Co – Isolated Locating Domination Number For Unicyclic Graphs

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Abstract: Let G (V, E) be a simple, finite, undirected connected graph. A non – empty set S ⊆ V of a graph G is a dominating set, if every vertex in V – S is adjacent to at least one vertex in S. A dominating set S ⊆ V is called a locating dominating set, if for any two vertices v, w ∈ V – S, N(v) ∩ S ≠ N(w) ∩ S. A locating dominating set S ⊆ V is called a co – isolated locating dominating set, if there exists at least one isolated vertex in < V – S >. The co – isolated locating domination number γcid is the minimum cardinality of a co – isolated locating dominating set. In this paper, the number γcid is obtained for unicyclic graphs.

Keywords: Dominating set, locating dominating set, co – isolated locating dominating set, co – isolated locating domination number.

I. Introduction

Let G = (V, E) be a simple graph of order p. For v ∈ V(G), the neighborhood N_G(v) (or simply N(v)) of v is the set of all vertices adjacent to v in G. For a connected graph G, the eccentricity e_G(v) of a vertex v in G is the distance to a vertex farthest from v. Thus, e_G(v) = {d_G(u, v) : u ∈ V(G)}, where d_G(u, v) is the distance between u and v in G. The minimum and maximum eccentricities are the radius and diameter of G, denoted r(G) and diam(G) respectively. A pendant vertex in a graph G is a degree of vertex one and a vertex is called a support if it is adjacent to a pendant vertex. A unicyclic graph G is a graph with exactly one cycle. The concept of domination in graphs was introduced by Ore [1]. A non – empty set S ⊆ V(G) of a graph G is a dominating set, if every vertex in V(G) – S is adjacent to some vertex in S. A special case of dominating set S is a locating dominating set. It was defined by D. F. Rall and P. J. Slater in [2]. A dominating set S in a graph G is called a locating dominating set in G, if for any two vertices v, w ∈ V(G) – S, N_G(v) ∩ S, N_G(w) ∩ S are distinct. The locating dominating number of G is defined as the minimum number of vertices in a locating dominating set in G. A locating dominating set S ⊆ V(G) is called a co – isolated locating dominating set, if < V – S > contains at most one isolated vertex. The minimum cardinality of a co – isolated locating dominating set is called the co – isolated locating domination number γcid(G). In this paper, the unicyclic graphs having co – isolated locating domination number γcid(G) = 3, 4, and 5 are characterized.

II. Prior Results

The following results are obtained in [3], [4], [5] & [6]

Theorem 2.1[3]:
For every non – trivial simple connected graph G with p vertices, 1 ≤ γcid(G) ≤ p - 1.

Theorem 2.2[3]:
γcid(G) = 1 if and only if G = K_2.

Observation 2.3 [3]:
If S is a co – isolated locating dominating set of G(V, E) with | S | = k, then V(G) – S contains at most pC_1 + pC_2 + ... + pC_k vertices.

Theorem 2.4 [3]:
γcid(G) = p – 1(p ≥ 4) if and only if V(G) can be partitioned into two sets X and Y such that one of the sets X and Y say, Y is independent and each vertex in Y and the subgraph < X > of G induced by X is one of the following
(a) < X > is a complete graph
(b) < X > is totally disconnected
(c) Any two non – adjacent vertices in V(< X >) have common neighbours in < X >.

Theorem 2.5 [4]:
γcid(G) = 2 if and only if G is one of the following graphs
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(a) $P_p$ ($p = 3, 4, 5$), where $P_p$ is a path on $p$ vertices.
(b) $C_p$ ($p = 3, 5$), where $C_p$ is a path on $p$ vertices.
(c) $C_5$ with a chord.
(d) $G$ is the graph obtained by attaching a pendant edge at a vertex of $C_3$ (or) at a vertex of degree 2 in $K_4 - e$.
(e) $G$ is the graph obtained by attaching a path of length 2 at a vertex of $C_3$.
(f) $G$ is the Bull graph.

**Theorem 2.6 [5]:**

For a path $P_p$ on $p$ vertices,

$$\gamma_{cld}(P_p) = \frac{2p+4}{5}, \quad p \geq 3.$$ 

**Theorem 2.7 [6]:**

If $C_p$ ($p \geq 3$) is a cycle on $p$ vertices, then

$$\gamma_{cld}(C_p) \leq \frac{2p}{5}.$$ 

**III. Main Results**

In the following, the unicyclic graphs having co-isolated locating domination number $\gamma_{cld}(G) = 3, 4$ and 5 are characterized.

**Notations 3.1:**

1. $C_p \oplus P_1$ is a graph obtained by attaching a path of length $k$ at exactly one vertex of $C_p$.

**Example 3.1.1:** The graph $G \cong C_4 \oplus P_3$ is given in Fig. 3.1.

2. $C_p \oplus P_{k_1} \oplus P_{k_2}$ is a graph obtained by attaching paths of length $k_1$ and $k_2$ respectively at vertices $u$ and $v$ of $C_p$ such that $d(u, v) = r$.

**Example 3.1.2:** The graph $G \cong C_5 \oplus P_3 \oplus P_2$ is given in Fig. 3.2.

3. $C_p \oplus P_{k_1} \oplus P_{k_2} \oplus P_{k_3}$ is a graph obtained by attaching paths of length $k_1$, $k_2$ and $k_3$ respectively at vertices $u$, $v$ and of $C_p$ such that $d(u, v) = r; d(v, w) = s$.

**Example 3.1.3:** The graph $G \cong C_8 \oplus P_2 \oplus P_3 \oplus P_3$ is given in Fig. 3.3.

4. $C_p \oplus P_{k_1} \oplus P_{k_2} \oplus P_{k_3} \oplus P_{k_4}$ ($n \geq 4$) is a graph obtained by attaching paths of length $k_1$, $k_2$, $k_3$ and $k_4$ respectively at vertices $u$, $v$, $w$ and $x$ on $C_p$ such that $d(u, v) = q; d(v, w) = r; d(w, x) = s$.

**Example 3.1.4:** The graph $G \cong C_8 \oplus P_2 \oplus P_4 \oplus P_3 \oplus P_1$ is given in Fig. 3.4.

5. $C_p \oplus P_1$ is a graph obtained by attaching a support of a path of length $k$ at a vertex of $C_p$.

**Example 3.1.5:** The graph $G \cong C_4 \oplus P_3$ is given in Fig. 3.5.
6. $C_p @ P_k$ is a graph obtained by attaching the central vertex of a path of length $k$ ($k$ is even) at a vertex of $C_p$.

Example 3.1.6: The graph $G \cong C_8 @ P_4$ is given in Fig. 3.6.

7. $C_p @ P_1 @ P_k$ is a graph obtained by attaching a path of length one at a vertex of $C_p$ and then attaching a support of a path of length $k$ to the pendant vertex of $P_1$.

Example 3.1.7: The graph $G \cong C_5 @ P_1 @ P_k$ is given in Fig. 3.7.

8. $C_p @ \left( \frac{P_1}{P_k @ P_1} \right)$ is a graph obtained by attaching a path of length 1 and also a path of length $k_1$ at a vertex of $C_p$ and then attaching a support of path of length $k$ at a pendant vertex of the path $P_{k_1}$.

Example 3.1.8: The graph $G \cong C_5 @ \left( \frac{P_1}{P_1 @ P_3} \right)$ is given in Fig. 3.8.

Example 3.1.9: $G \cong C_5 @ \left( \frac{P_1}{P_2 @ P_4} \right)$ is given in Fig. 3.9.

9. A graph can also be obtained by performing the combinations of the above operations.

Example 3.1.10: The graphs $G \cong C_6 @ P_1 @ P_2 P_1 @ P_3$ and $G \cong C_4 @ \left( \frac{P_1}{P_2 @ P_6} \right) @ P_2$ are given in Fig. 3.10.

Theorem 3.2:
For a connected unicyclic graph $G$, $\gamma_{\text{cldf}}(G) = 2$ if and only if $G$ is one of the graphs in the family $\mathcal{A}$, where $\mathcal{A} = \{ C_3, C_5, C_3 @ P_1, C_3 @ P_2 \}$.

