Symmetric Skew 4-Reverse Derivations on Semi Prime Rings

Dr. C. Jaya Subba Reddy¹, S. Vasantha Kumar², K.Subbarayudu³
¹,²,³ Department of Mathematics, S.V.University, Tirupati-517502, Andhra Pradesh, India.

Abstract: In this paper we introduce the notion of symmetric skew 4-reverse derivation of semiprime ring and we consider $R$ be a non-commutative 2,3-torsion free semiprime ring, I be a non zero two sided ideal of $R$, $\alpha$ be an anti-automorphism of $R$, and $D: R^3 \rightarrow R$ be a symmetric skew 4-reverse derivation associated with the anti-automorphism $\alpha$. Suppose that the trace function $f$ is commuting on $I$ and \([f(y), a(y)] \in Z\), for all $y \in I$, then \([f(y), a(y)] = 0\), for all $y \in I$.

Keywords: Semi prime ring, Derivation, Bi derivation, Reverse derivation, Symmetric Skew 4-derivation, Symmetric skew 4-Reverse Derivation and Anti-automorphism.

I. Introduction

Bresar and Vukman [2] have introduced the notion of a reverse derivations and Samman and Alamani [6] have studied some properties of semi prime rings with reverse derivations. The study of centralizing and commuting mappings on prime rings was initiated by the result of Posner [5] which states that the existence of a nonzero centralizing derivation on a prime ring implies that the ring has to be commutative. Vukman [7, 8] investigated symmetric bi derivation on prime and semi prime rings in connection with centralizing mappings. A Posner [1] have studied some results in symmetric skew 3-derivations with prime rings and semiprimerings. Recently, Faiza Shujat, Abuzaid Ansari [3] studied some results in symmetric skew 4-derivations in prime rings. C. Jaya Subba Reddy [4] have studied some results in symmetric skew 3-reverse derivations with semiprimerings. Motivated by the above work, in this paper we proved that under certain conditions of a semiprime ring with a nonzero symmetric skew 4-reverse derivations has to be commutative.

II. Preliminaries

Throughout the paper, $R$ will represent a ring with a center $Z$ and $\alpha$ an anti - automorphism of $R$. Let $n \geq 2$ be an integer. A ring $R$ is said to be $n$-torsion free if for $x \in R$, $nx = 0$ implies $x = 0$. For all $x, y \in R$ the symbol $[x, y]$ will denote the commutator $xy - yx$. We make extensive use of basic commutator identities $[x, y] = [y, x]$, and $[x, yz] = [x, y]z + y[x, z]$. Recall that a ring $R$ is semi prime if $xy = 0$ implies that $x = 0$. An additive map $D: R \rightarrow R$ is called derivation if $d(xy) = d(x)y + x d(y)$, for all $x, y \in R$, and it is called a skew derivation ($\alpha$-derivation ) of $R$ associated with the anti-automorphism $\alpha$ if $d(xy) = d(x)y + \alpha(x)d(y)$ for all $x, y \in R$. An additive map $D: R \rightarrow R$ is called reverse derivation if $d(xy) = x d(y) + y d(x)$ for all $x, y \in R$, and it is called a skew reverse derivation of $R$ associated with anti-automorphism $\alpha$ if $d(xy) = x d(y) + \alpha(y)d(x)$ for all $x, y \in R$.

Before starting our main theorem, we give some basic definitions and well known results which we will need in our further investigation.

Let $D$ be a symmetric 4-additive map of $R$, then obviously
$$D(−p, q, r, s) = −D(p, q, r, s), \text{forall } p, q, r, s \in R. \quad (1)$$

Namely, for all $y, z \in R$, the map $D(.,., y, z): R \rightarrow R$ is endo morphism of the additive group of $R$.

The map $f: R \rightarrow R$ defined by $f(x) = D(x, x, x, x)$, $x \in R$ is called trace of $D$.

