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# Random Attractors for Stochastic Wave Equations with Nonlinear Damping and With Multiplicative Noise *

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***Abstract:*** *In this paper, we study the existence of a random attractor for a stochastic dynamical system generated by nonlinear damped wave equation with multiplicative white noise defined on* $\mathbb{R}^3$*.First we proving the existence of the pullback absorbing set and the pullback asymptotic compactness of the cocycle in a certain parameter region by using uniform estimate then we prove the existence of a random attractor.*
***Key words:*** *stochastic wave equation; nonlinear damping; random attractor; pullback asymptotic compactness.*
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## 1. Introduction

We consider the asymptotic behavior of non-autonomous stochastic nonlinear damped wave equation with multiplicative noise defined in the space $\mathbb{R}^3$:

$$u_{tt} - \Delta u + \sigma(u)u_t + \lambda u + f(u) = g(x,t) + \varepsilon u \frac{dw}{dt} \qquad (1,1)$$

With initial data

$$u(\tau, x) = u_0(x) \quad , \ u_\tau(\tau, x) = u_1(x) \quad x \in \mathbb{R}^3, \tau \in \mathbb{R} \qquad (1.2)$$

Where $\Delta$ is the laplacian with respect to the variable $x \in \mathbb{R}^3$ , $u = u(t,x)$ is a real function of $\mathbb{R}^3$ and $t \geq \tau, \tau \in \mathbb{R}$ $\varepsilon$ and $\lambda$ are positive constants. The given function $g(x,t) \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^3)$ is external force dependent of t , W(t) are independent two sided real-valued wiener processes on probability space and define $\{\theta_t\}_{t\in\mathbb{R}}$ on $\mathbb{R}^3$ by $\theta_t\omega(.) = \omega(.+t) - \omega(t), \omega \in \Omega$

The following conditions are necessary to obtain our main results.

(a)The function $\sigma \in C^1(R)$ is not identically equal to zero and satisfying the following conditions.

$$-\alpha \leq \alpha_1 \leq \sigma(s) \leq \alpha_2 < +\infty \,; |\acute{\sigma}(s)| \leq \alpha_3 \qquad (1.3)$$

$\alpha_2 \geq |\alpha_1|$ , where $\alpha, \alpha_1, \alpha_2$ and $\alpha_3$ are positive constants.

(b) The nonlinear given function $f \in C^1(\mathbb{R})$ with $f(0)=0$ ,and it satisfies the following conditions.

$$f(s) \geq k|s|, \forall s \in \mathbb{R} \quad ; \qquad (1.4)$$

$$F(s) = \int_0^s f(r)dr \geq \beta_1(|s|^2 - 1), \forall s \in \mathbb{R} \; ; \qquad (1.5)$$

and

$$sf(s) \geq \beta_2(F(s) - 1), \forall s \in \mathbb{R} \quad ; \qquad (1.6)$$

$$\acute{f}(s) \geq k \ , \forall\, s \in \mathbb{R} \quad ; \qquad (1.7)$$

Where C, $\beta_1, \beta_2, k$ are positive constants.

It is well known that wave equation describe a great variety of wave phenomena occurring in the extensive applications of mathematical such as physics, engineering, biology and geosciences. In general there have been a lot of profound results on the dynamics of a variety of systems to equation(1,1)-(1,2),when $\sigma(s) = \alpha$ has studied by many authors,regraded the long time behavior of solution for deterministic and asymptotical behavior of solution for stochastic differential equation.For example,the asymptotical behavior of solutions for

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deterministic damped wave equation, regarding which kind of deterministic(global, uniform and exponential) has been investigated by many author(see[1,4,11,14,16-17,20,23-25,27,31,36]).

For the asymptotical behavior of solution of stochastic wave equation(1,1)-(1,2) when $\sigma(s) = \alpha$ that case reduce a stochastic damped wave equation autonomous and non-autonomous with additive noise have been studied by several author(see[3,10,13,18-19,22,29]).

Random attractor for non-autonomous stochastic damped wave equation(1,1)-(1,2)when $\sigma(s) = \alpha$ with multiplicative noise regarded in bounded domain and on unbounded domain have been investigated by (see[6,8,26,30,34-35]). But the case of equation (1,1)-(1,2)has investigated with additive noise by(see,[33].

In this paper we will combine the splitting technique in [35] with the idea of uniform estimates on the tails of solutions to investigate the existence of a random attractor for the stochastic damped wave equation with multiplicative noise defined on$\mathbb{R}^3$. So far as we know, there were no results on random attractors for stochastic damped wave equation (1,1)-(1,2) with random term and nonlinear damping term defined on bounded domain , which is more important and interesting.

This paper is organized as follows. Next section, we recall some preliminaries and properties for general random dynamical system and results on the existence of a pullback random attractor for random dynamical systems. In Section three, we define a continuous random dynamical system for (1.1) in $H^1$ $(\mathbb{R}^3) \times L^2$ $(\mathbb{R}^3)$ then we drive all necessary proving the existence of bounded absorbing sets and the asymptotic compactness of the equation to obtain the uniform estimates of solution as $t \to \infty$ In Section four,In Section five,we first establish the asymptotic compactness of the solution operator by giving uniform estimates on the tails of solutions, we prove the existence of a random attractor.

Written as$|.|_X$. The letters $C$ and $C_i$(i=1, 2) are generic positive constants which may change their values form line to line or even in the same line.

