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# A disproof of the Riemann hypothesis

Francesco Sovrano

***Abstract:*** *In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and the complex numbers with real part 1/2. It was proposed by Bernhard Riemann (1859), after whom it is named. Along with suitable generalizations, some mathematicians consider it the most important unresolved problem in pure mathematics (Bombieri 2000). The Riemann hypothesis, along with the Goldbach conjecture, is part of Hilbert's eighth problem in David Hilbert's list of 23 unsolved problems; it is also one of the seven Clay Mathematics Institute Millennium Prize Problems. [2]*
*In this paper we also prove that the Cramér's conjecture is false.*
***Keywords:*** *Riemann hypothesis, prime numbers, Prime Numbers Theorem, Mertens' 3rd theorem, Euler-Mascheroni constant, Zhang's bound, Maynard's bound, logarithmic integral function, Skewes' number, Littlewood's theorem, Koch's result, Schoenfeld's result, Cramér's conjecture*

## I. Legend

Some words used in this paper have been abbreviated. Below, you can find the abbreviations list with the equivalent meanings.

- eq. $\equiv$ equation
- neq. $\equiv$ inequation
- th. $\equiv$ theorem
- hyp. $\equiv$ hypothesis
- lim. $\equiv$ limit
- pg. $\equiv$ page

## II. Brief introduction to prime numbers

Assuming true the following hypothesis

**Hypothesis 1** *A prime number (or a prime) is a natural number greater than 1 that has no positive divisors other than 1 and itself.*

a natural number greater than $1$ that is not a prime number is called a composite number. For example, $5$ is prime because $1$ and $5$ are its only positive integer factors, whereas $6$ is composite because it has the divisors $2$ and $3$ in addition to $1$ and $6$. The fundamental theorem of arithmetic establishes the central role of primes in number theory: any integer greater than $1$ can be expressed as a product of primes that is unique up to ordering. The uniqueness in this theorem requires excluding $1$ as a prime because one can include arbitrarily many instances of $1$ in any factorization, e.g., $3$, $1 \cdot 3$, $1 \cdot 1 \cdot 3$, etc. are all valid factorizations of $3$. [1]

## III. The Mertens' 3rd theorem

Mertens' theorems are a set of classical estimates concerning the asymptotic distribution of the prime numbers [3]. For our purposes we enunciate only the third.

Let $x \in \mathbb{N}$ and

$$\Phi(x) = \prod_{p \le x} \left(1 - \frac{1}{p}\right) \qquad (1)$$

where $p \le x$ are the different prime numbers $p$ lower or equal to $x$.

The Mertens' 3rd theorem states that:

**Theorem 1** *Assuming the validity of hyp.(1)*

$$\lim_{x \to \infty} \Phi(x) \cdot \ln x = e^{-\gamma}$$

where $\gamma$ is the Euler-Mascheroni constant and has the numerical value:

$$\gamma = 0.57721566490153286060651209008240243104 2\ldots \quad (2)$$

## IV. Zhang's bound

Let $U = 7 \cdot 10^7$ and $p_n$ the $n$-th prime number, the result of Zhang

$$\liminf_{n\to\infty}(p_{n+1}-p_n) < U \quad (3)$$

is a major improvement on the Goldston-Graham-Pintz-Yldrm result. [5]

## V. Maynard's bound

In November 2013, Maynard gave a different proof of Yitang Zhang's theorem that there are bounded gaps between primes, and resolved a longstanding conjecture by showing that for any $m$ there are infinitely many intervals of bounded length containing $m$ prime numbers. [6]

Maynard's approach yielded the upper bound $U = 600$, thus

$$\liminf_{n\to\infty}(p_{n+1}-p_n) \le U \quad (4)$$

One year after Zhang's announcement, according to the Polymath project wiki [7], $U$ has been reduced to $246$.

Further, assuming the Elliott-Halberstam conjecture and its generalized form, the Polymath project wiki [7] states that $U$ has been reduced to $12$ and $6$, respectively.

