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A Study on Differential Equations on the Sphere Using Leapfrog Method

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Abstract: *In this article, a new method of analysis of the differential equations on the sphere using Leapfrog method is presented. To illustrate the effectiveness of the Leapfrog method, an example of the differential equations on the sphere has been considered and the solutions were obtained using methods taken taken from the literature [17] and Leapfrog method. The obtained discrete solutions are compared with the exact solutions of the differential equations on the sphere. Solution graphs for the differential equations on the sphere have been presented in the graphical form to show the efficiency of this Leapfrog method. This Leapfrog method can be easily implemented in a digital computer and the solution can be obtained for any length of time.*

Keywords: *Differential equations, system of differential equations, Differential equations on the sphere, Single-term Haar wavelet series, Leapfrog Method.*

I. Introduction

Mathematical modeling aims to describe the different aspects of the real world, their interaction, and their dynamics through mathematics. It constitutes the third pillar of science and engineering, achieving the fulfillment of the two more traditional disciplines, which are theoretical analysis and experimentation. Nowadays, mathematical modeling has a key role also in fields such as the environment and industry, while its potential contribution in many other areas is becoming more and more evident. One of the reasons for this growing success is definitely due to the impetuous progress of scientific computation; this discipline allows the translation of a mathematical model, which can be explicitly solved only occasionally, into algorithms that can be treated and solved by ever more powerful computers.

Since 1960 numerical analysis, the discipline that allows mathematical equations (algebraic, functional, differential, and integrals) to be solved through algorithms, had a leading role in solving problems linked to mathematical modeling derived from engineering and applied sciences. Following this success, new disciplines started to use mathematical modeling, namely information and communication technology, bioengineering, financial engineering, and so on. As a matter of fact, mathematical models offer new possibilities to manage the increasing complexity of technology, which is at the basis of modern industrial production. They can explore new solutions in a very short time period, thus allowing the speed of innovation cycles to be increased. This ensures a potential advantage to industries, which can save time and money in the development and validation phases. We can state therefore that mathematical modeling and scientific computation are gradually and relentlessly expanding in manifold fields, becoming a unique tool for qualitative and quantitative analysis. In the following paragraphs we will discuss the role of mathematical modeling and of scientific computation in applied sciences; their importance in simulating, analyzing, and decision making; and their contribution to technological progress. We will show some results and underline the perspectives in different fields such as industry, environment, life sciences, and sports.

The goal of this article is to construct a numerical method for addressing the differential equations on the sphere problem by an application of the Leapfrog method which was studied by Sekar and team of his researchers [2-15]. Recently, Sekar et al. [17] discussed the differential equations on the sphere problem using STHW. In this paper, the same differential equations on the sphere problem was considered (discussed by Sekar et al. [17]) but present a different approach using the Leapfrog method with more accuracy for the differential equations on the sphere problem.

II. Leapfrog Method

The most familiar and elementary method for approximating solutions of an initial value problem is Euler's Method. Euler's Method approximates the derivative in the form of $y' = f(t, y),\ y(t_0) = y_0,\ y \in R^d$ by a finite difference quotient $y'(t) \approx (y(t+h) - y(t))/h$. We shall usually discretize the independent variable in equal increments:

$$t_{n+1} = t_n + h,\ n = 0,1,\ldots,t_0.$$

Henceforth we focus on the scalar case, N = 1. Rearranging the difference quotient gives us the corresponding approximate values of the dependent variable:

$$y_{n+1} = y_n + hf(t_n, y_n),\ n = 0,1,\ldots,t_0$$

To obtain the leapfrog method, we discretize t_n as in $t_{n+1} = t_n + h,\ n = 0,1,\ldots,t_0$, but we double the time interval, h, and write the midpoint approximation $y(t+h) - y(t) \approx hy'\left(t + \frac{h}{2}\right)$ in the form

$$y'(t+h) \approx (y(t+2h) - y(t))/h$$

and then discretize it as follows:

