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On The Tensor Product Of Representations For The Symmetric GroupsS_n

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Abstract: *This paper covers our computational work and the algorithms designed and implemented in the construction of the main program designated for the determination of the tensor product of representations for the symmetric groups* $S_n, 1 \le n \le 14$ *including a flow – diagram of the main program. We have also been able to design the program of the diagram of the tensor product of permutations. Some algorithms are followed by simple examples for illustrations.*

I. Introduction

The tensor product of representation groups S_n has Morris [10] for , $1 \le n \le 14$. It seems to be that there is no ready – program available to calculate those tables. We give algorithms of programs to calculate tensor product table for $S_n, 1 \le n \le 14$. In our work we adopt the tensor product of two matrix as follows:

Definition

Let $A \in M_n(K), B \in M_m(K)$ we defined a matrix
$A \otimes B \in M_{nm}(K)$ Put

$$A \otimes B = \begin{pmatrix} \alpha_{11}B & \alpha_{12}B & \dots & \alpha_{1n}B \\ \alpha_{21}B & \alpha_{22}B & \dots & \alpha_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1}B & \alpha_{n2}B & \dots & \alpha_{nn}B \end{pmatrix}_{nm \times nm}$$

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{pmatrix}_{n \times n}$$

$$B = \begin{pmatrix} \beta_{11} & \beta_{12} & \dots & \beta_{1m} \\ \beta_{21} & \beta_{22} & \dots & \beta_{2m} \\ \vdots & \vdots & & \vdots \\ \beta_{m1} & \beta_{m2} & \dots & \beta_{mm} \end{pmatrix}_{m \times m}$$

Thus

$$A \otimes B = \begin{pmatrix} \delta_{11} & \delta_{12} & \dots & \delta_{1K} \\ \delta_{21} & \delta_{22} & \dots & \delta_{2K} \\ \vdots & \vdots & \vdots & \vdots \\ \delta_{K1} & \delta_{K2} & \dots & \delta_{KK} \end{pmatrix}_{m \times m}$$

Where

$$\delta_{11} = \begin{pmatrix} \alpha_{11}\beta_{11} & \alpha_{11}\beta_{12} & \dots & \alpha_{11}\beta_{1m} \\ \alpha_{11}\beta_{21} & \alpha_{11}\beta_{22} & \dots & \alpha_{11}\beta_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{11}\beta_{m1} & \alpha_{11}\beta_{m2} & \dots & \alpha_{11}\beta_{mm} \end{pmatrix}_{m \times m}$$

$$\delta_{1K} = \begin{pmatrix} \alpha_{1n}\beta_{11} & \alpha_{1n}\beta_{12} & \cdots & \alpha_{1n}\beta_{1m} \\ \alpha_{1n}\beta_{21} & \alpha_{1n}\beta_{22} & \cdots & \alpha_{1n}\beta_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{1n}\beta_{m1} & \alpha_{1n}\beta_{m2} & \cdots & \alpha_{1n}\beta_{mm} \end{pmatrix}_{m\times m}$$

$$\delta_{KK} = \begin{pmatrix} \alpha_{nn}\beta_{11} & \alpha_{nn}\beta_{12} & \cdots & \alpha_{nn}\beta_{1m} \\ \alpha_{nn}\beta_{21} & \alpha_{nn}\beta_{22} & \cdots & \alpha_{nn}\beta_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{nn}\beta_{m1} & \alpha_{nn}\beta_{m2} & \cdots & \alpha_{nn}\beta_{mm} \end{pmatrix}_{m\times m}$$

And $K = nm$

Example

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}_{2\times 2} \quad , \quad B = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 1 & 2 \\ 5 & 6 & 4 \end{pmatrix}_{3\times 3} \quad ,$$

$$A \otimes B = \left(\begin{array}{ccc|ccc} 1 & 3 & -1 & 2 & 6 & -2 \\ 2 & 1 & 2 & 4 & 5 & 4 \\ 5 & 6 & 4 & 10 & 12 & 8 \\ \hline -1 & -3 & 1 & & & \\ -2 & -1 & -2 & & 0 & \\ -5 & -6 & -4 & & & \end{array}\right)_{6\times 6}$$

Proposition

We prove that:

Let $A, A^{'} \in M_n(K), B, B^{'} \in M_m(K)$ then

1. $\left(A + A^{'}\right) \otimes B = (A \otimes B) + \left(A^{'} \otimes B\right)$
2. $(A \otimes B)\left(A^{'} \otimes B\right) = AA^{'} \otimes BB^{'}$

Prove:

(2) Let

$A = \left(a_{ij}\right)_{n\times n}$ $, A^{'} = \left(a_{ij}^{'}\right)_{n\times n}$

$B = \left(b_{ij}\right)_{m\times m}$ $, B^{'} = \left(b_{ij}^{'}\right)_{m\times m}$

$\therefore AA^{'} \otimes BB^{'} = \left(a_{ij}\right)\left(a_{ij}^{'}\right) \otimes \left(b_{ij}\right)\left(b_{ij}^{'}\right)$

$$C_{ij} = \sum_{K=1}^{n} a_{ik}a_{kj}$$

Where

$$\begin{aligned} AA^{'} \otimes BB^{'} &= \left(c_{ij}\right) \otimes BB^{'} \\ &= \left(C_{ij}BB^{'}\right) \\ &= \sum a_{ik}^{'}a_{kj}^{'}BB^{'} \\ &= \left[a_{i1}a_{ij}^{'}B + a_{i2}a_{2j}^{'}B + \cdots + a_{in}a_{2n}^{'}B\right]B^{'} \\ &= \left[a_{i1}Ba_{ij}^{'}B^{'} + a_{i2}Ba_{2j}^{'}B^{'} + \cdots + a_{in}Ba_{2n}^{'}B^{'}\right] \\ &= \left(\sum_{K=1}^{n} a_{ik}Ba_{kj}^{'}B^{'}\right)_{n.m\times n.m} \end{aligned}$$

$$= \sum_{k=1}^{n.m} a_{ik}^{*} a_{kj}^{'}$$

And $(A \otimes B)(A' \otimes B') = (a_{ij}B)(a'_{ij}B)$

$$= (a_{ik}^{*})(a_{kj}^{*})$$
$$= \sum_{k=1}^{n.m} a_{ik}^{*} a_{kj}^{'}$$

Where $a_{ik}^{*} = (a_{ik}B)$

(1) $(A + A') \otimes B = (A \otimes B) + (A' \otimes B)$

Let $A = (a_{ij})_{n\times n}$,$A' = (a'_{ij})_{n\times n} B = (b_{ij})_{m\times m}$

$$\therefore (A + A') = (a_{ij})_{n\times n} + (a'_{ij})_{n\times n}$$
$$(A + A') \otimes B = \left((a_{ij} + a'_{ij})B\right)_{nm.nm} = (a_{ij}B + a'_{ij}B)_{nm.nm}$$
$$A \otimes B = (a_{ij}B)_{nm.nm}$$
$$A' \otimes B = (a'_{ij}B)_{nm.nm}$$
$$(A \otimes B) + (A' \otimes B) = (a_{ij}B)_{nm.nm} + (a'_{ij}B)_{nm.nm}$$
$$\therefore (A + A') \otimes B = (A \otimes B) + (A' \otimes B)$$

Definition

1. $T$is called representation Group of G if $T(x)T(y) = T(x,y)T: G \longrightarrow M(F)$, $T(e) = I \forall x, y \in G, I$: is the identity matrix of degree n.
2. Let S and T be two representations of degree n and m of the group S_n for each $P \in S_n$, define $U(P) = S(P) \otimes T(P)$

Then U is a representation of degree $n.m$ we write $U = S \otimes T$. Now let χ_S, χ_T be two character of S and T respectively then $\chi_U = \chi_S \cdot \chi_T$.

The Algorithm

This part contains a collection of the computer – ready Fortran algorithms for many standard methods of number theory installed in our main program.

Algorithm1	The Number of degree of Presentation for groups S_n.
Algorithm2	The tensor product of two representation for groups S_n.
Algorithm 3	The tensor product of three representation for groups S_n.
Algorithm 4	The character of representation for groups S_n.
Algorithm 5	The main algorithm of tensor product. This algorithm is designed to evaluate the main program of the tensor product of representation for groups $S_n, n \geq 3$.

Algorithm (1) the number of degree of representation for groups S_n

Input: in (the degree of groups S_n)

Step 1: to evaluate m where

$$T: S_n \longrightarrow M_m(k)$$

$m$is the degree of matrix $M(K)$:

$$M_m(K) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{pmatrix}_{m\times m}$$

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Step 2: Do I=1 to m
        Do J=1 to m
        PRINT IA(I,J)
        END J-Loop
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END I-Loop

Output: the number of degree of representation for groups $S_n(m)$.

Example (1)

$T: S_3 \longrightarrow M_2(\mathbb{R})$

The degree of representation groups $S_3 = 2$

$$T_{(e)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$T_{(12)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$T_{(13)} = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$$

$$T_{(23)} = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$$

$$T_{(123)} = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$$

$$T_{(132)} = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$$

$T$is the representation of degree 2 for S_3.