Note that $f$ is not additive on $R$. But for all $x, y \in R$, we have
$$f(x + y) = [f(x) + 4D(x, x, x, y) + 6D(x, x, y, y) + 4D(x, y, y, y) + f(y)]$$

Recall that by (1), $f$ is even function

More precisely, for all $p, q, r, s, u, v, w, x \in R$, we have
$$D(pu, q, r, s) = pD(u, q, r, s) + \alpha(u)D(p, q, r, s),$$
$$D(pu, q, r, s) = \delta D(p, q, r, s),$$
$$D(p, qv, r, s) = qD(p, v, r, s) + \alpha(v)D(p, q, r, s),$$
$$D(p, qv, r, s) = \delta D(p, q, w, s) + \alpha(w)D(p, q, r, s),$$
$$D(p, q, r, sx) = sD(p, q, r, x) + \alpha(x)D(p, q, r, s).$$

Of course, if $D$ is symmetric, then the above four relations are equivalent to each other.

Lemma 1: Let $R$ be a prime ring and $a, b \in R$. If $a[x, b] = 0$ for all $x \in R$, then either $a = 0$ or $b \in Z$.

Proof: Note that \(0 = a[x, b] = ax[b, y] + a[x, b]y = ax[y, b]\) for all $x, y \in R$.

Thus \(a[x, b] = 0, y \in R\), and, since $R$ is prime, either $a = 0$ or $b \in Z$. 

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Theorem 1: Let $R$ be a 2,3-torsion free non commutative semi prime ring and $I$ be a nonzero ideal of $R$. Suppose $\alpha$ is an anti automorphism of $R$ and $D: R^3 \rightarrow R$ is a symmetric skew 4-reverse derivation associated with $\alpha$. Suppose that the trace function $f$ is commuting on $I$ and $[f(y), \alpha(y)] \in Z$, for all $y \in I$, then $[f(y), \alpha(y)] = 0$, for all $y \in I$.

Proof: Let $(f(y), \alpha(y)) \in Z$, for all $y \in I$. Linearization of (2) yields that

$$[f(x + y), \alpha(x + y)] \in Z,$$

for all $x, y \in I$.

By skew 4- derivation, we have

$$[f(x), \alpha(x) + \alpha(y)] = [f(x) + 4D(x, x, y) + 6D(x, y, x, y) + 4D(x, x, y, y) + f(y), \alpha(x) + \alpha(y)],$$

for all $x, y \in I$.

From (2) & (3), we get

$$4[D(x, x, y), \alpha(x)] + 6[D(x, x, y, y), \alpha(x)] + 4[D(x, y, x, y), \alpha(x)] + [f(y), \alpha(x)] + [f(x), \alpha(y)] + 4[D(x, x, y, y), \alpha(y)] + 4[D(x, y, y, y), \alpha(y)] \in Z,$$

for all $x, y \in I$.

Replacing $y$ by $-y$ in (4), we have

$$-4[D(x, x, y), \alpha(x)] + 6[D(x, x, y, y), \alpha(x)] - 4[D(x, y, x, y), \alpha(x)] + [f(y), \alpha(x)] - [f(x), \alpha(y)] + 4[D(x, x, y, y), \alpha(y)] - 6[D(x, x, y, y), \alpha(y)] + 4[D(x, y, y, y), \alpha(y)] \in Z,$$

for all $x, y \in I$.