$$y_{n+1} = y_{n-1} + 2hf(t_n, y_n),\ n = 0,1,\ldots,t_0$$

The leapfrog method is a linear m = 2-step method, with $a_0 = 0, a_1 = 1, b_{-1} = -1, b_0 = 2$ and $b_1 = 0$. It uses slopes evaluated at odd values of n to advance the values at points at even values of n, and vice versa, reminiscent of the children's game of the same name. For the same reason, there are multiple solutions of the leapfrog method with the same initial value $y = y_0$. This situation suggests a potential instability present in multistep methods, which must be addressed when we analyze them—two values, y_0 and y_1, are required to initialize solutions of $y_{n+1} = y_{n-1} + 2hf(t_n, y_n),\ n = 0,1,\ldots,t_0$ uniquely, but the analytical problem $y' = f(t, y),\ y(t_0) = y_0,\ y \in R^d$ only provides one. Also for this reason, one-step methods are used to initialize multistep methods.

III. Differential Equations on the Sphere

This section features differential equations on the sphere of ordinary differential equations to be solved numerically. The results from the Leapfrog method are compared with the results from some of the classical integrators such as STHW method, which is mentioned in Section 2, and the build-in integrator ODE45 from Matlab. For more information on ODE45, please refer to Shampine [16]. With the exception of ODE45, the experiments performed in this chapter will involve integrators based on fixed step size h and fixed order.

Differential equation on the sphere was designed to be solved by the Leapfrog method and STHW method, and it is another example featured in Hairer and Wanner [1].

$$\frac{dy}{dt} = \begin{bmatrix} 0 & t & -0.4\cos(t) \\ -t & 0 & 0.1t \\ 0.4\cos(t) & -0.1t & 0 \end{bmatrix} \quad y(t) = A(t)y(t),\ \ y(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{1}$$

The solution $y(t) \in \mathbf{R}^3$ in Eq. (1) is a vector of unit length, evolving on the unit sphere. That is, the solution vector y(t) rotates around the spherical surface with respect to its origin, and naturally, the homogeneous manifold in this case is the unit sphere. Then, the general presentation of the differential equation, with which we want to solve for the solution y(t) is given by,

$$\dot{Q}(t) = A(t)Q(t), \quad Q(0) = Id,$$

where Q(t) is the corresponding 3 × 3 matrix in the differential equation. This is exactly the form of the differential equation. To perform the numerical experiments, let us suppose that we want to integrate Eq. (1) with the given initial condition y(0), over the time interval t = [0, 64] with a constant stepsize of h = 1/20, and order p = 4 for the 4 integrators Leapfrog and STHW methods. Once again, we also integrate the same problem with Matlab's ODE45 for comparison.

In the two graphs in Figure 1, we plotted the three dimensional solutions y(t) over the integration period, on the unit sphere for the four methods under consideration, using the same stepsize h. It is apparent that solutions from the RK integrator eventually drift away from the surface of the sphere, while solutions from the other three integrators appear to adhere to the same surface. Again assuming that ODE45 produces the most accurate solutions y(t) when solving the same initial value problem, then we can take a look at the difference of solutions between ODE45's and the other four integrators. Such results are plotted in the bottom four graphs in Figure 1. Note that the difference between ODE45's and the method such as Leapfrog method are significantly smaller, than the differences between ODE45's and STHW method.

One of the advantage of using the Leapfrog method for solving this type of differential equations on the sphere is that the Leapfrog method action produces special orthogonal matrices, which, when multiplied to the vector $y(t) \in \mathbf{R}^3$, preserves its length in the rotation, so that solutions y(t) will always remain on the sphere. This is not true if we use classical integrators such as STHW method. We can see this by observing the variation in the length of the solution vectors y(t) with respect to the initial length, at each time step for all four integrators tested, in Figure 2.