Algorithm (2) Two representation for group S_n

Input: n (the degree of groups S_n)

Step 1: DC is the matrix of dimension $mn \times mn$

$$C(0,0) = 0$$

Do I = 1 to n
Do J = 1 to n
$T(p)$ =A(I,J)
END J-Loop
END I-Loop

Step 2: Do I= 1 to m
Do J = 1 to m
Set T(p)=B(I,J)
END J-Loop
END I-Loop

Step 3: call algorithm 1

Step 4: to evaluate C where
C(I, J)=A(I,J)*B

Step 5: Set C(1,1) = A(1,1)*B
C(1,2) = A(1,2)*B
⋮
C(1,n) = A(1,n)*B

$$B = \begin{pmatrix} B_{11} & B_{12} & \dots & B_{1m} \\ B_{21} & B_{22} & \dots & B_{2m} \\ \vdots & \vdots & & \vdots \\ B_{m1} & B_{m2} & \dots & B_{mm} \end{pmatrix}_{m\times m}$$

Step 5: Set

$$C = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1m} \\ C_{21} & C_{22} & \dots & C_{2m} \\ \vdots & \vdots & & \vdots \\ C_{m1} & C_{m2} & \dots & C_{mm} \end{pmatrix}_{m\times m}$$

Output: the tensor product of representation of $S_n G(mn, mn)$

Example

$$T_{(12)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{2\times 2}, \quad T_{(23)} = \begin{pmatrix} -1/2 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1/2 \end{pmatrix}_{2\times 2}, T_{(13)} = \begin{pmatrix} -1/2 & \frac{-\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & 1/2 \end{pmatrix}_{2\times 2}$$

$$T_{(13)} \otimes T_{(23)} = \left(\begin{array}{cc|cc} -1/2 & -\sqrt{3}/2 & & \\ -\sqrt{3}/2 & 1/2 & & 0 \\ \hline & & 1/2 & \sqrt{3}/2 \\ 0 & & \sqrt{3}/2 & -1/2 \end{array}\right)_{4\times 4}$$

$$T_{(13)} \otimes T_{(23)} = \left(\begin{array}{cc|cc} 1/4 & \sqrt{3}/4 & \sqrt{3}/4 & -3/4 \\ -\sqrt{3}/4 & -1/4 & -3/4 & -\sqrt{3}/4 \\ \hline \sqrt{3}/4 & -3/4 & -1/4 & \sqrt{3}/4 \\ -3/4 & -\sqrt{3}/4 & \sqrt{3}/4 & 1/4 \end{array}\right)_{4\times 4}$$

$$T_{(23)} \otimes T_{(12)} = \left(\begin{array}{cc|cc} 1/2 & 0 & \sqrt{3}/4 & 0 \\ 0 & -1/2 & 0 & -\sqrt{3}/4 \\ \hline \sqrt{3}/2 & 0 & 1/2 & 0 \\ 0 & -\sqrt{3}/2 & 0 & -1/2 \end{array}\right)_{4\times 4}$$

$$T_{(12)} \otimes T_{(23)} = \left(\begin{array}{cc|cc} -1/2 & \sqrt{3}/2 & & \\ \sqrt{3}/2 & 1/2 & & 0 \\ \hline & & 1/2 & -\sqrt{3}/2 \\ 0 & & -\sqrt{3}/2 & -1/2 \end{array}\right)_{4\times 4}$$

$$T_{(13)} \otimes T_{(12)} = \left(\begin{array}{cc|cc} -1/2 & 0 & -\sqrt{3}/4 & 0 \\ 0 & 1/2 & 0 & -\sqrt{3}/4 \\ \hline -\sqrt{3}/2 & 0 & 1/2 & 0 \\ 0 & \sqrt{3}/2 & 0 & -1/2 \end{array}\right)_{4\times 4}$$

$$T_{(13)} \otimes T_{(23)} = \left(\begin{array}{cc|cc} 1/4 & \sqrt{3}/4 & -\sqrt{3}/4 & -3/4 \\ \sqrt{3}/4 & -1/4 & -3/4 & \sqrt{3}/4 \\ \hline \sqrt{3}/4 & -3/4 & -1/4 & -\sqrt{3}/4 \\ -3/4 & -\sqrt{3}/4 & -\sqrt{3}/4 & 1/4 \end{array}\right)_{4\times 4}$$

