Decomposition of Line Graph into Paths and Cycles

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Abstract: Let \( P_{k+1} \) denote a path of length \( k \) and let \( C_k \) denote a cycle of length \( k \). As usual \( K_n \) denotes the complete graph on \( n \) vertices. In this paper we investigate decompositions of line graph of \( K_n \) into \( p \) copies of \( P_k \) and \( q \) copies of \( C_s \) for all possible values of \( p \geq 0 \) and \( q \geq 0 \).

Keywords: Path, Cycle, Graph Decomposition, Complete graph, Line graph.

I. Introduction

Unlike stated otherwise all graphs considered here are finite, simple, and undirected. For the standard graph-theoretic terminology the readers are referred to [7]. Let \( P_{k+1} \) denote a path of length \( k \) and let \( C_k \) denote a cycle of length \( k \). Let \( S_k \) denotes a star on \( k \) vertices, i.e. \( S_k = K_{1,k-1} \). As usual \( K_n \) denotes the complete graph on \( n \) vertices. Let \( K_{m,n} \) denote the complete bipartite graph with \( m \) and \( n \) vertices in the parts. If \( G = (V,E) \) is a simple graph then the line graph of \( G \) is the graph \( L(G) = (E,L) \), where \( L = \{ \{e,f\} | e,f \subseteq E, e \cap f \neq \emptyset \} \) = 1. For the sake of convenience, we use \( uw \) to denote an edge \{\( u,v \}\) and \( u,v \) are called the ends of the edge \{\( u,v \}\}. In general, if one allows more than one edge (but a finite number) between same pair of vertices, the resulting graph is called a multigraph. In particular, if \( G \) is a simple graph then for any \( \lambda \geq 1 \), \( G (\lambda) \) and \( \lambda G \) respectively denote the multigraph with edge-multiplicity \( \lambda \) and the disjoint union of \( \lambda \) copies of \( G \).

By a decomposition of a graph \( G \), we mean a list of edge-disjoint subgraphs \( H_1, \ldots, H_k \) of \( G \) whose union is \( G \) (ignoring isolated vertices). When each subgraph in a decomposition is isomorphic to \( H \), we say that \( G \) has an \( H \)-decomposition. It is easily seen that \( \sum e(H_m) = e(G) \) is one of the obvious necessary condition for the existence of a \( \{H_1, H_2, \ldots, H_k\} \) – decomposition of \( G \). A \( \{H_1, H_2\} \) – decomposition of \( G \) is a decomposition of \( G \) into copies of \( H_1 \) and \( H_2 \) using at least one of each. If \( G \) has a \( \{H_1, H_2\} \)-decomposition, we say that \( G \) is \( \{H_1, H_2\} \)-decomposable.

The problem of \( H \)-decomposition of \( K_n (\lambda) \) is the well-known Alspach’s conjecture [6] when \( H \) is any set of cycles of length at most \( n \) satisfying the necessary sum conditions and \( 2 \mid \lambda \) (n-1) . For the case \( \lambda = 1 \), Alspach conjecture is also stated for even values of \( n \), where in this case the cycles should decompose \( K_n \) minus a factor. There are many related results, but only special cases of this conjecture are solved completely. When \( H \) is a set of paths, in this case the problem of \( H \)-decomposition has been investigated by Tarsi [19] who showed that if \( (n-1) \lambda \) is even and \( H \) is any set of paths of length at most \( n - 3 \) satisfying the necessary sum condition, then \( K_n (\lambda) \) has an \( H \)-decomposition. The problem of \( H \)-decomposition of \( K_{m,n} (\lambda) \) has been investigated by Truszczyński [20] when \( m \) and \( n \) are even and \( H \) is any set of the paths with some constraints on length satisfying the necessary sum condition. It is natural to consider the problem of \( H \)-decomposition of \( K_n \), where \( H \) is a combination of paths, cycles, and some other subgraphs. We will restrict our attention to \( H \) which is any set of paths and cycles satisfying the necessary sum condition. There is several similarly known results as follows. A graph-pair of order \( t \) consists of two non-isomorphic graphs \( G \) and \( H \) on \( t \) non-isolated vertices for which \( G \cup H \) is isomorphic to \( K_{t} \).

