The Euler-Mascheroni Constant is trascendental

Francesco Sovrano

Abstract: Until now, it was not known if this constant is irrational, let alone transcendental (Wells 1986, p. 28). The famous English mathematician G. H. Hardy is alleged to have offered to give up his Savilian Chair at Oxford to anyone who proved \( \gamma \) (the Euler-Mascheroni constant) to be irrational (Havil 2003, p. 52), although no written reference for this quote seems to be known. Hilbert mentioned the irrationality of \( \gamma \) as an unsolved problem that seems "unapproachable" and in front of which mathematicians stand helpless (Havil 2003, p. 97). This paper has been written to prove the trascendency of \( \gamma \).

Keywords: Euler-Mascheroni constant, gamma, Harmonic series, trascendental numbers, Lindemann-Weierstrass theorem, Mertens’ 3rd theorem

I. Legend

Some words used in this paper have been abbreviated. Below, you can find the abbreviations list with the equivalent meanings.

- eq. \( \equiv \) equation
- th. \( \equiv \) theorem

II. What is \( \gamma \)

\( \gamma \) is the Euler-Mascheroni constant and has the numerical value:

\[
\gamma = 0.577215664901532860606512090082402431042\ldots (1)
\]

Let \( n \in \mathbb{N} \), then \( \gamma \) is defined as the limit of the sequence:

\[
\gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \ln{n} \right) = \lim_{n \to \infty} H(n) - \ln{n} \tag{2}
\]

\( H(n) \) is called Harmonic series.

III. Brief introduction to irrational and trascendental numbers

3.1 Irrational

In mathematics, an irrational number is any real number that cannot be expressed as a ratio of integers. Informally, this means that an irrational number cannot be represented as a simple fraction. Irrational numbers are those real numbers that cannot be represented as terminating or repeating decimals. As a consequence of Cantor’s proof that the real numbers are uncountable (and the rationals countable) it follows that almost all real numbers are irrational.

3.2 Trascendental

In mathematics, a transcendental number is a real or complex number that is not algebraic or, using different words, it is not a root of a non-zero polynomial equation with rational coefficients. The best-known transcendental numbers are \( \pi \) and \( e \). Though only a few classes of transcendental numbers are known (in part because it can be extremely difficult to show that a given number is transcendental), transcendental numbers are not rare. Indeed, almost all real and complex numbers are transcendental, since the algebraic numbers are countable while the sets of real and complex numbers are both uncountable. All real transcendental numbers are irrational, since all rational numbers are algebraic. The converse is not true: not all irrational numbers are transcendental; e.g., the square root of 2 is irrational but not a transcendental number, since it is a solution of the polynomial equation \( x^2 - 2 = 0 \).

IV. The Lindemann-Weierstrass theorem

In transcendental number theory, the Lindemann-Weierstrass theorem is a result that is very useful in establishing the transcendence of numbers. It states that if \( a_1, \ldots, a_n \) are algebraic numbers which are linearly independent over the rational numbers \( \mathbb{Q} \), then \( e^{a_1}, \ldots, e^{an} \) are algebraically independent over \( \mathbb{Q} \); in other words the extension field \( \mathbb{Q}(e^{a_1}, \ldots, e^{an}) \) has transcendence degree \( n \) over \( \mathbb{Q} \).

In 1882, Ferdinand von Lindemann proved that \( e^\delta \) is transcendental for each non-zero algebraic number \( \delta \), while in 1885 Karl Weierstrass proved the more general version enunciated before.
V. The Mertens’ 3rd theorem

Mertens’ theorems are a set of classical estimates concerning the asymptotic distribution of the prime numbers. For our purposes we enunciate only the third.

Let $x \in \mathbb{N}$ and

$$\Phi(x) = \prod_{p \leq x} \left(1 - \frac{1}{p}\right) \quad (3)$$

where $p \leq x$ are the different prime numbers $p$ lower or equal to $x$.

The Mertens’ 3rd theorem states that:

$$\lim_{x \to \infty} \ln x \cdot \Phi(x) = e^{-\gamma} \quad (4)$$

VI. Proof by contradiction

We start our proof considering the eq.(4) and the eq.(2) combined together.

Thank to their own definition, we know that $\Phi: \mathbb{N} \to \mathbb{Q}$ and $H: \mathbb{N} \to \mathbb{Q}$.

In the asymptotic limit $x \to \infty$, we have:

$$\Phi(x) \cdot [H(x) - \gamma] = e^{-\gamma} \quad (5)$$

The output of $\Phi(x)$ and $H(x)$ will always be rational.

Let $a = \Phi(x)$ and $b = H(x)$, then $a, b \in \mathbb{Q}$.

The eq.(5) became:

$$a(b - \gamma) = e^{-\gamma} \quad (6)$$

Hypothesis 1 $\gamma$ is rational.

Let $\alpha = a(b - \gamma)$ and $\beta = e^{-\gamma}$, then, because of the previous equation, we have that:

$$\alpha = \beta \quad (7)$$

Because of the Lindemann-Weierstrass theorem, if $\gamma$ is rational then $\beta$ is transcendental.

$\alpha$ must be rational because $a, b$ and $\gamma$ are rational and $\mathbb{Q}$ is closed under multiplication, division, addition and subtraction.

But, a transcendental number cannot be equal to a rational number! Thus, there is a contradiction assuming the validity of the Hypothesis (1).

Theorem 1 $\gamma$ is irrational.

VII. Proofs of transcendency

7.1 $\gamma$ is transcendental

For this proof, we have to consider the eq.(6) and the eq.(2) combined together.

Let $x \in \mathbb{N}$, in the asymptotic limit $x \to \infty$:

$$a(b - \gamma) = e^{-\gamma} = e^{\ln x - b} = xe^{-b} \quad (8)$$

We know that $b$ is rational and, because of the Lindemann-Weierstrass theorem, we know that $e^{-b}$ is transcendental.

Let $c = \frac{x}{a}$, then it is true that $c \in \mathbb{Q}$, because $x \in \mathbb{N}$ and $a \in \mathbb{Q}$.

The eq.(8) can be written as:

$$\gamma = b - \frac{xe^{-b}}{a} = b - ce^{-b} \quad (9)$$

A transcendental number multiplied by or subtracted to a rational number gives a new transcendental number, thus $b - ce^{-b}$ is transcendental and it is equal to $\gamma$.

Theorem 2 $\gamma$ is transcendental.

7.2 $e^{\gamma}$ is transcendental

For this proof, we have to consider the eq.(6) and the eq.(2) combined together.

As written before, a transcendental number multiplied by or subtracted to a rational number gives a new transcendental number.

If $\gamma$ is transcendental, then $\alpha$ (defined in section 6) is transcendental.

Because of eq.(7):

$$e^{\gamma} = \alpha \quad (10)$$

thus, $e^{-\gamma}$ is transcendental and this implies that $e^{\gamma}$ is transcendental too.

Theorem 3 $e^{\gamma}$ is transcendental.
The Euler-Mascheroni Constant is transcendental

References