Proof:
If $G$ is one of the graphs of $\mathcal{A}$, then $\gamma_{\text{cldf}}(G) = 2$. 

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Conversely, assume that $\gamma_{cd}(G) = 2$. Let $S = \{a, b\}$ be a $\gamma_{cd}$-set of $G$ with $|S| = 2$. Then $|V - S| \leq 2^2 - 1 = 3$ and $<V - S>$ contains at least one isolated vertex.

**Case (1):** $|V - S| = 1$

If $<S> \cong 2K_1$, then $G$ is not unicyclic.

If $<S> \cong K_2$, then $G \cong C_3$.

**Case (2):** $|V - S| = 2$

Let $V - S = \{x_1, x_2\}$. Then $<V - S> \cong 2K_1$.

If $N(x_1) \cap S = \{a, b\} \cap S$, then $G \cong C_3 \oplus P_1$.

In all the other cases, $G$ is not unicyclic.

**Case (3):** $|V - S| = 3$

Let $V - S = \{x_1, x_2, x_3\}$ and $N(x_1) \cap S = \{a\}; N(x_2) \cap S = \{b\}$ and $N(x_2) \cap S = \{a, b\}$.

**Subcase (3.a):** $x_3x_3 \in <V - S>$ and $x_3$ is isolated in $<V - S>$

If $ab \notin E(G)$, then $G \cong C_3 \oplus P_2$ and if $ab \in E(G)$, then $G$ is not unicyclic.

**Subcase (3.b):** $x_1, x_2$ and $x_3$ are all isolated in $<V - S>$.

If $ab \notin E(G)$, then $G \cong C_3$ and if $ab \in E(G)$, then $G$ is not unicyclic.

Hence the theorem follows.

**Notation 3.3:**

The family of graphs $\mathcal{B} = \{B_1, B_2, ..., B_8\}$ are defined as follows, where

$B_1 = \{B_{1,1}, B_{1,2}, ..., B_{1,5}, B_{1,6}\} = \{C_3, C_6 \oplus P_1, C_7, C_6 \oplus P_2, C_7 \oplus P_1, C_6 \oplus P_3\}$

$B_2 = \{B_{2,1}, B_{2,2}, B_{2,3}, B_{2,4}\} = \{C_3 \oplus P_1, C_3 \oplus P_2, C_3 \oplus P_3, C_3 \oplus P_1 \oplus P_1\}$

$B_3 = \{B_{3,1}, B_{3,2}, ..., B_{3,6}\} = \{C_4, C_4 \oplus P_1, C_4 \oplus P_2, C_4 \oplus P_1 \oplus P_2, C_4 \oplus P_1 \oplus P_1\}$

$B_4 = \{B_{4,1}\} = \{C_3 \oplus P_1 \oplus P_1, C_3 \oplus P_1\}$

$B_5 = \{B_{5,1}, B_{5,2}, ..., B_{5,7}\} = \{C_4 \oplus P_3, C_3 \oplus P_1 \oplus P_3, C_3 \oplus P_2 \oplus P_2, C_3 \oplus P_3\}$

$B_6 = \{B_{6,1}, B_{6,2}\} = \{C_4 \oplus P_3, C_4 \oplus P_1\}$

$B_7 = \{B_{7,1}, B_{7,2}, ..., B_{7,7}\} = \{C_4 \oplus P_3, C_4 \oplus P_1, C_4 \oplus P_2, C_4 \oplus P_1 \oplus P_2\}$

$B_8 = \{B_{8,1}, B_{8,2}, B_{8,3}\} = \{C_3 \oplus P_1 \oplus P_2, C_3 \oplus P_1 \oplus P_3, C_3 \oplus P_1 \oplus P_3\}$

$B_9 = \{B_{9,1}, B_{9,2}, B_{9,3}\} = \{C_4 \oplus P_2, C_3 \oplus P_3, C_3 \oplus P_1 \oplus P_3\}$

**Theorem 3.4:**

For a connected unicyclic graph $G$, $\gamma_{cd}(G) = 3$ if and only if $G$ is one of the graphs in the family $\mathcal{B}$.

**Proof:**

If $G$ is one of the graphs in the family $\mathcal{B}$, then $\gamma_{cd}(G) = 3$.

Conversely, let $S$ be a $\gamma_{cd}$-set of a unicyclic graph $G$ with $|S| = 3$ and therefore $|V - S| \leq 3^3 - 1 = 7$.

**Case (1):** All the vertices of $S$ lie on the cycle.

Then $<S> \cong 3K_1 \cup K_2 \cup P_3$ (or) $C_3$.

**Subcase (1.a):** $<S> \cong 3K_1$

Since all the vertices of $S$ lie on the cycle and $<S> \cong 3K_1$, the cycle in this case is $C_6$ or $C_7$. Hence, $3 \leq |V - S| \leq 6$. 
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(i) \([V - S] = 3\)
If \(<V - S> \cong 3K_1\), then \(G \cong B_{1,1}\)
If \(<V - S> \cong K_1 \cup K_2\), then \(G\) is not unicyclic.
(ii) \([V - S] = 4\)
If \(<V - S> \cong 4K_1\), then \(G \cong B_{1,2}\)
If \(<V - S> \cong 2K_1 \cup K_2\), then \(G \cong B_{1,3}\).
(iii) \([V - S] = 5\)
If \(<V - S> \cong 5K_1\), then \(G \cong B_{1,4}\).
If \(<V - S> \cong 3K_1 \cup K_2\), then \(G \cong B_{1,5}\).
(iv) \([V - S] = 6\)
If \(<V - S> \cong 6K_1\), then \(G \cong B_{1,6}\).
If \(<V - S> \cong 4K_1 \cup K_2\), then \(G \cong C_7 \oplus P_2\) and for this graph \(\gamma_{cld}(G) = 4\).
(v) \([V - S] = 7\), then either \(G\) is not unicyclic or \(S\) will not be a \(\gamma_{cld}\) set of \(G\).

Subcase(1.b): \(<S> \cong K_1 \cup K_2\)
The cycle in this case is \(C_3\) (or) \(C_6\).
(i) \([V - S] = 3\)
If \(<V - S> \cong 3K_1\), then \(G \cong B_{2,1}\).
If \(<V - S> \cong K_1 \cup K_2\), then \(G \cong B_{1,1}\).
(ii) \([V - S] = 4\)
If \(<V - S> \cong 4K_1\), then \(G \cong B_{2,2}\) (or) \(B_{2,3}\).
If \(<V - S> \cong 2K_1 \cup K_2\), then \(G \cong B_{1,2}\).
(iii) \([V - S] = 5\)
If \(<V - S> \cong 5K_1\), then \(G \cong B_{2,4}\).
If \(<V - S> \cong 3K_1 \cup K_2\), then \(S\) will not be a \(\gamma_{cld}\) set of \(G\).
(iv) \([V - S] = 6\) (or) \(7\), then \(G\) is not unicyclic.

Subcase(1.c): \(<S> \cong P_1\)
The cycle in this case is \(C_3\) (or) \(C_4\).
(i) \([V - S] = 1\)
If \(<V - S> \cong K_1\), then \(G \cong B_{3,1}\).
(ii) \([V - S] = 2\)
If \(<V - S> \cong 2K_1\), then \(G \cong B_{3,2}\).
(iii) \([V - S] = 3\)
If \(<V - S> \cong 3K_1\), then \(G \cong B_{3,3} \cup B_{3,4}\) (or) \(B_{3,5}\).
If \(<V - S> \cong K_1 \cup K_2\), then \(G \cong B_{2,1}\).
(v) \([V - S] = 4\)
If \(<V - S> \cong 4K_1\), then \(G \cong B_{3,6}\).
If \(<V - S> \cong 2K_1 \cup K_2\), then \(G\) is not unicyclic.
(vi) \([V - S] = 5\) (or) \(6\) (or) \(7\), then \(G\) is not unicyclic.