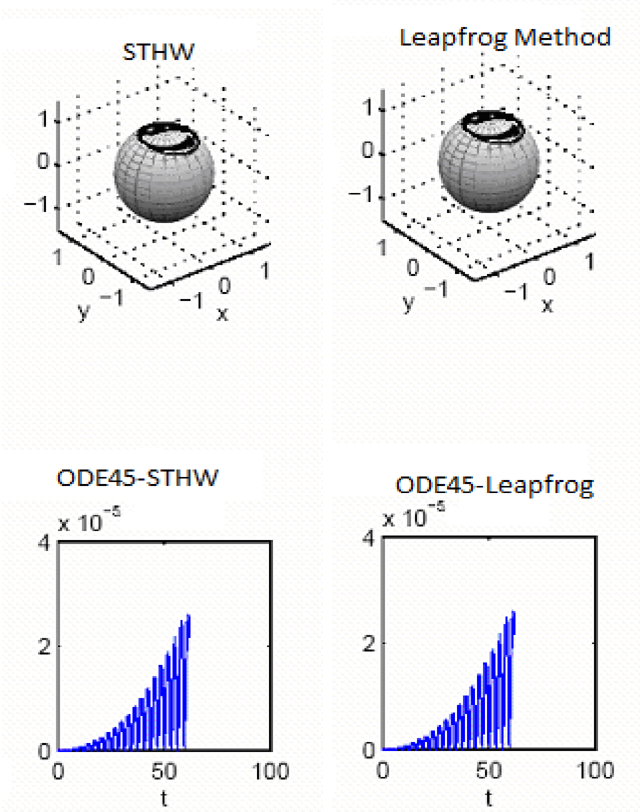

Figure 1: From solving Equation (1), the above 2 graphs are the solutions from each integrator plotted on the surface of a unit sphere. In Figure 2 graphs are the difference between solutions from ODE45 and each of the 2 integrators. $y_0 = [0, 0, 1]$, $t_0 = 0$, $t_f = 64$, and $h = 1/20$.

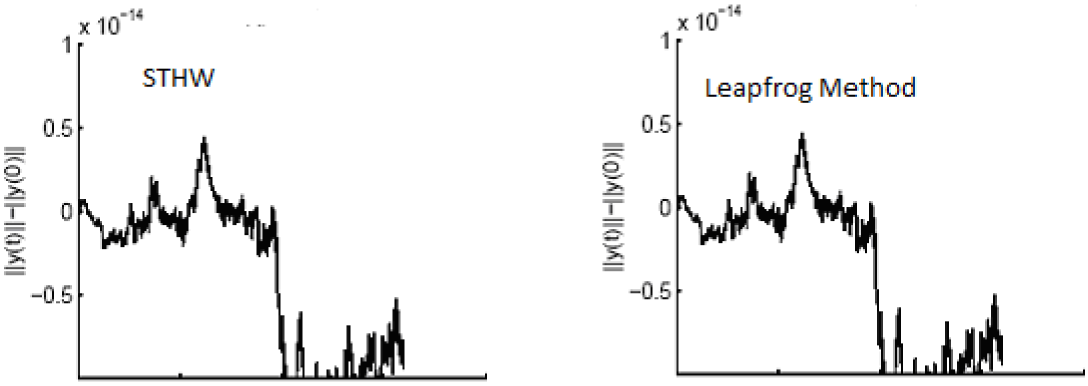

Figure 2: From solving Equation (1), the variation in the length of solution vectors against time t is plotted for each integrator. $y_0 = [0, 0, 1]$, $t_0 = 0$, $t_f = 64$, and $h = 1/20$.

IV. Conclusion

The obtained discrete solution of the numerical examples shows the efficiency of the Leapfrog for finding the solution of differential equations on the sphere. From the Figures 1 and 2, we can observe that the solution graph is come closer in Leapfrog method when compared to other methods from ODE45 taken from the literature []. Hence, the Leapfrog method is more suitable for studying the differential equations on the sphere.

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