Study on \( \{H_1, H_2\} \)-decomposition of graphs is not new. Abueva and Daven [1,3] completely determined the values of \( n \) for which \( K_n (\lambda) \) admits the \( \{H_1, H_2\} \)-decomposition such that \( H_1 \cup H_2 = K_n \), when \( \lambda \geq 1 \) and \( |V(H_1)| = |V(H_2)| = t \), where \( t \in \{4, 5\} \). Abuya and Daven [2] proved that there exists a \( \{K_k, S_{k+1}\} \)-decomposition of \( K_n \) for \( k \geq 3 \) and \( n \equiv 0 \) (mod \( k \)). Abueva and O’Neill [4] proved that for \( k \in \{3, 4, 5\} \), the \( \{C_k, S_k\} \)-decomposition of \( K_n (\lambda) \) exists, whenever \( n \geq k + 1 \) except for the ordered triples \( (k,n,\lambda) \in \{(3,4,1), (4,5,1), (5,6,1), (5,6,2), (5,6,4), (5,7,1), (5,8,1)\} \). Abueva and Daven [2] obtained necessary and sufficient conditions for the \( \{C_5, (2K_3)\} \)-decomposition of the Cartesian product and tensor product of paths, cycles, and complete graphs. Shyu [14] obtained a necessary and sufficient condition for the existence of a \( \{P_a, C_b\} \)-decomposition of \( K_n \). Shyu [15] proved that \( K_n \) has a \( \{P_a, S_b\} \)-decomposition if and only if \( n \geq 6 \) and \( 3(p+q) \geq 25 \). Also he proved that \( K_n \) has a \( \{P_a, S_b\} \)-decomposition with a restriction \( p \geq k/2 \), when \( k \) even (resp., \( p \geq k \), when \( k \) odd). Shyu [16] obtained a necessary and sufficient condition for the existence of a \( \{P_a, K_b\} \)-decomposition of \( K_n \). Shyu [17] proved that \( K_n \) has a \( \{C_4, S_3\} \)-decomposition if and only if \( 4(p+q) = \binom{a}{2} \).
$q \neq 1$, when $n$ is odd and $q \geq \max \{\frac{3n}{4}\}$, when $n$ is even. Shyu [18] proved that $K_{m,n}$ has a $\{P_s, S_k\}$-decomposition for some $m$ and $n$ and also obtained some necessary and sufficient condition for the existence of a $\{P_s, S_k\}$-decomposition of $K_{m,n}$. Sarvate and Zhang [13] obtained necessary and sufficient conditions for the existence of a $\{pP_q, qK_t\}$-decomposition of $K_n(\lambda)$, when $p = q$.

Chou et al. [8] proved that for a given triple $(p,q,r)$ of nonnegative integers, $G$ decompose into $p$ copies of $C_4$, $q$ copies of $C_6$, and $r$ copies of $C_8$ such that $4p+6q+8r = |E(G)|$ in the following two cases: (a) $G = K_m,n$ with $m$ and $n$ both even and greater than four (b) $G = K_m,n - I$, where $n$ is odd. Chou and Fu [9] proved that the existence of a $\{C_m,C_{2k}\}$-decomposition of $K_{p,2n}$, where $t/2 \leq u,v < t$, when $t$ even (resp., $t + 1)/2 \leq u,v \leq (3t - 1)/2$, when $t$ odd) implies such decomposition in $K_{m,n}$, when $m,n \geq t$ (resp., $m,n \geq (3t + 1)/2$). Lee and Chu [10,11] obtained a necessary and sufficient condition for the existence of a $\{P_s, S_k\}$-decomposition of $K_{m,n}$ and $K_{m,n}$. Lee and Lin [12] obtained a necessary and sufficient condition for the existence of a $\{C_k, S_{k+1}\}$-decomposition of $K_{m,n}$ when $n = 1$. Abueida and Lian [5] obtained necessary and sufficient conditions for the existence of a $\{C_k, S_{k+1}\}$-decomposition of $K_n$ for some $n$.