Subcase(1.d): \(<S> \cong C_3\)
If \([V - S] = 1\) (or) \(2\), then \(S\) will not be a \(\gamma_{cld}\) set of \(G\). If \([V - S] > 3\), then \(G\) is not unicyclic. Hence \([V - S] = 3\).
If \(<V - S> \cong 3K_1\), then \(G \cong B_{4,1}\).

Case(2): One vertex of \(S\) lie on the cycle and the other two vertices does not lie on the cycle.
The only cycle with this property is \(C_3\). Also, \(<S> \cong 3K_1\) (or) \(K_1 \cup K_2\).

Subcase(2.a): \(<S> \cong 3K_1\)
Then \(<V - S>\) must contain \(K_2\) to form \(C_3\). Also, \(<V - S>\) must have atleast one isolated vertex. Therefore \([V - S] \geq 3\).
(i) \([V - S] = 3\)
If \( <V - S> \cong K_1 \cup K_2 \), then \( G \cong B_{5,1} \).

(ii) \( |V - S| = 4 \)
If \( <V - S> \cong 2K_1 \cup K_2 \), then \( G \cong B_{5,2} \) (or) \( B_{5,3} \) (or) \( B_{5,4} \).

(iii) \( |V - S| = 5 \)
If \( <V - S> \cong 3K_1 \cup K_2 \), then \( G \cong B_{5,5} \) (or) \( B_{5,6} \) (or) \( B_{5,7} \).

(iv) \( |V - S| = 6 \) (or) 7, then \( G \) is not unicyclic.

**Subcase(2.b):** \( <S> \cong K_1 \cup K_2 \)
By a similar argument as in Subcase(2.a), \( |V - S| \geq 3 \).

(i) \( |V - S| = 3 \)
If \( <V - S> \cong K_1 \cup K_2 \), then \( G \cong B_{6,1} \).

(ii) \( |V - S| = 4 \)
If \( <V - S> \cong 2K_1 \cup K_2 \), then \( G \) is not unicyclic.

(iii) \( |V - S| = 5 \) (or) 6 (or) 7, then \( G \) is not unicyclic.

**Case(3):** Two vertices of \( S \) lie on the cycle and the other vertex does not lie on the cycle.
In this case, \( <S> \cong 3K_1 \) (or) \( K_1 \cup K_2 \) (or) \( P_3 \).

**Subcase(3.a):** \( <S> \cong 3K_1 \)

(i) \( |V - S| = 3 \)
If \( <V - S> \cong 3K_1 \), then \( G \cong B_{7,1} \).
If \( <V - S> \cong K_1 \cup K_2 \), then is not unicyclic.

(ii) \( |V - S| = 4 \)
If \( <V - S> \cong 4K_1 \), then \( G \cong B_{7,2} \).
If \( <V - S> \cong 2K_1 \cup K_2 \), then \( G \cong B_{7,3} \) (or) \( B_{7,4} \).

(iii) \( |V - S| = 5 \)
If \( <V - S> \cong 5K_1 \), then \( G \cong B_{7,5} \).
If \( <V - S> \cong 3K_1 \cup K_2 \), then \( G \cong B_{7,6} \) (or) \( B_{7,7} \).

(iv) \( |V - S| = 6 \) (or) 7, then \( G \) is not unicyclic.

**Subcase(3.b):** \( <S> \cong K_1 \cup K_2 \)
If \( <V - S> \) contains \( K_2 \), then \( G \) is not unicyclic. The only cycle in this case is \( C_3 \). If \( |V - S| = 1 \) (or) 2, then S will not be a \( \gamma_{cd} \) set of \( G \).

(i) \( |V - S| = 3 \)
If \( <V - S> \cong 3K_1 \), then \( G \cong B_{8,1} \).

(ii) \( |V - S| = 4 \)
If \( <V - S> \cong 4K_1 \), then \( G \cong B_{8,2} \) (or) \( B_{8,3} \).

(iii) \( |V - S| = 5 \) (or) 6 (or) 7, then \( G \) is not unicyclic.

**Subcase(3.c):** \( <S> \cong P_3 \)
The only cycle in this case is \( C_3 \).

(i) \( |V - S| = 1 \)
If \( <V - S> \cong K_1 \), then \( G \) is not unicyclic

(ii) \( |V - S| = 2 \)
If \( <V - S> \cong 2K_1 \), then \( G \cong B_{9,1} \)

(iii) \( |V - S| = 3 \)
If \( <V - S> \cong 3K_1 \), then \( G \cong B_{9,2} \) (or) \( B_{9,3} \)

(iv) \( |V - S| = 4 \)
If \( <V - S> \cong 4K_1 \), then \( G \cong B_{9,3} \)

(v) \( |V - S| = 5 \) (or) 6 (or) 7, then \( G \) is not unicyclic.

Hence the theorem follows.
Notation 3.5:  
The family of graphs \( C = \{ C_1, C_2 \} \) are defined as follows, where  
\( C_1 = \{ C_{1,1}, C_{1,2}, \ldots, C_{1,44}, C_{1,45} \} \); and  
\( C_2 = \{ C_{2,1}, C_{2,2}, \ldots, C_{2,11}, C_{2,12} \} \).

\[
\begin{align*}
C_{1,1} &= C_1 \oplus P_2 \oplus P_3, & C_{1,20} &= C_2 \oplus P_1 \oplus P_1 \oplus P_1 \oplus \epsilon_1 P_3, & C_{1,39} &= C_4 \oplus P_1 \oplus \epsilon_3 P_3 \oplus P_3, \\
C_{1,2} &= C_1 \oplus P_1 \oplus P_1 \oplus P_3, & C_{1,21} &= C_1 \oplus P_1 \oplus P_2 \oplus P_3, & C_{1,40} &= C_5 \oplus P_1 \oplus P_2 \oplus P_4, \\
C_{1,3} &= C_2 \oplus P_1 \oplus P_4, & C_{1,22} &= C_2 \oplus P_1 \oplus P_2 \oplus P_3, & C_{1,41} &= C_2 \oplus P_1 \oplus P_2 \oplus \epsilon_1 P_3, \\
C_{1,4} &= C_3 \oplus P_1 \oplus P_1 \oplus \epsilon_3 P_3, & C_{1,23} &= C_3 \oplus P_1 \oplus \epsilon_1 P_4, & C_{1,42} &= C_3 \oplus P_2 \oplus \epsilon_2 P_4 \oplus P_2, \\
C_{1,5} &= C_4 \oplus P_1 \oplus P_1 \oplus P_3, & C_{1,24} &= C_4 \oplus P_1 \oplus \epsilon_3 P_4, & C_{1,43} &= C_5 \oplus \epsilon_1 P_1 \oplus \epsilon_3 P_4, \\
C_{1,6} &= C_3 \oplus P_1 \oplus P_3, & C_{1,25} &= C_3 \oplus P_1 \oplus P_1 \oplus \epsilon_3 P_4, & C_{1,44} &= C_4 \oplus P_3 \oplus P_5, \\
C_{1,7} &= C_2 \oplus P_1 \oplus P_2 \oplus P_3, & C_{1,26} &= C_4 \oplus P_1 \oplus P_1 \oplus \epsilon_3 P_4, & C_{1,7} &= C_5 \oplus P_1 \oplus P_1 \oplus P_2, \\
C_{1,8} &= C_2 \oplus P_1 \oplus P_3, & C_{1,27} &= C_4 \oplus P_1 \oplus P_2 \oplus P_4, & C_{2,2} &= C_5 \oplus P_1 \oplus P_1 \oplus P_2, \\
C_{1,9} &= C_2 \oplus P_1 \oplus P_1 \oplus P_1, & C_{1,28} &= C_4 \oplus P_1 \oplus P_2 \oplus P_3, & C_{2,3} &= C_5 \oplus P_1 \oplus P_2 \oplus P_2, \\
C_{1,10} &= C_5 \oplus P_1 \oplus P_1 \oplus P_2, & C_{1,29} &= C_5 \oplus P_1 \oplus P_2 \oplus \epsilon_3 P_3 \oplus P_2, & C_{2,4} &= C_5 \oplus P_1 \oplus P_1 \oplus P_2, \\
C_{1,11} &= C_2 \oplus P_1 \oplus P_4, & C_{1,30} &= C_5 \oplus P_1 \oplus P_2 \oplus P_3, & C_{2,5} &= C_5 \oplus P_1 \oplus \epsilon_3 P_3 \oplus P_2, \\
C_{1,12} &= C_2 \oplus P_1 \oplus P_1, & C_{1,31} &= C_5 \oplus P_1 \oplus P_2 \oplus P_3, & C_{2,6} &= C_5 \oplus P_1 \oplus P_1 \oplus P_2, \\
C_{1,13} &= C_4 \oplus P_1 \oplus P_2, & C_{1,32} &= C_5 \oplus P_1 \oplus P_2 \oplus P_3, & C_{2,7} &= C_6 \oplus P_1 \oplus P_1 \oplus P_2, \\
C_{1,14} &= C_2 \oplus P_1 \oplus P_1 \oplus \epsilon_3 P_3, & C_{1,33} &= C_5 \oplus P_1 \oplus \epsilon_3 P_4, & C_{2,8} &= C_5 \oplus P_2 \oplus P_2, \\
C_{1,15} &= C_2 \oplus P_1 \oplus P_1 \oplus P_4, & C_{1,34} &= C_5 \oplus P_1 \oplus \epsilon_3 P_4, & C_{2,9} &= C_1 \oplus P_2 \oplus P_2, \\
C_{1,16} &= C_3 \oplus P_1 \oplus P_2, & C_{1,35} &= C_5 \oplus P_1 \oplus P_2, & C_{2,10} &= C_1 \oplus P_1 \oplus \epsilon_3 P_2 \oplus P_2, \\
C_{1,17} &= C_3 \oplus P_1 \oplus P_1 \oplus P_4, & C_{1,36} &= C_3 \oplus P_1 \oplus P_1 \oplus \epsilon_3 P_4, & C_{2,11} &= C_3 \oplus P_4 \oplus P_2, \\
C_{1,18} &= C_3 \oplus P_1 \oplus \epsilon_3 P_2 \oplus P_2, & C_{1,37} &= C_3 \oplus P_1 \oplus P_3, & C_{2,12} &= C_3 \oplus P_1 \oplus \epsilon_3 P_2 \oplus P_3, \\
C_{1,19} &= C_3 \oplus P_1 \oplus P_3, & C_{1,38} &= C_3 \oplus P_2 \oplus P_3, & C_{2,13} &= C_3 \oplus P_1 \oplus \epsilon_3 P_2 \oplus P_3.
\end{align*}
\]

