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On Prime Ideals, Radical of a Ring and M-Systems Prime Ideals and M-Systems

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Abstract: *A group is a nonempty set equipped with one binary operation satisfying contain axioms while a ring is an algebraic structure with two binary operations namely addition and multiplication. We have tried to discuss Prime ideals, radical of rings and M-system in this paper.*
Keywords: *Prime ideals, m-system, Semi-prime ideal, n-system, Prime radical of the ring.*

I. Introduction

In this paper, we have tried to discuss about prime ideals, radical of a ring and few properties of m-systems. Besides these some theorems and lemma have been raised such as" If $r \in B(A)$, then there exists a positive integer n such that $r^n \in A$."

"If A is an ideal in the ring R , then $B(A)$ coincides with the intersection of all the prime ideals which contain A ." Besides these a few theorems and lemma have been established here which are related with m-system and n-system."

1.1 Prime Ideal:

An ideal P in a ring R is said to be a prime ideal if and only if it has the following property. If A and B are ideals in R such that $AB \subset P$, then $A \subseteq P$ or $B \subseteq P$.

1.2 m-systems:

A set M of element of a ring R is said to be an m-system if and only if it has the following property: If $a, b \in M$, there exists $x \in R$ such that $axb \in M$.

Remarks:

(1) If P is an ideal in a ring R . Let us denote by $C(P)$ .The complement of P in R , that is $C(P)$ is the set of element of R which are not elements of P .

(2) From the above theorem (i) and (ii) asserts that an ideal P in a ring R is a prime ideal in R if and only if $C(P)$ is an m-system.

(3) If R is itself a prime ideal in R then clearly $C(R) = \phi$.

1.3 Prime Radical $B(A)$:

The prime radical $B(A)$ of the ideal A in a ring R is the set consisting of those elements r of R with the property that every m-system in R which contains r meets A (that is has non empty intersection with A).

Remarks:

(1) Obviously $B(A)$ is an ideal in R . Also $A \subseteq B(A)$, so any prime ideal which contains $B(A)$ necessarily contains A .

(2) Let P be a prime ideal in R such that $A \subseteq P$. and $\mu r \in B(A)$. If $r \in P$, $C(P)$ would be an m-system containing r , and so we have $C(P) \cap A \neq \phi$ .However since $A \subseteq P$, $c(P) \cap P = \phi$, and this contradiction shows that $r \in P$. Hence $B(A) \subseteq P$.

Note: If $r \in R$, then the set $\{ r^i \text{I} i = 1,2,3,.........\ ... \}$ is a multiplicative system and hence also an m-system.

1.4 Semi-Prime Ideals:

An ideal Q in a ring R is said to be a semi-prime ideal if and only if it has the following property. If A is an ideal in R such that $A^2 \subseteq Q$, then $A \subseteq Q$.

1.5 N-Systems:

A set N of elements of a ring R is said to be an n-system iff it has the following property:
If $a \in N$, there exists an element $x \in R$ such that $axa \in N$.
Note: Clearly m-system is also an n-system.

II. The Prime Radical Of A Ring

The prime radical of the ideal in a ring R may be called the prime radical of the ring R.

2.1 Definition:

The radical of a ring R is defined as $B(R) = \{ r \mathrm{I} r \in R,$ every m-system in R which contains r also contains 0}.

2.2 Prime ring:

A ring R is said to be a prime ring if and only if the zero ideal is a prime ideal in R. That is, a ring R is a prime iff either of the following conditions hold , if A and B are ideals in R such that $AB = (0)$ then $A = (0)$ or $B = (0)$. $a, b \in R$ such that $aRb = 0$, then $a = 0$ or $b = 0$.
Remark:
If R is a commutative ring, then R is a prime ring iff it has no non zero divisors of zero.

III. Principal Left (Right) Ideal

3.1 If P is an ideal in a ring R all of the following conditions are equivalent:
(i) P is a prime ideal
(ii) If $a, b \in R$ such that $aRb \subseteq P$, then $a \in P$ or $b \in P$.
(iii) If (a) and (b) are principal ideals in R such that $(a)(b) \subseteq P$,Then $a \in P$ or $b \in P$.
(iv) If U and V are right ideals in R such that $UV \subseteq P$, then $U \subseteq P$ or $V \subseteq P$.
(v) If U and V are left ideals in R such that $UV \subseteq P$, then $U \subseteq P$ or $V \subseteq P$.