## VI. The prime numbers theorem

In number theory, the prime number theorem (PNT) describes the asymptotic distribution of the prime numbers among the positive integers. It formalizes the intuitive idea that primes become less common as they become larger by precisely quantifying the rate at which this occurs. The theorem was proved independently by Jacques Hadamard and Charles Jean de la Vallée-Poussin in 1896 using ideas introduced by Bernhard Riemann (in particular, the Riemann zeta function). [4]

This theorem states that

$$\lim_{n\to\infty}\Pi(n) = \frac{n}{\ln(n)} \quad (5)$$

where $\Pi(n)$ is the prime-counting function and $\ln(n)$ is the natural logarithm of $n$.

## VII. The logarithmic integral function

In mathematics, the logarithmic integral function or integral logarithm $li(x)$ is a special function.

It is relevant in problems of physics and has number theoretic significance, occurring in the prime number theorem as an estimate of the number of prime numbers less than a given value [8]. The logarithmic integral has an integral representation defined for all positive real numbers $x \neq 1$ by the definite integral:

$$li(x) = \int_0^x \frac{dt}{\ln t} \quad (6)$$

The function $\frac{1}{\ln(t)}$ has a singularity at $t = 1$, and the integral for $x > 1$ has to be interpreted as a Cauchy principal value:

$$li(x) = \lim_{\varepsilon\to 0+}\left(\int_0^{1-\varepsilon}\frac{dt}{\ln t} + \int_{1+\varepsilon}^{x}\frac{dt}{\ln t}\right) \quad (7)$$

The form of this function appearing in the prime number theorem and sometimes referred to as the "European" definition is defined so that $Li(2) = 0$:

$$Li(x) = \int_2^x \frac{du}{\ln u} = li(x) - li(2). \quad (8)$$

This integral is strongly suggestive of the notion that the 'density' of primes around $u$ should be $\frac{1}{\ln u}$.

This function is related to the logarithm by the full asymptotic expansion

$$Li(x) \sim \frac{x}{\ln x}\sum_{k=0}^{\infty}\frac{k!}{\ln^k x} = \frac{x}{\ln x} + \frac{x}{\ln^2 x} + \frac{2x}{\ln^3 x} + \cdots \quad (9)$$

Note that, as an asymptotic expansion, this series is not convergent: it is a reasonable approximation only if the series is truncated at a finite number of terms, and only large values of $x$ are employed. This expansion follows directly from the asymptotic expansion for the exponential integral [8].

This gives the following more accurate asymptotic behaviour:

$$Li(x) - \frac{x}{\ln x} = O\left(\frac{x}{\ln^2 x}\right) \quad (10)$$

where $O(\dots)$ is the big $O$ notation. So, the prime number theorem can be written as $\Pi(x) \approx Li(x)$. [8]

## VIII. Skewes' number

In number theory, Skewes' number is any of several extremely large numbers used by the South African mathematician Stanley Skewes as upper bounds for the smallest natural number $x$ for which

$$\Pi(x) > li(x) \quad (11)$$

These bounds have since been improved by others: there exists one value $x$ in the interval $[e^{727.95132478}, e^{727.95134681}]$ such that

$$\Pi(x) - li(x) > 9.1472 \cdot 10^{10149} \quad (12)$$

It is not known whether that $x$ is the smallest.