Substitute $z \neq x$ in (2) and rewrite (2) as

$$4[D(x, x, y), \alpha(x)] + 4[D(x, x, y), \alpha(x)] = [f(x), \alpha(x)] + [f(y), \alpha(x)] + 6[D(x, x, y, y), \alpha(y)] \in Z, \text{ for all } x, y \in I.$$

Using 2-torsion freeness of $R$, we get

$$4[D(x, x, y), \alpha(x)] + 4[D(x, x, y, y), \alpha(x)] + [f(x), \alpha(x)] + [f(y), \alpha(x)] + 6[D(x, x, y, y), \alpha(y)] \in Z, \text{ for all } x, y \in I.$$

Replacing $y$ by $-y$ in (7), we obtain

$$12[D(x, x, y, z), \alpha(x)] + 12[D(x, x, z, y), \alpha(x)] + 12[D(x, x, z, y), \alpha(x)] + 6[D(x, x, z, y), \alpha(y)] + 6[D(x, x, y, z), \alpha(y)] + 6[D(x, x, y, z), \alpha(y)] \in Z, \text{ for all } x, y, z, a \in I.$$

Using of 2-torsion freeness ring, we have

$$12[D(x, x, y, z), \alpha(x)] + 12[D(x, x, y, z), \alpha(y)] + 6[D(x, x, y, z), \alpha(z)] \in Z, \text{ for all } x, y, z, \alpha \in I.$$
Therefore, from equation (11), we get
\[ 4[D(x, x, x), f(x)] + [f(x), f(x)] = 0, \text{ for all } x, z \in I. \] (10)

On the other hand, taking \( z = x^2 \) in equation (10), we get
\[ 4[D(x, x, x), f(x)] + [f(x), f(x)] = 0, \text{ for all } x, z \in I. \]

Again replaced \( z \) by \( xz \) in (10) and using (10) we obtain
\[ 4[D(x, x, z), f(x)] + [f(x), f(x)] = 0, \text{ for all } z \in I. \]

It follows that (11) holds for all \( x, z \in I \).

Replacing \( z \) by \( [f(x), f(x)] \) in (13), we get
\[ \alpha ([f(x), f(x)] f(x)) + 8[f(x), f(x)] f(x) = 0, \text{ for all } x, z \in I. \] (13)

Since \( f \) is commutative in \( I \), and we have 2, 3- torsion freeness of \( R \), we have
\[ 2[f(x), f(x)]^3 = 0. \]

It follows that \( (2[f(x), f(x)]^2) R 2[f(x), f(x)]^2 = 0. \)

Since \( R \) is semiprime, we have
\[ 2[f(x), f(x)]^2 = 0, \text{ for all } x \in I. \] (14)

On the other hand, taking \( z = x^2 \) in equation (10), we get
\[ 4[D(x, x, x), f(x)] + [f(x), f(x)] = 0, \text{ for all } x, z \in I. \]

Since \( f \) is commuting on \( I \) and using equation (16), we get
6[f(\alpha(x),\alpha(x))]^2 = 0, for all x \in I.

We have 2-torsion freeness, we get
3[f(x),\alpha(x)]^2 = 0, for all x \in I.

Comparing (14) and (17), we get
\[ f(x),\alpha(x) = 0, \text{for all } x \in I. \]  

(17)

Note that zero is the only nilpotent element in the center of semiprime ring. Thus,
\[ f(x),\alpha(x) = 0, \text{for all } x \in I. \]

This completes the proof.

**Corollary 1:** Let R be a 3!-torsion free prime ring, I be a non-zero ideal of R and \( \alpha \) be an anti-automorphism of R. Suppose that there exists a non-zero symmetric skew 4-derivation \( D: R^2 \to R \) associated with the anti-automorphism \( \alpha \) such that the trace function \( f \) is commuting on \( I \) and \( (f(x),\alpha(x)) \in Z, \) for all \( x \in I \), then \( D = 0 \).

**Proof:** From theorem 1, \( (f(x),\alpha(x)) \in Z, \) for all \( x \in I \), then we have \( (f(x),\alpha(x)) = 0, \) for all \( x \in I. \)

From [4, Theorem 1] states that \( (f(x),\alpha(x)) = 0, \) for all \( x \in I, \) then \( D = 0. \)

Observing above two relations we concludes that \( (f(x),\alpha(x)) \in Z, \) for all \( x \in I, \) then \( D = 0. \)

**References**