3.2 Lemma:

If $r \in B(A)$, then there exists a positive integer n such that $r^n \in A$.

3.3 Theorem:

If A is an ideal in the ring R, then $B(A)$ coincides with the intersection of all the prime ideals which contain A.

Proof: By our remarks, $B(A)$ is contained in every prime ideal which contains A .If $r \notin B(A)$, then there exists a prime ideal P in R such that $r \notin P$ and $A \subseteq P$. Since $r \notin B(A)$, by definition of $B(A)$, there exists an m-system M in R such that $r \in M$ and $M \cap A = \Phi$ .Now consider the set of all ideals K in R such that $A \subseteq K$ and $M \cap K = \Phi$.
This set is not empty since A is one such ideal . The existence of a maximal ideal, say P, is to be shown. Clearly, $r \notin P$ since $r \in M$ and $M \cap P = \Phi$ .We shall show that P is a prime ideal in R. Suppose that $a \notin P$ and $b \notin P$. The maximal property of P shows that the ideal $P + (a)$ contains an element m_1 of M and similarly, $P + (b)$ contains an element m_2 of M. Since M is an m-system, there exists an element x of R such that
$m_1 x m_2 \in M$. Moreover, $m_1 x m_2$ is an element of the ideal ($P + (a)$)($P + (b)$).Now , if $(a)(b) \subseteq P_1$ we would have $(P + (a))(P + (b)) \subseteq P$ and it would follow that $m_1 x m_2 \in P$. But this is impossible since $m_1 x m_2 \in M$. and $MnP = \Phi$ .Hence $(a)(b) \not\subset P$ and is therefore a prime ideal.

IV. Semi-Primal And Commutative Ideals

If A is an ideal in the commutative ring R, then $B(A) = \{r \mathrm{I} r^n \in A$ for some positive integers n }.

4.1 Theorem:

An ideal Q in a ring R is a semi-prime ideal in R iff the residue class ring R/Q contains no non zero nilpotent ideals.

Proof:

Let θ be the natural homomorphism of R onto R/Q with kernel Q. Suppose that Q is a semi-prime ideal in R and that U is a nilpotent ideal in R/Q, say $U^n = (0)$. Then $U^n \theta^{-1} = Q$ and this follows that $(UQ^{-1})^n \subseteq U^n \theta^{-1} = Q$. Since, Q is semi-prime ideal. This implies that $UQ^{-1} \subseteq Q$, and hence that $U = (0)$.Conversely, suppose that R/Q contains no non zero nilpotent ideals and that A is an ideal in R such that $A^2 \subseteq Q$. Then $(A\theta)^2 = A^2\theta = (0)$ and hence $A\theta = (0)$ and $A \subseteq Q$.

4.2 Theorem:

If Q is an ideal in a ring R, then all of the following conditions are equivalent.

(i) Q is a semi prime ideal.

(ii) If $a \in R$ such that $aRa \subseteq Q$, then $a \in Q$.

(iii) If (a) is a principal ideal in R such that $(a)^2 \subseteq Q$, then $a \in Q$.

(iv) If U is a right ideal in R such that $U^2 \subseteq Q$ then $U \subseteq Q$.

(v) If U is a left ideal in R such that $U^2 \subseteq Q$ then $U \subseteq Q$.

4.3 Lemma:

If N is an n-system in the ring R and $a \in N$, then there exists an m-system M in R such that $a \in M$ and $M \subseteq N$.

Proof:

Let $M = \{a_1, a_2, a_3, \ldots\ldots\ldots \ \ldots\ldots\}$, where the elements of this sequence are defined inductively as follows: First we define $a_1 = a$. Since now $a_1 \in N$, then $a_1 R a_1 \cap N \neq \phi$, and we choose a_2 as some element of $a_1 R a_1 \cap N$. In general, if a_i has been defined with $a_i \in N$, we choose a_{i+1} as an element of $a_i R a_i \cap N$. Thus a set M is defined such that $a \in M$ and $M \leq N$.To complete the proof, we only need to show that M is an m-system. Suppose that $a_i, a_j \in M$ and, for convenience, let us assume that $i \leq j, a_{j+1} \in a_j R a_j \subseteq a_i R a_j$ and $a_{j+1} \in M$ .A similar argument takes care of the case in which $i > j$, so we conclude that M is indeed an m-system and this completes the proof.