Theorem 3.6:  
Let \( G \) be a connected unicyclic graph in which one vertex of a \( \gamma_{cd}(G) \) set lies on the cycle. Then \( \gamma_{cd}(G) = 4 \) if and only if \( G \) is one of the graphs in the family \( C \).

Proof:  
If \( G \) is one of the graphs in the family \( C \), then \( \gamma_{cd}(G) = 4 \). Conversely, let \( S \) be a \( \gamma_{cd} \) set of the unicyclic graph \( G \) with \( |S| = 4 \) and therefore \( |V - S| \leq 2^4 - 1 = 15 \) and \( \gamma_{cd}(G) = 4 \).

Therefore, \( |V - S| = 7 \).

Case (1):  
\( |S| \equiv 4K_1 \).

Then \( V - S \) must contain \( K_2 \). Since \( V - S \) contains at least one isolated vertex, \( |V - S| \geq 3 \).

Subcase (1.a):  
\( |S - V| = 3 \).

If \( |V - S| \equiv K_1 \cup K_2 \), then \( G \equiv C_{1,1} \).

Subcase (1.b):  
\( |V - S| = 4 \).

If \( |V - S| \equiv 2K_1 \cup K_2 \), then \( G \equiv C_{1,5} \).

Subcase (1.c):  
\( |V - S| = 5 \).

If \( |V - S| \equiv K_1 \cup P_5 \), then \( G \equiv C_{1,10} \).

Subcase (1.d):  
\( |V - S| = 6 \).

If \( |V - S| \equiv 3K_1 \cup P_3 \), then \( G \equiv C_{1,27} \).

Subcase (1.e):  
\( |V - S| = 7 \).

If \( |V - S| \equiv 5K_1 \cup K_2 \), then \( G \) is the set of graphs from \( C_{1,4} \) to \( C_{1,44} \).
Case (2): \( <S> \cong 2K_1 \cup K_3 \)

By a similar argument as in Case (1), \(|V - S| \geq 3\).

Subcase (2.a): \(|V - S| = 3\)

If \(<V - S> \cong K_1 \cup K_2\), then \( G \cong C_{2,1} \) and \( C_{2,2} \).

Subcase (2.b): \(|V - S| = 4\)

If \(<V - S> \cong 2K_1 \cup K_2\), then \( G \) is one of the graphs from \( C_{2,3} \) to \( C_{2,7} \) and \( C_{1,2} \).

Subcase (2.c): \(|V - S| = 5\)

If \(<V - S> \cong 3K_1 \cup K_2\), then \( G \) is one of the graphs from \( C_{2,3} \) to \( C_{2,12} \) and \( C_{1,19} \).

If \(<V - S> \cong 2K_1 \cup P_3\), then \( G \cong C_{1,14} \).

Subcase (2.d): \(|V - S| = 6\)

If \(<V - S> \cong 4K_1 \cup K_2\), then \( G \cong C_{1,31} \).

Subcase (2.e): \(|V - S| = 7\)

Then either \( G \) is not unicyclic or \( S \) will not be a \( \gamma_{cld} \) set of \( G \).

Case (3): \( <S> \cong K_{1,3} \)

In this case, it is observed that all vertices in \(<V - S>\) are isolated vertices. Therefore, \(|V - S| = 4\).

Subcase (3.a): \(|V - S| = 4\)

If \(<V - S> \cong K_1 \cup K_2\), then \( G \cong C_{1,1} \).

Case (4): \(<S> \cong K_1 \cup P_3\)

By a similar argument as in Case (1), \(|V - S| \geq 3\).

Subcase (4.a): \(|V - S| = 3\)

If \(<V - S> \cong K_1 \cup K_2\), then \( G \cong C_{1,1} \).

Subcase (4.b): \(|V - S| = 4\)

If \(<V - S> \cong 2K_1 \cup K_2\), then \( G \cong C_{1,6} \) and \( C_{1,3} \).

Subcase (4.c): \(|V - S| = 5\)

If \(<V - S> \cong 3K_1 \cup K_2\), then \( G \cong C_{1,16} \).

Subcase (4.d): \(|V - S| = 6 \text{ (or) } 7\)

If \(<V - S> \cong 3K_1 \cup K_2\), then \( G \cong C_{1,16} \).

Then either \( G \) is not unicyclic or \( S \) will not be a \( \gamma_{cld} \) set of \( G \).

This completes the proof of the theorem.