John Edensor Littlewood, who was Skewes' research supervisor, had proved that there is such a number (and so, a first such number); and indeed found that the sign of the difference $\Pi(x) - li(x)$ changes infinitely often. All numerical evidence then available seemed to suggest that $\Pi(x)$ was always less than $li(x)$. Littlewood's proof did not, however, exhibit a concrete such number $x$. [9]

The following theorems will prove that the sign of $\Pi(x) - Li(x)$ changes infinitely too. Thank to Littlewood, we know that

**Theorem 2** *Let* $x > 1$ *and* $y > 0$. *Then we have*

$$li(x+y) - li(x) = \int_x^{x+y} \frac{dt}{\ln t} < \frac{y}{\ln x}$$

that is equivalent to say that:

$$Li(x+y) - Li(x) = \int_x^{x+y} \frac{dt}{\ln t} < \frac{y}{\ln x} \qquad (13)$$

from the previous th., it follows:

**Theorem 3** *Let* $x$ *be a real number such that* $\Pi(x) - Li(x) = A$ *where* $A > 0$. *Then, if* $y$ *is a real number such that* $0 \le y < A \cdot lnx$, *we have* $\Pi(x+y) - Li(x+y) > 0$.

For the demonstration of those two theorems, please read the paper: "On the positive region of $\Pi(x) - li(x)$" (pg. 59) by Stefanie Zegowitz [10].

## IX. The Koch's result

Von Koch (1901) proved that the Riemann hypothesis implies the "best possible" bound for the error of the prime number theorem. A precise version of Koch's result, due to Schoenfeld says that [2]:

**Theorem 4** *The correctness of the Riemann hypothesis implies that for all* $x \ge 2657$

$$|\Pi(x) - Li(x)| < \frac{1}{8\pi} \cdot \sqrt{x} \ln x$$

## X. The Cramér's conjecture

In number theory, Cramér's conjecture, formulated by the Swedish mathematician Harald Cramér in 1936, is an estimate for the size of gaps between consecutive prime numbers: intuitively, that gaps between consecutive primes are always small, and the conjecture quantifies asymptotically just how small they must be. It states that

$$p_{n+1} - p_n = O(\ln^2 p_n) \qquad (14)$$

where $p_n$ denotes the $n$th prime number.

While this is the statement explicitly conjectured by Cramér, his argument actually supports the stronger statement

$$\limsup_{n\to\infty} \frac{p_{n+1} - p_n}{\ln^2 p_n} = 1 \qquad (15)$$

and this formulation is often called Cramér's conjecture in the literature.

Neither form of Cramér's conjecture has yet been proven or disproven [11].

Cramér gave a conditional proof of the much weaker statement that

**Theorem 5** *If the Riemann hypothesis is true, then*

$$p_{n+1} - p_n = O(\sqrt{p_n} \cdot \ln p_n)$$

## XI. The prime numbers function

Let $x > 1$ and $x \in \mathbb{N}$. The function $\Upsilon(x)$ is defined as:

$$\Upsilon(x) = \frac{p_x - \frac{x}{2\Phi(p_x)} + p_{x-1}}{2p_{x-1}} \qquad (16)$$

We want to study the following limit:

$$\lim_{x\to\infty} \Upsilon(x) \qquad (17)$$

**The following considerations will be done in the asymptotic limit** $x \to \infty$.

We have that:

$$\frac{x}{2\Phi(p_x)} = \frac{p_x}{2\Phi(p_x) \cdot \ln p_x} \qquad (18)$$

in fact, the Prime Numbers Theorem states that:

$$x = \Pi(p_x) = \frac{p_x}{\ln p_x} \qquad (19)$$

For the Mertens' 3rd theorem

$$\frac{x}{2\Phi(p_x)} = \frac{p_x}{2\Phi(p_x) \cdot \ln p_x} = \frac{p_x}{2e^{-\gamma}} = \frac{e^{\gamma} p_x}{2} \qquad (20)$$