Algorithm (3): Three representation for Groups S_n
Input: n (the degree of groups S_n)
Step 1: Call algorithm 2
Step 2: Do I = 1 to K
Do J = 1 to K
D(I,J)
END J-Loop
END I-Loop
Step 3: to evaluate R where
R(I, J)=C(I,J)*D
Step 4: Set
R(1,1) = C(1,1)*D
R(1,2) = C(1,2)*D
Where

$$D = \begin{pmatrix} D_{11} & D_{12} & \dots & D_{1K} \\ D_{21} & D_{22} & \dots & D_{2K} \\ \vdots & \vdots & & \vdots \\ D_{K1} & D_{K2} & \dots & D_{KK} \end{pmatrix}_{K\times K}$$

Step 5: Set

$$R = \begin{pmatrix} R_{11} & R_{12} & \dots & R_{1S} \\ R_{21} & R_{22} & \dots & R_{2S} \\ \vdots & \vdots & & \vdots \\ R_{S1} & R_{S2} & \dots & R_{SS} \end{pmatrix}_{S\times S}$$

Where $S = nm \times k$
Step 6: Do I= 1 to S
Do J = 1 to S
PRINT R(I,J)
END J-Loop
END I-Loop

Output: the Three representation for Groups $S_n R(S,S)$

Example

$$T_{(12)} \otimes T_{(23)} \otimes T_{(12)} = \begin{pmatrix} \begin{matrix} -1/2 & 0 & -\sqrt{3}/4 & 0 \\ 0 & 1/2 & 0 & -\sqrt{3}/4 \\ -\sqrt{3}/2 & 0 & 1/2 & 0 \\ 0 & \sqrt{3}/2 & 0 & -1/2 \end{matrix} & 0 \\ 0 & \begin{matrix} 1/2 & 0 & -\sqrt{3}/4 & 0 \\ 0 & -1/2 & 0 & -\sqrt{3}/4 \end{matrix} \end{pmatrix}$$

$$T_{(e)} \otimes T_{(23)} \otimes T_{(12)}$$

$$= \begin{pmatrix} \begin{matrix} -1/2 & 0 & -\sqrt{3}/4 & 0 \\ 0 & 1/2 & 0 & -\sqrt{3}/4 \\ -\sqrt{3}/2 & 0 & 1/2 & 0 \\ 0 & \sqrt{3}/2 & 0 & -1/2 \end{matrix} & 0 \\ 0 & \begin{matrix} -1/2 & 0 & -\sqrt{3}/4 & 0 \\ 0 & 1/2 & 0 & -\sqrt{3}/4 \end{matrix} \end{pmatrix}$$

$$T_{(e)} \otimes T_{(23)} \otimes T_{(12)}$$

$$= \begin{pmatrix} \begin{matrix} -1/2 & 0 & -\sqrt{3}/4 & 0 \\ 0 & 1/2 & 0 & \sqrt{3}/4 \\ \sqrt{3}/2 & 0 & 1/2 & 0 \\ 0 & -\sqrt{3}/2 & 0 & -1/2 \end{matrix} & 0 \\ 0 & \begin{matrix} -1/2 & 0 & \sqrt{3}/4 & 0 \\ 0 & 1/2 & 0 & -\sqrt{3}/4 \end{matrix} \end{pmatrix}$$

$$`T_{(e)} \otimes T_{(23)} \otimes T_{(12)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Algorithm (4) The character of representation for S_n
Input: n (the Degree of S_n)
Step 1: $\chi(0) = 0$
Step 2: Do I= 1 to m
    $\chi(I)$
    End I-Loop
Step 2: Do J = 1 to n
    $\chi(J)$
    End J-Loop
Step 4: Do I= 1 to m
    Do J = 1 to n
    $\chi(k) = \chi(I) * \chi(J)$
    END J-Loop
    END I-Loop
    PRINT$\chi(k)$

Step 5: set $\chi(k) = \begin{bmatrix} \chi_{(1)} \\ \chi_{(2)} \\ \chi_{(3)} \\ \vdots \\ \chi_{(S)} \end{bmatrix}$

$$S = (n.m)/2$$

Step 6: Call algorithm 3
Step 7: Call algorithm 4
Output: the character of representation for S_n.

$$(\chi(k), k = 1\ to S)$$

Class	1^3	12	3^1
Order	1	3	2
$\chi(1)$	1	1	1
$\chi(2)$	1	-1	1
$\chi(3)$	2	0	-1

In 13

$$\chi(1) \otimes \chi(2) = 1 * 1 = 1$$
$$\chi(1) \otimes \chi(3) = 1 * 2 = 2$$
$$\chi(2) \otimes \chi(3) = 1 * 2 = 2$$