Notation 3.7:

The family of graphs \( \mathcal{D} = \{ D_1, D_2, D_3, D_4 \} \) are defined as follows, where

\[
\begin{align*}
\mathcal{D}_1 &= \{ D_{1,1}, D_{1,2}, ..., D_{1,49}, D_{1,50} \}; \\
\mathcal{D}_2 &= \{ D_{2,1}, D_{2,2}, ..., D_{2,37}, D_{2,38} \}; \\
\mathcal{D}_3 &= \{ D_{3,1}, D_{3,2}, ..., D_{3,59}, D_{3,60} \}; \\
\mathcal{D}_4 &= \{ D_{4,1}, D_{4,2}, ..., D_{4,57} \};
\end{align*}
\]

\[
\begin{align*}
D_{1,1} &= C_4 \cup P_1 \cup P_2, \\
D_{1,2} &= C_4 \cup P_1 \cup P_3, \\
D_{1,3} &= C_4 \cup P_1 \cup P_3, \\
D_{1,4} &= C_4 \cup P_1 \cup P_3, \\
D_{1,5} &= C_4 \cup P_1 \cup P_3, \\
D_{1,6} &= C_4 \cup P_1 \cup P_3, \\
D_{1,7} &= C_4 \cup P_1 \cup P_3, \\
D_{1,8} &= C_4 \cup P_1 \cup P_3, \\
D_{1,9} &= C_4 \cup P_1 \cup P_3, \\
D_{1,10} &= C_4 \cup P_1 \cup P_3, \\
D_{1,11} &= C_4 \cup P_1 \cup P_3, \\
D_{1,12} &= C_4 \cup P_1 \cup P_3, \\
D_{1,13} &= C_4 \cup P_1 \cup P_3, \\
D_{1,14} &= C_4 \cup P_1 \cup P_3, \\
D_{1,15} &= C_4 \cup P_1 \cup P_3, \\
D_{1,16} &= C_4 \cup P_1 \cup P_3.
\end{align*}
\]
Theorem 3.8:

Let G be a connected unicyclic graph in which two vertices of \( \gamma_{\text{clid}} \) set lie on the cycle. Then \( \gamma_{\text{clid}}(G) = 4 \) if and only if G is one of the graphs in the family \( \emptyset \).

Proof:

If G is one of the graphs in the family \( \emptyset \), then \( \gamma_{\text{clid}}(G) = 4 \).

Conversely, let S be a \( \gamma_{\text{clid}} \) set of the unicyclic graph G with \( |S| = 4 \) and two vertices of S lie on the cycle of G.

By Theorem 3.6, 3 \leq |V - S| \leq 7. Since \( <V - S> \) contains at least one isolated vertex, \( <S> \cong 4K_1 \), \( 2K_1 \cup K_2 \), \( 2K_2 \) (or) \( K_1 \cup P_3 \).

Case (1): \( <S> \cong 4K_1 \)

Subcase (1.a): \(|V - S| = 3\)

If \( <V - S> \cong 3K_1 \), then G \( \cong D_{1,3} \)

Subcase (1.b): \(|V - S| = 4\)

If \( <V - S> \cong 4K_1 \), then G \( \cong D_{1,2} \) and \( D_{1,3} \)

If \( <V - S> \cong 2K_1 \cup K_2 \), then G \( \cong D_{1,4} \)

If \( <V - S> \cong K_2 \cup K_3 \), then G \( \cong D_{1,5} \)

Subcase (1.c): \(|V - S| = 5\)

If \( <V - S> \cong 5K_1 \), then G is one of the graphs from \( D_{1,6} \) to \( D_{1,13} \)

If \( <V - S> \cong 3K_1 \cup K_2 \), then G is one of the graphs from \( D_{1,14} \) to \( D_{1,21} \)

If \( <V - S> \cong K_1 \cup 2K_2 \), then G \( \cong D_{1,22} \) and \( D_{1,23} \)

If \( <V - S> \cong 2K_1 \cup P_3 \), then G \( \cong D_{1,24} \), \( D_{1,22} \) and \( D_{1,23} \)

Subcase (1.d): \(|V - S| = 6\)

If \( <V - S> \cong 6K_1 \), then G is one of the graphs from \( D_{1,25} \) to \( D_{1,31} \)

If \( <V - S> \cong 4K_1 \cup K_2 \), then G is one of the graphs from \( D_{1,32} \) to \( D_{1,41} \)

If \( <V - S> \cong 2K_1 \cup K_2 \), then G \( \cong D_{1,42} \) and \( D_{1,22} \)

If \( <V - S> \cong 3K_1 \cup K_3 \), then G \( \cong D_{1,43} \)

Subcase (1.e): \(|V - S| = 7\)

If \( <V - S> \cong 7K_1 \), then G is one of the graphs from \( D_{1,44} \) to \( D_{1,48} \)

If \( <V - S> \cong 5K_1 \cup K_2 \), then G \( \cong D_{1,49} \) and \( D_{1,50} \)

If \( <V - S> \cong 3K_1 \cup 2K_2 \) (or) \( K_1 \cup 3K_2 \), then S will not a \( \gamma_{\text{clid}} \) set of G.

Case (2): \( <S> \cong 2K_1 \cup K_2 \)
Subcase (2.a): $|V - S| = 3$
If $<V - S> \cong 3K_1$, then $G \cong D_{2,1}, D_{2,2}$ and $D_{1,1}$
If $<V - S> \cong 2K_1 \cup 2K_2$, then $G \cong D_{2,2}$

Subcase (2.b): $|V - S| = 4$
If $<V - S> \cong 4K_1$, then $G$ is one of the graphs from $D_{2,3}$ to $D_{2,7}$ and $D_{1,12}$
If $<V - S> \cong 2K_3 \cup K_2$, then $G \cong D_{2,4}$ to $D_{2,10}$ $D_{1,8}$ and $D_{1,19}$

Subcase (2.c): $|V - S| = 5$
If $<V - S> \cong 5K_1$, then $G$ is one of the graphs from $D_{2,11}$ to $D_{2,23}$, $D_{1,38}$ and $D_{1,31}$
If $<V - S> \cong 3K_1 \cup K_2$, then $G$ is one of the graphs from $D_{2,24}$ to $D_{2,29}$, $D_{1,14}$, $D_{1,19}$ and $D_{2,15}$

Subcase (2.d): $|V - S| = 6$
If $<V - S> \cong 6K_1$, then $G$ is one of the graphs from $D_{2,30}$ to $D_{2,34}$ and $D_{1,11}$
If $<V - S> \cong 4K_1 \cup K_2$, then $G$ is one of the graphs from $D_{2,35}$ to $D_{2,37}$, $D_{1,30}$, $D_{1,34}$ and $D_{1,36}$

Subcase (2.e): $|V - S| = 7$
If $<V - S> \cong 7K_1$, then $G \cong D_{2,38}$
If $<V - S> \cong 5K_1 \cup K_2$, then $G$ is not unicyclic.

Case (3): $<S> \cong 2K_2$

Subcase (3.a): $|V - S| = 3$
If $<V - S> \cong 3K_1$, then $G \cong D_{3,1}$ and $D_{3,2}$
If $<V - S> \cong K_1 \cup K_2$, then $G \cong D_{1,11}$

Subcase (3.b): $|V - S| = 4$
If $<V - S> \cong 4K_1$, then $G$ is one of the graphs from $D_{3,1}$ to $D_{3,6}$ and $D_{2,13}$
If $<V - S> \cong 2K_1 \cup K_2$, then $G \cong D_{2,7}$

Subcase (3.c): $|V - S| = 5$
If $<V - S> \cong 5K_1$, then $G$ is one of the graphs $D_{3,5}$, $D_{3,6}$, $D_{2,13}$ and $D_{2,16}$
If $<V - S> \cong 3K_1 \cup K_2$, then $G \cong D_{3,7}$

Subcase (3.d): $|V - S| = 6$
If $<V - S> \cong 6K_1$, then $G \cong D_{2,35}$

Subcase (3.d): $|V - S| = 7$
Then either $G$ is not unicyclic or $S$ will not be a $\gamma_{olv}$ – set of $G$.

Case (4): $<S> \cong K_1 \cup P_3$

Subcase (4.a): $|V - S| = 3$
If $<V - S> \cong 3K_1$, then $G \cong D_{4,1}$ and $D_{4,2}$

Subcase (4.b): $|V - S| = 4$
If $<V - S> \cong 4K_1$, then $G \cong D_{4,3}$, $D_{4,4}$ and $D_{4,5}$

Subcase (4.c): $|V - S| = 5$
If $<V - S> \cong 5K_1$, then $G \cong D_{4,5}$

Subcase (4.d): $|V - S| = 6$ (or) 7
Then either $G$ is not unicyclic or $S$ will not be a $\gamma_{olv}$ – set of $G$.

This completes the proof of the theorem.