V. Semi Prime Ideals

An ideal Q in a ring R is a semi-prime ideal in R iff $B(Q) = Q$.

Proof:

Let Q be a semi – prime ideal in R. Then clearly $Q \leq B(Q)$. So let us assume that $Q \subset B(Q)$ and seck a contradiction. Suppose, $a \in B(Q)$ with $a \notin Q$. Hence, $C(Q)$ is an n- system and $a \in C(Q)$. By the previous lemma, there exists an m-system M such that $a \in M \subseteq C(Q)$.

Now, $a \in B(Q)$ and by definition of $B(Q)$ every m-system which contains a meets Q. But $Q \cap C(Q) = \phi$, and therefore $M \cap Q = \phi$ which is a contradiction and this contradiction gives the proof of the theorem.

5.1 Corollary:

An ideal Q in a ring R is a semi prime ideal iff Q is an intersection of prime ideals in R.

5.2 Corollary:

If A is an ideal in the ring R, then $B(A)$ is the smallest semi-prime ideal in R which contains A.

5.3. Theorem:

If $B(R)$ is the prime radical of the ring R, then

(i) $B(R)$ Coincides with the intersection of all prime ideals in R.

(ii) $B(R)$ is a semi prime ideal which is contained in every semi prime ideal in R.

5.4. Theorem:

$B(R)$ is a nil ideal which contains every nilpotent right (left) ideal in R.

5.5. Corollary:

If R is a commutative ring . Then $B(R)$ is the ideal consisting of all nilpotent elements of R.

VI. Condition For Prime Radical Of The Ring

6.1 Theorem:

If S is an ideal in the ring R, the prime radical of the ring S is $S \cap B(R)$.

Proof:

Here we consider the radical of S as a ring and not the radical of the ideal of S in R.

Let us denote the radical of the ring S by K, so that K is the intersection of all the prime ideals in S.

However, if P is a prime ideal in R, then $P \cap S$ is a prime ideal in S, and hence $K \subseteq S \cap B(R)$.

Conversely, if $a \in S \cap B(R)$, then every m-system in R which contains a also contains 0(zero).In particular, every m-system in S which contains a also contains 0.Hence $a \in K$ and $S \cap B(R) \subseteq K$. We have therefore shown that $k = S \cap B(R)$ and the proof is completed.

6.2 Lemma:

A ring R has zero prime radical if and only if it contains no non-zero nilpotent ideal(right ideal, left ideal).

6.3 Lemma:

If $B(R)$ is the prime radical of the ring R, then $(R / B(R)) = (0)$.

Proof:

Let θ be the natural homomorphism of R onto $R / B(R)$, with kernel $B(R)$ and suppose that $a \in R$ such that $a\theta \in (R / B(R))$ .Then $a\theta$ is contained in every prime ideal in the ring $R / B(R)$ .If P is an arbitrary prime ideal in R, then it contains the kernel of the homomorphism θ and hence, we have $P = (P\theta)\theta^{-1}$. But $P\theta$ is a prime ideal in $R / B(R)$, so $a\theta \in P\theta$ and $a \in (P\theta)\theta^{-1} = P$. This shows that a is contained in every prime ideal in R and hence that $a \in B(R)$. That is, $a\theta$ is the zero of the ring $R / B(R)$. This shows that $B(R / B(R)) = (0)$ which completes the proof of the theorem.

VII. Conclusion

After establishing **these** problems in difficult way 6.1 portrays that If S is an ideal in the ring R, the prime radical of the ring S is $S \cap B(R)$. Lemma 6.2 makes certain that A ring R has zero prime radical if and only if it contains no non-zero nilpotent ideal(right ideal, left ideal).

Lemma 6.3 also makes certain that If $B(R)$ **is the prime radical of the ring R, then** $(R / B(R)) = (0)$.

Hence theorem 6.1 , Lemma 6.2 and 6.3 together with m-system and n-system prove the Radical of a Ring.

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