In12

$$\chi(1) \otimes \chi(2) = 1 * -1 = -1$$
$$\chi(1) \otimes \chi(3) = 1 * 0 = 0$$
$$\chi(2) \otimes \chi(3) = -1 * 0 = 0$$

In 3^1

$$\chi(1) \otimes \chi(2) = 1 * 1 = 1$$
$$\chi(1) \otimes \chi(3) = 1 * 1 = 1$$
$$\chi(2) \otimes \chi(3) = 1 * (-1) = -1$$

In 1^3

$$\chi(1) \otimes \chi(2) \otimes \chi(3) = 1 * 1 * 2 = 2$$

$$\chi = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \\ 0 \\ 0 \\ -1 \\ 2 \end{bmatrix}$$

The Algorithm of the Main Program
The Tensor Product of Representation Groups S_n

$$1 \le n \le 12$$

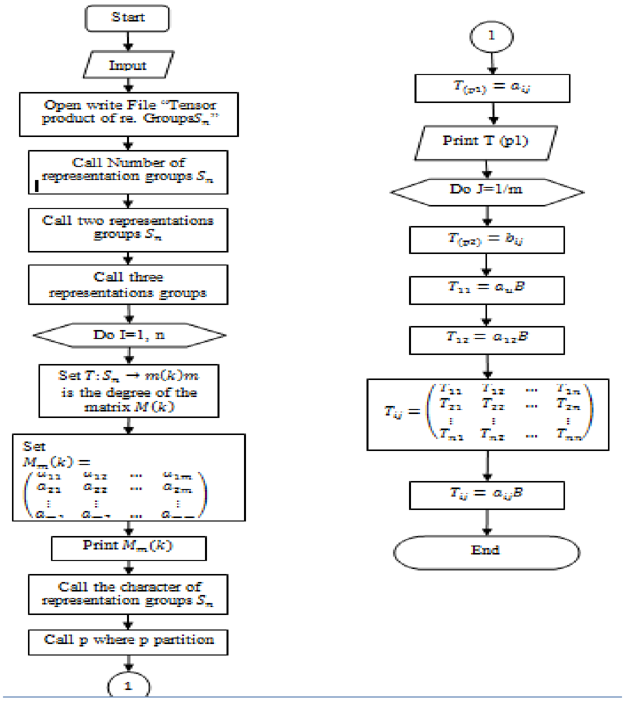
Input: n (the degree of the Symmetric groups)
Step 1: Call algorithm 1
Step 2: Call algorithm 2
Step 3: Call algorithm 3
Step 4: Call algorithm 4
Output : (T(I), I=1 to m) To evaluate the tensor product of representation groups S_n.
End

II. Recommendation

Our recommendation for future worke are

1. Evaluating the tensor product of representation for groups $S_n, n \ge 12$.
2. Evaluating the Tensor product of representation for groups $C_n = \langle g : g^n = e \rangle$.
3. Evaluating the Tensor product of regular representation for a group S_n.

Flow Diagram of the Main Program



References

[1]. K. Chen and w. Lee: "A general method to calculate all irreduciblerepresentations of a finite groups" Britain J. phys. A. math. Gen-19 (1989).

[2]. ML. Curtis: "Matrix Groups "2 nd. Ed Springer- verlag New york1984.

[3]. C. Curtis and I. Reiner: "Representation theory of finite groups andassociative algebras", John wiley and sone, New York (1962).

[4]. L. Dornhoff: "Group representation theory". Part A, Marcel Dekker,Inc-New York (1971).

[5]. H. DynyaAbdulkamid: "Some Identities relation to partitions andrepetitions of parts" M. SC. Thesis (2000).

[6]. J. B. Fraleigh: "A first course in abstract algebra", Wesleypublishingcompany (1982).

[7]. G. James and A. Kerber: "The Representation theory of the Symmetricgroups-Encyclopedia of Mathematics and its Application", Addision-wesiey, publishing company, (1981).

[8]. R-Keown:"An Introduction to group representation theory"Academic presses, New York (1975).

[9]. V. Krishnamurthy: "Combinatorics", New Delhi-Madras (1985).

[10]. A. O. Moms: "the Spin representation of the symmetric group, prac-London-Math. Soc (3) 12 (1992) 55-76.

[11]. K. K. Nwabuze: "Some definition of the Artin exponent of finite groups", MSRI, preprint (1996-077)

[12]. J. Owen Brison. "Alternating groups and a conjecture about rationalvalued characters", Linear and multi linear algebra, vol. 24, pp: 199. 207 (1989).

[13]. J. J. Rotman: "An Introduction to the theory of groups" 3 ed. Wn. C.Brown publishers (1988)