Notation 3.9:
The family of graphs $\mathcal{E} = \{ \mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_6, \mathcal{E}_6 \}$ are defined as follows, where

$\mathcal{E}_1 = \{ E_{1,1}, E_{1,2}, ..., E_{1,13}, E_{1,14} \}; \mathcal{E}_2 = \{ E_{2,1}, E_{2,2}, ..., E_{2,27}, E_{2,28} \}; \mathcal{E}_3 = \{ E_{3,1}, E_{3,2}, E_{3,3}, E_{3,4} \}; \mathcal{E}_4 = \{ E_{4,1}, E_{4,2}, ..., E_{4,14}, E_{4,15} \}; \mathcal{E}_5 = \{ E_{5,1}, E_{5,2}, E_{5,3}, E_{5,4} \};$ and $\mathcal{E}_6 = \{ E_{6,1} \}$.

$E_{1,1} = C_6 \circ P_2$ $E_{1,7} = C_4 \circ P_1 \circ P_2$ $E_{1,13} = C_6 \circ P_4$
$E_{1,2} = C_6 \circ P_3$ $E_{1,8} = C_4 \circ P_1 \circ P_1$ $E_{1,14} = C_6 \circ P_1 \circ P_2$ $E_{1,3} = C_6 \circ P_3$ $E_{1,9} = C_4 \circ P_1 \circ P_1$ $E_{1,15} = C_4 \circ P_1 \circ P_1$ $E_{1,4} = C_6 \circ P_4$ $E_{1,10} = C_4 \circ P_2$ $E_{1,16} = C_6 \circ P_1 \circ P_1$ $E_{1,5} = C_6 \circ P_2$ $E_{1,11} = C_4 \circ P_1$ $E_{1,17} = C_6 \circ P_1 \circ P_1$ $E_{1,6} = C_6 \circ P_2$ $E_{1,12} = C_4 \circ P_1 \circ P_1$ $E_{1,18} = C_6 \circ P_1 \circ P_1 \circ P_1$
Co – Isolated Locating Domination Number For Unicyclic Graphs

\[ E_{1,20} = C_7 \oplus P_3 \]
\[ E_{1,21} = C_7 \oplus P_1 \oplus P_2 \]
\[ E_{1,22} = C_7 \oplus P_1 \oplus 2 P_2 \]
\[ E_{1,23} = C_7 \oplus P_1 \oplus P_3 \]
\[ E_{1,24} = C_7 \oplus P_1 \oplus 2 P_1 \oplus P_2 \]
\[ E_{1,25} = C_7 \oplus P_2 \]
\[ E_{1,26} = C_6 \oplus P_1 \oplus P_2 \]
\[ E_{1,27} = C_6 \oplus P_1 \oplus 2 P_2 \]
\[ E_{1,28} = C_6 \oplus P_2 \oplus P_1 \oplus 2 P_2 \]
\[ E_{1,29} = C_6 \oplus P_2 \oplus 2 P_1 \oplus P_2 \]
\[ E_{1,30} = C_7 \oplus P_1 \]
\[ E_{1,31} = C_7 \oplus P_1 \oplus P_2 \]
\[ E_{1,32} = C_7 \oplus P_1 \oplus 2 P_2 \]
\[ E_{1,33} = C_7 \oplus P_2 \oplus 2 P_1 \]
\[ E_{1,34} = C_7 \oplus P_2 \]
\[ E_{1,35} = C_6 \oplus P_1 \oplus P_1 \]
\[ E_{1,36} = C_6 \oplus P_1 \oplus 2 P_1 \]
\[ E_{1,37} = C_6 \oplus P_2 \oplus P_2 \]
\[ E_{1,38} = C_6 \oplus P_2 \]
\[ E_{1,39} = C_5 \oplus P_1 \]
\[ E_{1,40} = C_5 \oplus P_2 \]
\[ E_{1,41} = C_5 \oplus P_3 \]
\[ E_{1,42} = C_5 \oplus P_4 \]
\[ E_{1,43} = C_4 \oplus P_1 \]
\[ E_{1,44} = C_4 \oplus P_2 \]
\[ E_{1,45} = C_4 \oplus P_3 \]
\[ E_{1,46} = C_4 \oplus P_4 \]
\[ E_{1,47} = C_3 \oplus P_1 \]
\[ E_{1,48} = C_3 \oplus P_2 \]
\[ E_{1,49} = C_3 \oplus P_3 \]
\[ E_{1,50} = C_3 \oplus P_4 \]
\[ E_{1,51} = C_2 \oplus P_1 \]
\[ E_{1,52} = C_2 \oplus P_2 \]
\[ E_{1,53} = C_2 \oplus P_3 \]
\[ E_{1,54} = C_2 \oplus P_4 \]
\[ E_{1,55} = C_1 \oplus P_1 \]
\[ E_{1,56} = C_1 \oplus P_2 \]
\[ E_{1,57} = C_1 \oplus P_3 \]
\[ E_{1,58} = C_1 \oplus P_4 \]

Theorem 3.10:
Let G be connected unicyclic graph in which three vertices of \( \gamma_{cd} \) – set lie on the cycle. Then \( \gamma_{cd}(G) = 4 \) if and only if G is one of the graphs in the family \( E \).

Proof:
If G is one of the graphs in the family \( E \), then \( \gamma_{cd}(G) = 4 \).

Conversely, let S be a \( \gamma_{cd} \) – set of the unicyclic graph G with \( |S| = 4 \). Since three vertices of S lie on the cycle and G is unicyclic, \( 3 \leq |V - S| \leq 8 \).

Since \( <V - S> \) contains at least one isolated vertex, 
\( <S> \cong 4K_1, 2K_1 \cup K_2, 2K_2, P_5, C_5 \) (or) \( K_{1,5} \).

Case (1): \( |V - S| \cong 4K_1 \)

Subcase (1.a): \( |V - S| = 3 \)
Then S is not a \( \gamma_{cd} \) – set of G.

Subcase (1.b): \( |V - S| = 4 \)
If \( |V - S| \cong 4K_1 \), then G \( \cong E_{1,1} \)
If \( |V - S| \cong 2K_1 \cup K_2 \), then G is not unicyclic.

Subcase (1.c): \( |V - S| = 5 \)
If \( |V - S| \cong 5K_1 \), then G is one of the graphs from \( E_{1,7} \) to \( E_{1,19} \)
If \( |V - S| \cong 3K_1 \cup K_2 \), then G is one of the graphs from \( E_{1,8} \) to \( E_{1,20} \)
If \( |V - S| \cong K_1 \cup 2K_2 \), then G \( \cong E_{1,11} \)

Subcase (1.d): \( |V - S| = 6 \)
If \( |V - S| \cong 6K_1 \), then G is one of the graphs from \( E_{1,11} \) to \( E_{1,22} \)
If \( |V - S| \cong 4K_1 \cup K_2 \), then G is one of the graphs from \( E_{1,19} \) to \( E_{1,24} \)
If \( |V - S| \cong 2K_1 \cup 2K_2 \), then G \( \cong E_{1,25} \)

Subcase (1.e): \( |V - S| = 7 \)
If \( |V - S| \cong 7K_1 \), then G is one of the graphs from \( E_{1,26} \) to \( E_{1,29} \) and \( E_{1,24} \)
If \( |V - S| \cong 5K_1 \cup K_2 \), then G \( \cong E_{1,30} \)
If \( |V - S| \cong 3K_1 \cup 2K_2 \), then G \( \cong E_{1,24} \)
If \( |V - S| \cong 3K_1 \cup 3K_2 \), then S will not be a \( \gamma_{cd} \) – set of G.

Subcase (1.f): \( |V - S| = 8 \)

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If \(|V - S| \equiv 8K_1\), then \(G \cong E_{12}\)
If \(|V - S|\) contains \(K_2\) as one of its components, then \(S\) will not be a \(\gamma_{cld}\) set of \(G\).