Again, for the PNT we can say that $x \cdot \ln p_x = p_x$, so:

$$\begin{aligned}\frac{p_x}{p_{x-1}} &= \frac{x\cdot\ln p_x}{(x-1)\cdot\ln p_{x-1}} \\ &= \frac{x\cdot\ln(x\cdot\ln p_x)}{(x-1)\cdot\ln((x-1)\cdot\ln p_{x-1})} \\ &= \frac{x\cdot(\ln x+\ln\ln p_x)}{x\cdot(\ln(x-1)+\ln\ln p_{x-1})} \qquad (21) \\ &= \frac{\ln x+\ln\ln p_x}{\ln(x-1)+\ln\ln p_{x-1}} \\ &= 1\end{aligned}$$

this result had been object of a famous dispute between Erdös and Selberg [12].
Finally, we have proved that:

$$\begin{aligned}\Upsilon(x) &= \frac{p_x - \frac{x}{2\Phi(p_x)} + p_{x-1}}{2p_{x-1}} \\ &= \frac{p_x - \frac{e^{\gamma}p_x}{2} + p_{x-1}}{2p_{x-1}} \\ &= \frac{1-\frac{e^{\gamma}}{2}+1}{2} \qquad (22) \\ &= 1-\frac{e^{\gamma}}{4}\end{aligned}$$

**Theorem 6** *Assuming the validity of hyp.(1)*

$$\lim_{x\to\infty}\Upsilon(x) = 1-\frac{e^{\gamma}}{4}$$

The previous theorem have been also verified experimentally.

## XII. Disproofs of the Riemann hypothesis

Let $x > 1$ and $x \in \mathbb{N}$ From eq.(16) we obtain:

$$x = 2\Phi(p_x)\cdot[p_x + p_{x-1} - 2\Upsilon(x)\cdot p_{x-1}] \qquad (23)$$

Let function $Er$ be:

$$Er(p_x) = \Pi(p_x) - Li(p_x) \qquad (24)$$

We know that $x = \Pi(p_x)$, so:

$$x = Li(p_x) + Er(p_x) \qquad (25)$$

Thus

$$2\Phi(p_x)\cdot[p_x + p_{x-1} - 2\Upsilon(x)\cdot p_{x-1}] = Li(p_x) + Er(p_x) \qquad (26)$$

and

$$Er(p_x) = 2\Phi(p_x)\cdot[p_x + p_{x-1} - 2\Upsilon(x)\cdot p_{x-1}] - Li(p_x) \qquad (27)$$

**The following considerations will be done in the asymptotic limit $x \to \infty$.**

For the Mertens' 3rd theorem, we have that:

$$\begin{aligned}\Phi(p_x) &= \frac{e^{-\gamma}}{\ln p_x} \\ &= \frac{1}{e^{\gamma}\ln p_x} \qquad (28)\end{aligned}$$

thus

$$Er(p_x) = 2\cdot\frac{p_x+p_{x-1}-2\Upsilon(x)\cdot p_{x-1}}{e^{\gamma}\ln p_x} - Li(p_x) \qquad (29)$$

Considering eq.(10), we have:

$$Er(p_x) = 2\cdot\frac{p_x+p_{x-1}-2\Upsilon(x)\cdot p_{x-1}}{e^{\gamma}\ln p_x} - \frac{p_x}{\ln p_x} - \frac{p_x}{\ln^2 p_x} - O\left(\frac{p_x}{\ln^3 p_x}\right) \qquad (30)$$

Considering th.(6), we have:

$$\begin{aligned} Er(p_x) &= 2\cdot\frac{p_x+p_{x-1}-2\cdot(1-\frac{e^\gamma}{4})\cdot p_{x-1}}{e^\gamma \ln p_x}-\frac{p_x}{\ln p_x}-\frac{p_x}{\ln^2 p_x}-O\left(\frac{p_x}{\ln^3 p_x}\right)\\ &= 2\cdot\frac{p_x+p_{x-1}-2p_{x-1}+\frac{e^\gamma}{2}p_{x-1}}{e^\gamma \ln p_x}-\frac{p_x}{\ln p_x}-\frac{p_x}{\ln^2 p_x}-O\left(\frac{p_x}{\ln^3 p_x}\right)\\ &= 2\cdot\frac{p_x-p_{x-1}}{e^\gamma \ln p_x}+\frac{p_{x-1}}{\ln p_x}-\frac{p_x}{\ln p_x}-\frac{p_x}{\ln^2 p_x}-O\left(\frac{p_x}{\ln^3 p_x}\right)\\ &= 2\cdot\frac{p_x-p_{x-1}}{e^\gamma \ln p_x}-\frac{p_x-p_{x-1}}{\ln p_x}-\frac{p_x}{\ln^2 p_x}-O\left(\frac{p_x}{\ln^3 p_x}\right)\\ &= \left(\frac{2}{e^\gamma}-1\right)\frac{p_x-p_{x-1}}{\ln p_x}-\frac{p_x}{\ln^2 p_x}-O\left(\frac{p_x}{\ln^3 p_x}\right) \end{aligned} \quad (31)$$