In this paper we investigate decompositions of line graph of $K_n$ into $p$ copies of $P_5$ and $q$ copies of $C_4$ for all possible values of $p \geq 0$ and $q \geq 0$.

**II. $\{P_5,C_4\}$-decomposition of $L(K_n)$**

In this section, we investigate the existence of $\{P_5, C_4\}$-decomposition of $L(K_n)$.

**III. Construction**

Let $C_4^A$ and $C_4^B$ be two cycles of length 4, where $C_4^A = \{abca\}$ and $C_4^B = \{wxyz\}$. If $v$ is a only common vertex of $C_4^A$ and $C_4^B$, say $c = y = v$, then we have two paths of length 4 as follows: $abvw, wxvd$.

**Lemma 2.1.** There exists a $\{P_5, C_4\}$-decomposition of $K_{4,4}$.

**Proof.** Let $V(K_{4,4}) = \{x_1, x_2, x_3, x_4\} \cup \{y_1, y_2, y_3, y_4\}$. We exhibit the $\{P_5, C_4\}$-decomposition of $K_{4,4}$ as follows:

1. $p = 0$ and $q = 4$. The required cycles are
   - $(x_1y_1x_2y_2x_1)$, $(x_1y_1x_2y_2x_3)$, $(x_2y_1x_3y_2x_3)$.
2. $p = 2$ and $q = 2$. The required paths and cycles are
   - $(x_1y_1x_2y_2y_1)$, $(x_2y_1x_3y_2y_3)$, $(x_3y_1x_4y_2x_3)$, $(x_4y_1x_3x_2y_3)$.
3. $p = 3$ and $q = 1$. The required paths and cycles are
   - $(x_2y_1x_3y_2y_1)$, $(x_3y_1x_4y_2y_3)$, $(x_4y_1x_3y_2y_3)$, $(x_3y_1x_4x_2y_3)$.
4. $p = 4$ and $q = 0$. The required paths are
   - $(x_1y_1x_2y_2y_4)$, $(x_2y_1x_3y_2y_1)$, $(x_3y_1x_4y_2y_4)$, $(x_4y_1x_3y_2y_3)$.

**Lemma 2.2.** There exists a $\{P_5, C_4\}$-decomposition of the complete bipartite graph $K_{8,8m}$ for all $m \geq 1$.

**Proof.** Let $V(K_{8,8m}) = \{X, Y\}$ with $X = \{x_1, \ldots, x_8\}$ and $Y = \{y_1, \ldots, y_{8m}\}$. Partition $(X, Y)$ into 4-subsets $\{A_1^X, A_2^X\}$ such that $\bigcup_{i=1}^{m} A_1^X = X$, $\bigcup_{j=1}^{m} A_2^Y = Y$. Then $G[\{A_1^X, A_2^Y\}] \cong K_{4,4}$. Thus $K_{8,8m} = 4m(K_{4,4})$. By Lemma 2.1, the graph $K_{4,4}$ has a $\{P_5, C_4\}$-decomposition. Hence the graph $K_{8,8m}$ has the desired decomposition.

**Lemma 2.3.** There exists a $\{P_5, C_4\}$-decomposition of the graph $K_{8m} - F$ for all $m \geq 1$, where $F$ is a 1-factor of $K_{8m}$.

**Proof.** Let $8m = 4k$, where $k$ is a positive integer and $V(K_{4k} - I) = \{x_0, \ldots, x_{4k-1}\}$, where $I = \{x_2, x_{2i+1} : 0 \leq i \leq 2k-1\}$ and $AI = \{x_2, x_{2i+1}\}$, $0 \leq i \leq 2k-1$. We obtain a new graph $G$ from $K_{4k} - I$, by identifying each set $Ai$ with a single vertex $ai$; join two vertices $ai$ and $aj$ by an edge if the corresponding sets $Ai$ and $Aj$ form a $\{K[A]_i, A_j\}$ in $K_{4k}$. Then the graph $G = K_{2k}$. The graph $K_{2k}$ has a Hamilton path decomposition $\{G_0, \ldots, G_k\}$, where each $G_i$, $0 \leq i \leq k$ is a Hamilton path. Each $G_i$ decomposes some copies of $P_5$ or $P_3$. When we go back to $K_{4k} - I$, each $P_3$ (resp., $P_5$) of $G_i$, $0 \leq i \leq k$ will give rise to $2C_4$ or $2P_5$ (resp., $2P_5$, $1C_4$) or $3P_3$ in $K_{4k} - I$. Hence the graph $K_{8m} - I$ has the desired decomposition.