**Case (2):** \(|S| \equiv 2K_1 \cup K_2\)

**Subcase (2.a):** \(|V - S| = 3\)
Then \(S\) is not a \(\gamma_{cld}\) set of \(G\).

**Subcase (2.b):** \(|V - S| = 4\)
If \(|V - S| \equiv 4K_1\), then \(G\) is one of the graphs from \(E_{2.3}\) to \(E_{2.4}\) and \(E_{4.1}\)
If \(|V - S| \equiv 2K_1 \cup K_2\), then \(G \cong E_{2.5}\)

**Subcase (2.c):** \(|V - S| = 5\)
If \(|V - S| \equiv 5K_1\), then \(G\) is one of the graphs from \(E_{2.6}\) to \(E_{2.14}\) and \(E_{1.3}\)
If \(|V - S| \equiv 5K_1 \cup K_2\), then \(G\) is one of the graphs \(E_{2.15}\), \(E_{2.16}\), \(E_{1.10}\), \(E_{2.3}\) and \(E_{2.10}\)

**Subcase (2.d):** \(|V - S| = 6\)
If \(|V - S| \equiv 6K_1\), then \(G\) is one of the graphs from \(E_{2.17}\) to \(E_{2.25}\)
If \(|V - S| \equiv 4K_1 \cup K_2\), then \(G \cong E_{2.26}, E_{1.16}\) and \(E_{1.19}\)

**Subcase (2.e):** \(|V - S| = 7\)
If \(|V - S| \equiv 7K_1\), then \(G\) is one of the graphs \(E_{2.27}\), \(E_{2.28}\), \(E_{1.26}\) and \(E_{1.27}\)
If \(|V - S| \equiv 5K_1 \cup K_2\), then \(G\) is not unicyclic.

**Subcase (2.f):** \(|V - S| = 8\)
Then either \(G\) is not unicyclic or \(S\) will not be a \(\gamma_{cld}\) set of \(G\).

**Case (3):** \(|S| \equiv 2K_2\)

**Subcase (3.a):** \(|V - S| = 3\)
Then \(S\) is not a \(\gamma_{cld}\) set of \(G\).

**Subcase (3.b):** \(|V - S| = 4\)
If \(|V - S| \equiv 4K_1\), then \(G \cong E_{3.1}\)
If \(|V - S| \equiv 2K_1 \cup K_2\), then \(S\) will not be a \(\gamma_{cld}\) set of \(G\).

**Subcase (3.c):** \(|V - S| = 5\)
If \(|V - S| \equiv 5K_1\), then \(G \cong E_{3.2}\) and \(E_{3.3}\)
If \(|V - S| \equiv K_1 \cup K_2\) (or) \(3K_1 \cup K_2\), then \(S\) will not be a \(\gamma_{cld}\) set of \(G\).

**Subcase (3.d):** \(|V - S| = 6\)
If \(|V - S| \equiv 6K_1\), then \(G \cong E_{3.4}\)
If \(|V - S| \equiv 4K_1 \cup K_2\) (or) \(2K_1 \cup 2K_2\), then \(G\) is not unicyclic.

**Subcase (3.e):** \(|V - S| = 7\) (or) \(8\)
Then either \(G\) is not unicyclic or \(S\) will not be a \(\gamma_{cld}\) set of \(G\).

**Case (4):** \(|S| \equiv K_1 \cup P_3\)

**Subcase (4.a):** \(|V - S| = 3\)
If \(|V - S| \equiv 3K_1\), then \(G \cong E_{4.1}\)

**Subcase (4.b):** \(|V - S| = 4\)
If \(|V - S| \equiv 4K_1\), then \(G\) is one of the graphs from \(E_{4.2}\) to \(E_{4.5}\) and \(E_{2.3}\)
If \(|V - S| \equiv 2K_1 \cup K_2\), then \(S\) will not be a \(\gamma_{cld}\) set of \(G\).

**Subcase (4.c):** \(|V - S| = 5\)
If \(|V - S| \equiv 5K_1\), then \(G\) is one of the graphs from \(E_{4.6}\) to \(E_{4.11}\) and \(E_{1.3}\)
If \(|V - S| \equiv K_1 \cup K_2\) (or) \(3K_1 \cup K_2\), then \(S\) will not be a \(\gamma_{cld}\) set of \(G\).

**Subcase (4.d):** \(|V - S| = 6\)
If \(|V - S| \equiv 6K_1\), then \(G \cong E_{4.12}\) and \(E_{4.13}\)
If \(|V - S| \equiv 4K_1 \cup K_2\) (or) \(2K_1 \cup 2K_2\), then \(G\) is not unicyclic.

**Subcase (4.e):** \(|V - S| = 7\) (or) \(8\)
Then either \(G\) is not unicyclic or \(S\) will not be a \(\gamma_{cld}\) set of \(G\).

**Case (5):** \(|S| \equiv K_1 \cup C_3\)

**Subcase (5.a):** \(|V - S| = 3\)
If \(|V - S| \equiv 3K_1\), then \(G \cong E_{5.1}\)

**Subcase (5.b):** \(|V - S| = 4\)
If \( \langle V - S \rangle \cong 4K_1 \), then \( G \cong E_{4,2} \) and \( E_{3,3} \).
If \( \langle V - S \rangle \cong 2K_1 \cup K_2 \), then \( S \) will not be a \( \gamma_{cd} \) – set of \( G \).

**Subcase (5.c):** \( |V - S| = 5 \)
- If \( \langle V - S \rangle \cong 5K_1 \), then \( G \cong E_{5,4} \).
- If \( \langle V - S \rangle \cong K_1 \cup K_2 \) (or) \( 3K_1 \cup K_2 \), then \( S \) will not be a \( \gamma_{cd} \) – set of \( G \).

**Subcase (5.d):** \( |V - S| = 6, 7 \) (or) 8

Then either \( G \) is not unicyclic or \( S \) will not be a \( \gamma_{cd} \) – set of \( G \).

**Case (6):** \( \langle S \rangle \cong P_4 \)

**Subcase (6.a):** \( |V - S| = 3 \)
Then \( S \) is not a \( \gamma_{cd} \) – set of \( G \).

**Subcase (6.b):** \( |V - S| = 4 \)
- If \( \langle V - S \rangle \cong 4K_1 \), then \( G \cong E_{6,1} \).

**Subcase (6.c):** \( |V - S| = 5 \)
- If \( \langle V - S \rangle \cong 5K_1 \), then \( G \cong E_{6,8} \).
- If \( \langle V - S \rangle \cong 2K_1 \cup K_2 \), then \( S \) will not be a \( \gamma_{cd} \) – set of \( G \).

**Subcase (6.d):** \( |V - S| = 5, 6, 7 \) (or) 8

Then either \( G \) is not unicyclic or \( S \) will not be a \( \gamma_{cd} \) – set of \( G \).
This completes the proof of the theorem.

**Notation 3.11:**
The family of graphs \( F = \{ F_1, F_2, ..., F_5 \} \) are defined as follows, where

- \( F_{1,1} = C_8 \)
- \( F_{1,2} = C_8 \cup P_1 \)
- \( F_{1,3} = C_9 \)
- \( F_{1,4} = C_8 \cup P_1 \cup P_2 \)
- \( F_{1,5} = C_9 \cup P_1 \)
- \( F_{1,6} = C_{10} \)
- \( F_{1,7} = C_8 \cup P_1 \cup P_2 \)
- \( F_{1,8} = C_9 \cup P_1 \cup P_2 \)

- \( F_{1,9} = C_8 \cup P_1 \cup P_2 \cup P_3 \)
- \( F_{3,1} = \gamma_{cd} \)
- \( F_{3,2} = \gamma_{cd} \cup P_1 \cup P_3 \)
- \( F_{3,3} = \gamma_{cd} \cup P_1 \cup P_2 \cup P_3 \)
- \( F_{3,4} = \gamma_{cd} \cup P_1 \cup P_2 \cup P_3 \)
- \( F_{3,5} = \gamma_{cd} \cup P_1 \cup P_3 \)
- \( F_{3,6} = \gamma_{cd} \cup P_1 \cup P_2 \cup P_3 \)

**Theorem 3.12:**
Let \( G \) be a connected unicyclic graph \( G \) in which four vertices of \( \gamma_{cd} \) – set lie on the cycle. Then \( \gamma_{cd}(G) = 4 \) if and only if \( G \) is one of the graphs in the family \( F \).