### 12.1 The Littlewood theorem has been respected

As we can see, we have that, in the asymptotic limit $x \to \infty$, the sign of $Er(p_x)$ may change infinitely often as proved by th.(3). In chapter 13 we are going to discuss deeply about it.
In fact $\left(\frac{2}{e^\gamma}-1\right) = 0.1229189 \ldots > 0$ and $p_x - p_{x-1} > 0$ for each $x > 1$ and $x \in \mathbb{N}$.

### 12.2 1st disproof

We are now considering lim.(21), lim.(31) and th.(5) combined together. Assuming the validity of the following hypothesis

**Hypothesis 2** *The Riemann hypothesis is true*
we have that:

$$\begin{aligned} \limsup_{x\to\infty} Er(p_x) &= \limsup_{x\to\infty}\left[\left(\frac{2}{e^\gamma}-1\right)\frac{\sqrt{p_{x-1}}\cdot\ln p_{x-1}}{\ln p_x}-\frac{p_x}{\ln^2 p_x}-O\left(\frac{p_x}{\ln^3 p_x}\right)\right]\\ &= \limsup_{x\to\infty}\left[\left(\frac{2}{e^\gamma}-1\right)\sqrt{p_{x-1}}-\frac{p_x}{\ln^2 p_x}-O\left(\frac{p_x}{\ln^3 p_x}\right)\right] \end{aligned} \quad (32)$$

We are now going to study the following limit:

$$\begin{aligned} \lim_{x\to\infty}\frac{\sqrt{p_{x-1}}}{\frac{p_x}{\ln^2 p_x}} &= \lim_{x\to\infty}\frac{\sqrt{p_{x-1}}\cdot\ln^2 p_x}{\sqrt{p_x}\sqrt{p_x}}\\ &= \lim_{x\to\infty}\frac{\ln^2 p_x}{\sqrt{p_x}}\\ &= 0 \end{aligned} \quad (33)$$

This implies that, for a big enough $x$:

$$\frac{p_x}{\ln^2 p_x} > \sqrt{p_x} \quad (34)$$

and consequentely that:

$$\limsup_{x\to\infty}\frac{|Er(p_x)|}{\frac{p_x}{\ln^2 p_x}} = 1 \quad (35)$$

now, considering the previous limit, eq.(24) and th.(4), we have that:

$$\begin{aligned} \lim_{x\to\infty}\frac{\frac{1}{8\pi}\sqrt{p_x}\cdot\ln p_x}{|Er(p_x)|} &= \lim_{x\to\infty}\frac{\frac{1}{8\pi}\sqrt{p_x}\cdot\ln p_x}{\frac{p_x}{\ln^2 p_x}}\\ &= \lim_{x\to\infty}\frac{\ln^3 p_x}{\sqrt{p_x}}\\ &= 0 \end{aligned} \quad (36)$$

it follows that, for a big enough $x$:

$$\frac{p_x}{\ln^2 p_x} > \frac{1}{8\pi}\sqrt{p_x}\cdot\ln p_x \quad (37)$$

thus

$$|\Pi(p_x) - Li(p_x)| = |Er(x)| > \frac{1}{8\pi}\sqrt{p_x}\cdot\ln p_x \quad (38)$$

this implies that the inequation stated by th.(4) is surely wrong, and this is possible if and only if the Riemann hypothesis is false. Thus, we have a contradiction with hyp.(2), for this reason we claim that:

**Theorem 7** *The Riemann hypothesis is false*

### 12.3 2nd disproof

We are now considering lim.(31). Because of th.(4), we know that infinite prime numbers $p_{n+1}$ and $p_n$ exist such that $p_{n+1} - p_n \le U$. For this reason we can say that, **in the asymptotic <u>inferior</u> limit** $x \to \infty$, we have:

$$
\begin{aligned}
Er(p_x) &= \left(\frac{2}{e^{\gamma}} - 1\right)\frac{c}{\ln p_x} - \frac{p_x}{\ln^2 p_x} - O\left(\frac{p_x}{\ln^3 p_x}\right) \\
&= -\frac{p_x}{\ln^2 p_x} - O\left(\frac{p_x}{\ln^3 p_x}\right) \qquad (39)
\end{aligned}
$$

where $c \in \mathbb{N}$ and $0 < c \leq U$.

**Theorem 8** $liminf_{x\to\infty} \frac{Er(p_x)}{-\frac{p_x}{ln^2 p_x} - O\left(\frac{p_x}{ln^3 p_x}\right)} = \frac{\Pi(p_x) - Li(p_x)}{-\frac{p_x}{ln^2 p_x} - O\left(\frac{p_x}{ln^3 p_x}\right)} = 1$

Once again, as proved by lim.(36), the Riemann Hypothesis must be false.

## XIII. The prime gap

Let $x > 1$ and $x \in \mathbb{N}$. Because of th.(3) we know that the sign of $Er(p_x)$ changes infinitely, so it must be true that:

$$\left(\frac{2}{e^{\gamma}} - 1\right)\frac{p_x - p_{x-1}}{\ln p_x} - \frac{p_x}{\ln^2 p_x} - O\left(\frac{p_x}{\ln^3 p_x}\right) > 0 \qquad (40)$$

Let $b = \left(\frac{2}{e^{\gamma}} - 1\right)$, it is true that:

$$b(p_x - p_{x-1}) > \frac{p_x}{\ln p_x}\left[1 + O\left(\frac{1}{\ln p_x}\right)\right] \qquad (41)$$

$$p_x\left[b - \frac{1}{\ln p_x} - O\left(\frac{1}{\ln^2 p_x}\right)\right] - bp_{x-1} > 0 \qquad (42)$$

$$p_x > \frac{bp_{x-1}}{b - \frac{1}{\ln p_x} - O\left(\frac{1}{\ln^2 p_x}\right)} \qquad (43)$$

$$p_x > \frac{bp_{x-1}\ln p_x}{b\ln p_x - 1 - O\left(\frac{1}{\ln p_x}\right)} \qquad (44)$$

$$p_x > \frac{p_{x-1}\left(b\ln p_x - 1 - O\left(\frac{1}{\ln p_x}\right) + 1 + O\left(\frac{1}{\ln p_x}\right)\right)}{b\ln p_x - 1 - O\left(\frac{1}{\ln p_x}\right)} \qquad (45)$$

$$p_x > p_{x-1} + \frac{p_{x-1}\left(1 + O\left(\frac{1}{\ln p_x}\right)\right)}{b\ln p_x - 1 - O\left(\frac{1}{\ln p_x}\right)} \qquad (46)$$

this is a proof for the following theorem

**Theorem 9** $limsup_{x\to\infty}(p_x - p_{x-1}) > \frac{p_{x-1}\left(1 + O\left(\frac{1}{ln p_x}\right)\right)}{bln p_x - 1 - O\left(\frac{1}{ln p_x}\right)} \approx \frac{x}{b}$

and this implies that the Cramér's conjecture cannot be true.

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