**Lemma 2.4.** There exist a $\{P_5, C_4\}$-decomposition of the graph $L(K_9)$.
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Proof. For $0 \leq k \leq 8$, by Lemma 2.3 there exist a $[P_5,C_4]$-decomposition $G_i \cong K_8 - F_i$, where $G_i$ defined on the vertex set $\{i,j\}$ with $j \leq 8$, $j \neq i$ and where $F_i = \{\{i, 1+i\}, \{i, 2+i\}, \{i, 3+i\}, \{i, 7+i\}\} \cup \{\{i, 4+i\}, \{i, 8+i\}, \{i, 5+i\}, \{i, 6+i\}\}$, reducing all sums modulo 9. Then a $[P_3,C_4]$-decomposition of $L(K_8)$ on the vertex set $\{i,j\} | (i,j) \subseteq \{0,1,\ldots,8\}$ follows from the partition of the edges of $U_{j=0}^{8}$ into $P_3$ or $C_4$, $\{(j,j+1), (j+1,5+j), (5+j,2+j), (2+j,j)\} | 0 \leq j \leq 8$, again reducing all sums modulo 9.

Theorem 2.1. If $n \equiv 1 \pmod{8}$ then there exists a $[P_3,C_4]$-decomposition of $L(K_n)$.

Proof. The result is true for $n = 9$, so we proceed by induction on $n$. Let $n = 8m + 1$, $m \geq 2$, and let the vertex set of $K_n$ be $\{\omega\} \cup \{1, \ldots, 8m\}$. $L(K_n)$ can be partitioned into the following edge-disjoint subgraphs:

1. $L(K_{8m}), K_3$ being defined on the vertex set $\{\omega\} \cup \{1, \ldots, 8\}$;
2. $L(K_{8m}, 0), K_{8m,7}$ being defined on the vertex set $\{\omega\} \cup \{9, \ldots, 8m\}$;
3. For $1 \leq i \leq 8$, $G_i \cong K_{8m,8} - F_i$ on the vertex set $\{i, j\} | 9 \leq j \leq 8m$, where $F_i = \{\{i, 2k-1\}, \{i, 2k\} | 5 \leq k \leq 4m$;
4. For $9 \leq j \leq 8m, G_i \cong K_8 - F_j$ on the vertex set $\{i,j\} | 1 \leq i \leq 8$, where $F_j = \{\{2k-1,j\}, \{2k,j\} | 1 \leq k \leq 4$;
5. $\bigcup_{j=1}^{8} (F_j) \cup \bigcup_{j=0}^{8m} (F_j)$;
6. $K_{8m,n}$ with $b$-partition $\{\{\omega\}, i\} | 1 \leq i \leq 8$ and $\{\{\omega\}, j\} | 9 \leq j \leq 8m$;
7. For $1 \leq i \leq 8, H_i \cong K_{8,8m,8}$ with bi-partition $\{\{i,k\} | k \in \{\omega, 1, 2, \ldots, 8\}\ \{\{i\}\}$ and $\{\{i,j\} | 9 \leq j \leq 8m$; and
8. $9 \leq j \leq 8m, H_j \cong K_{8,8m,8}$ with bi-partition $\{\{i,j\} | 1 \leq i \leq 8$ and $\{\{k,j\} | k \in \{\omega, 9, 10, \ldots, 8m\}\ \{\{j\}\}$.

The result now follows, since there exist $[P_3,C_4]$-decomposition of graphs defined in (1) and (2) by induction and Lemma 2.4, (3) and (4) by Lemma 2.3, (5) since these edges form vertex-disjoint $C_4$, and (6),(7) and (8) by Lemma 2.2. Hence the graph $L(K_n)$ has the desired decomposition.

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