been derived. Also the efficiency factor for CNBD (2) corresponding to the optimal continuous block design was calculated.

References

[1]. AZA IS, J.M., BAILEY, R.A. and MONOD, H., (1993). Acatalogue of efficient Neighbor- designs with border plots. Biometrics 49.

[2]. AI, M.-y., GE, G.-n.andCHAN, L.-y. (2007). Circular neighbor-balanced designs universally optimal for total effects.Sci. China Ser. A 50 821–828.MR2353065

[3]. AI, M., YU, Y. and HE, S. (2009). Optimality of circular neighbor-balanced designs fortotal effects with autoregressive correlated observations. J. Statist. Plann.Inference **139**2293–2304.MR2507991

[4]. BAILEY, R.A. and DRUILHET, P., (2004). Optimality of Neighbor-balanced designs for total effects. Ann. Statist. 32(4),1650-1661.

[5]. BOX, G.E.P., HUNTER, J.S. and HUNTER, W.G., (2005).Statistics for Experimenters. John Wiley & Sons, New Jersey.

[6]. CHENG, C.S. and WU, C.F.J.,(1980). Balanced repeated measurements designs. Ann. Statist. 8, 1272-1283.

[7]. D. RICHARD CUTLER.(1990) Efficient block designs for comparing testtreatments to a control when the errors are correlated. Journal of Statistical Planning and Inference 37 (1993) 393-412 North-Holland

[8]. DRUILHET,P., (1999). Optimality of Neighbor balanced designs. J. Statist. Plann.Inference 81,141-152.

[9]. FILIPIAK, K. and MARKIEWICZ, A., (2005). Optimality and efficiency of circular Neighbor balanced designs for correlated observations.Metrika61,17-27.

[10]. FILIPIAK, K. (2012). Universally optimal designs under an interference model with equal left- and right-neighbor effects. Statist.Probab.Lett. 82 592–598.MR2887476

[11]. HEDAYAT,A. and AFSARINEJAD, K., (1978). Repeated measurements design II. Ann. Statist. 6, 619 - 628.

[12]. KUNERT, J. and MARTIN, R.J., (2000). On the determination of optimal designs for an interference model. Ann. Statist.28,1728-1742.

[13]. KUNERT, J., (1984a). Design balanced for circular residual effects. Comm.Statist.A-TheoryMethods13, 2665-2671.

[14]. KUNERT,J., (1984b). Optimality of balanced uniform repeated measurements de-signs. Ann. Statist.12, 1006-1017.

[15]. KUSHNER, H.B., (1997). Optimal repeated measurements designs: The linear optimality equations. Ann. Statist.25,2328-2344.
[16]. MAGDA, C.D., (1980). Circular balanced repeated measurements design. Comm. Statist. A- Theory Methods 9, 1901-1918.
[17]. PIERRE DRUILHET and TINSSON WALTER (2013). Efficient Circular neighbor designs for spatial interference model. Journal Of Statistical Planning and Inference. 1161 – 1169.
[18]. SANTHARAM, C, PONNUSAMY, K.N., AND CHANDRASEKAR B., 1996: Universal Optimality of nearest Neighbor balanced block designs using ARMA models. Biometrical J. 38, 725 – 730.
[19]. SANTHARAM, C, and PONNUSAMY, K.N., 1997: On the efficiencies of nearest Neighbor balanced block design with correlated error structure. Biometrical J. 39, 85 – 98.