**Proof:**
If \( G \) is one of the graphs in the family \( F \), then \( \gamma_{cd}(G) = 4 \).
Conversely, let \( S \) be a \( \gamma_{cd} \) – set of the unicyclic graph \( G \). Since four vertices of \( S \) lie on the cycle, \( 4 \leq |V - S| \leq 8 \).
Since \( \langle V - S \rangle \) contains at least one isolated vertex, \( \langle S \rangle \) is one of the graphs \( 4K_1, 2K_1 \cup K_2, K_1 \cup P_3, P_4 \) and \( C_4 \).

**Case (1):** \( \langle S \rangle \cong 4K_1 \)

**Subcase (1.a):** \( |V - S| = 4 \)
- If \( \langle V - S \rangle \cong 4K_1 \), then \( G \cong F_{1,1} \).
- If \( \langle V - S \rangle \cong 2K_1 \cup K_2 \), then \( G \) is not unicyclic.

**Subcase (1.b):** \( |V - S| = 5 \)
- If \( \langle V - S \rangle \cong 5K_1 \), then \( G \cong F_{1,2} \).
- If \( \langle V - S \rangle \cong 3K_1 \cup K_2 \), then \( G \cong F_{1,3} \).

**Subcase (1.c):** \( |V - S| = 6 \)
- If \( \langle V - S \rangle \cong 6K_1 \), then \( G \cong F_{1,4} \).
- If \( \langle V - S \rangle \cong 4K_1 \cup K_2 \), then \( G \cong F_{1,5} \).

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If \( |V - S| = 7 \), then \( G \cong F_{1,6} \).

**Subcase (1.d):** \(|V - S| = 7\)

If \( |V - S| = 7K_1 \), then \( G \cong F_{1,7} \).
If \( |V - S| = 7K_1 \cup K_2 \), then \( G \cong F_{1,8} \).
If \( |V - S| = 3K_1 \cup 2K_2 \) (or) \( K_1 \cup 3K_2 \), then \( S \) will not be a \( \gamma_{cd} \) set of \( G \).

**Subcase (1.e):** \(|V - S| = 8\)

If \( |V - S| = 8K_1 \), then \( G \cong F_{1,9} \).
If \( |V - S| \) contains \( K_2 \) as one of its components, then \( S \) will not be a \( \gamma_{cd} \) set of \( G \).

**Case (2):** \( |V - S| = 2K_1 \cup K_2 \)

**Subcase (2.a):** \(|V - S| = 4\)

If \( |V - S| = 4K_1 \), then \( G \cong F_{1,2} \).
If \( |V - S| = 2K_1 \cup K_2 \), then \( G \cong F_{1,1} \).

**Subcase (2.b):** \(|V - S| = 5\)

If \( |V - S| = 5K_1 \), then \( G \cong F_{2,1} \). \( F_{2,2} \) and \( F_{2,3} \).
If \( |V - S| = 3K_1 \cup K_2 \), then \( G \cong F_{1,3} \).

**Subcase (2.c):** \(|V - S| = 6\)

If \( |V - S| = 6K_1 \), then \( G \cong F_{2,4} \) and \( F_{2,5} \).
If \( |V - S| = 4K_1 \cup K_2 \), then \( G \cong F_{2,6} \).

**Subcase (2.d):** \(|V - S| = 7\)

If \( |V - S| = 7K_1 \), then \( G \cong F_{2,7} \).
If \( |V - S| = 5K_1 \cup K_2 \), then \( G \) is not unicyclic.

**Subcase (2.e):** \(|V - S| = 8\)

Then either \( G \) is not unicyclic or \( S \) will not be a \( \gamma_{cd} \) set of \( G \).

**Case (3):** \( |V - S| = 2K_2 \)

**Subcase (3.a):** \(|V - S| = 4\)

If \( |V - S| = 4K_1 \), then \( G \cong F_{3,1} \).
If \( |V - S| = 2K_1 \cup K_2 \), then \( S \) will not be a \( \gamma_{cd} \) set of \( G \).

**Subcase (3.b):** \(|V - S| = 5\)

If \( |V - S| = 5K_1 \), then \( G \cong F_{3,2} \).
If \( |V - S| = 3K_1 \cup K_2 \), then \( G \cong F_{2,2} \).
If \( |V - S| = K_1 \cup K_2 \), then \( S \) will not be a \( \gamma_{cd} \) set of \( G \).

**Subcase (3.c):** \(|V - S| = 6\)

If \( |V - S| = 6K_1 \), then \( G \cong F_{3,3} \).
If \( |V - S| = 4K_1 \cup K_2 \) (or) \( 2K_1 \cup 2K_2 \), then \( G \) is not unicyclic.

**Subcase (3.d):** \(|V - S| = 7 \) (or) \( 8 \)

Then either \( G \) is not unicyclic or \( S \) will not be a \( \gamma_{cd} \) set of \( G \).

**Case (4):** \( |V - S| = K_1 \cup P_1 \)

**Subcase (4.a):** \(|V - S| = 4\)

If \( |V - S| = 4K_1 \), then \( G \cong F_{3,1} \).
If \( |V - S| = 2K_1 \cup K_2 \), then \( S \) will not be a \( \gamma_{cd} \) set of \( G \).

**Subcase (4.b):** \(|V - S| = 5\)

If \( |V - S| = 5K_1 \), then \( G \cong F_{4,1} \) and \( F_{2,2} \).
If \( |V - S| = 3K_1 \cup K_2 \), then \( G \cong F_{2,1} \).
If \( |V - S| = K_1 \cup K_2 \), then \( S \) will not be a \( \gamma_{cd} \) set of \( G \).

**Subcase (4.c):** \(|V - S| = 6\)

If \( |V - S| = 6K_1 \), then \( G \cong F_{4,2} \).
If \( |V - S| = 4K_1 \cup K_2 \) (or) \( 2K_1 \cup 2K_2 \), then \( G \) is not unicyclic.

**Subcase (4.d):** \(|V - S| = 7 \) (or) \( 8 \)

Then either \( G \) is not unicyclic or \( S \) will not be a \( \gamma_{cd} \) set of \( G \).

**Case (5):** \( |V - S| = P_1 \)

**Subcase (5.a):** \(|V - S| = 4\)
If $<V - S> \cong 4K_1$, then $G \cong F_{3,1}$
If $<V - S> \cong 2K_1 \cup K_2$, then $G \cong F_{3,1}$

**Subcase (5.b):** $|V - S| = 5$
If $<V - S> \cong 5K_1$, then $G \cong F_{5,1}$
If $<V - S> \cong K_1 \cup K_2$, then $G$ is not unicyclic.

**Subcase (5.c):** $|V - S| = 6, 7$ (or) $8$
Then either $G$ is not unicyclic or $S$ will not be a $\gamma_{cl}$-set of $G$.

**Case (6):** $<S> \cong C_4$

**Subcase (6.a):** $|V - S| = 4$
If $<V - S> \cong 4K_1$, then $G \cong F_{3,3}$

**Subcase (6.b):** $|V - S| = 5, 6, 7$ (or) $8$
Then either $G$ is not unicyclic or $S$ will not be a $\gamma_{cl}$-set of $G$.

This completes the proof of the theorem.

**Remark 3.13:**
Let $G$ be a connected unicyclic graph. Then $\gamma_{cl}(G) = 4$ if and only if $G$ is isomorphic to one of the graphs in the family of graphs $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$.

**IV. Conclusion**
This paper results on finding the co–isolated locating domination number for unicyclic graphs. Determining the co–isolated locating domination number remain open. In particular the co–isolated locating domination number equal to 3 (or) 4 (or) 5 are of interest. For large values of $n \geq 6$ proof similar to those presented in this paper get too complicated. So a new approach seems necessary.

**References**