## 2. Preliminaries and Abstract Results

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space and $(X,d)$ be a polish space with the Borel $\sigma$-algebra $B(X)$.The distance between $x \in X$ and $B \subseteq X$ is denoted by $d(x,B)$.If $B \subseteq X$ and $C \subseteq X$, the hausdorff semi-distance B to C is denoted by $d(B, C) = \sup_{x \in B} d(x, C)$.

**Definition 2.1.** $(\Omega, \mathcal{F}, \mathrm{P}, \{\theta_t\}_{t\in\mathbb{R}})$ is called a metric dynamical system if $\theta: \mathbb{R} \times \Omega \to \Omega$ is $(\mathrm{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$ measurable, $\theta_0$ is the identity on $\Omega$, $\theta_{t+s} = \theta_t \circ \theta_t$ , $\forall\, s, t \in \mathbb{R}$ and $\theta_0 P = P$ , $\forall\, t \in \mathbb{R}$ .

**Definition 2.2.** A mapping $\Phi(\mathrm{t}, \tau, \omega, \mathrm{x}): \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is called continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathrm{P}, \{\theta_t\}_{t\in\mathbb{R}})$ , if for all $\tau \in \mathbb{R}$ , $\omega \in \Omega$ and $s, t \in \mathbb{R}^+$ the following conditions are satisfied:

i) $\Phi(., \tau, ., .): \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is a $(\mathrm{B}(\mathbb{R}^+) \times \mathcal{F}, \mathrm{B(x)})$ measurable mapping

ii) $\Phi(., \tau, ., .)$ is identity on X .

iii) $\Phi(\mathrm{t} + \mathrm{s}, \tau, \omega, .) = \Phi(t, \tau + s, \theta_s\omega, .)\mathrm{o}\Phi(s, \tau, \omega\,)$

iv) $\Phi(\mathrm{t}, \tau, \omega, \mathrm{x}): \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is continuous .

**Definition 2.3.** Let $2^X$ be the collection of all subsets of X ,a set valued mapping $\tau, \omega \to D(t, \omega): \mathbb{R} \times \Omega \to 2^X$ is called measurable with respect to $\mathcal{F}$ in $\Omega$ if $D(t, \omega)$ is a(usually closed )nonempty subset of X and the mapping $\omega \in \Omega \to d(X, B(\tau, \omega))$ is $(\mathcal{F}, \mathrm{B}(\mathbb{R}))$ ) -measurable for every fixed $x \in X$, and $\tau \in \mathbb{R}$ , then $B = B(\tau, \omega): \tau \in \mathbb{R}, \omega \in \Omega$ is called a random set .

**Definition 2.4.** A random bounded set $\{B = B(\tau, \omega): \tau \in \mathbb{R}, \omega \in \Omega\} \in D$ of X is called tempered with respect to$\{\theta_t\}_{t\in\mathbb{R}}$, if for p-a.e $\omega \in \Omega$ ,

$$\lim_{t\to\infty} e^{-\beta t} d(B(\theta_{-t}\omega)) = 0 \ , \forall\, \beta > 0$$

Where

$$d(B) = \sup_{\alpha \in B} \|x\|_X$$

.

**Definition 2.5.** Let D be a collection of random subset of X and $\{K = K(\tau, \omega): \tau \in \mathbb{R}, \omega \in \Omega\} \in D$ ,then K is called an absorbing set of $\Phi \in D$ if for all $\tau \in \mathbb{R}, \omega \in \Omega$ and $B \in D$ there exists $T = T(\tau, \omega, \mathrm{B}) > 0$ such that.

$$\Phi(\mathrm{t}, \tau - \mathrm{t}, \theta_{-\mathrm{t}}\omega, \mathrm{B}(\tau - \mathrm{t}, \theta_{-\mathrm{t}}\omega)) \subseteq K(\tau, \omega) \quad , \forall\, t \geq T$$

**Definition 2.6.** Let D be a collection of random subset of X the $\Phi$ is said to be D-pullback asymptotically compact in x if for p-a.e $\omega \in \Omega$ , $\left\{\Phi\left(\mathrm{t_n}, \tau - \mathrm{t_n}, \theta_{-\mathrm{t_n}}\omega, \mathrm{x_n}\right)\right\}_{n=1}^{\infty}$ has a convergent subsequence in x when ever $\mathrm{t_n} \to \infty$ and $\mathrm{x_n} \in \mathrm{B}(\theta_{-\mathrm{t}}\omega)$ with$\mathrm{B}(\omega)$.

**Definition 2.7.** Let D be a collection of random subset of X and $\{A = A(\tau, \omega): \tau \in \mathbb{R}, \omega \in \Omega\} \in D$ then A is called a D-random attractor (or D-pullback attractor) for $\Phi$ the following conditions are satisfied: for all $t \in \mathbb{R}^+, \tau \in \mathbb{R}$ and $\omega \in \Omega$ }

i) $A(\tau, \omega)$ is compact , and $\omega \in \Omega \to d(X, A(\omega))$ is measurable for every $\mathrm{x} \in \mathrm{X}$

ii) $A(\tau, \omega)$ is invariant ,that is

$$\Phi\big(\mathrm{t},\tau,\omega,A(\tau,\omega)\big) = A(\tau+t,\theta_t\omega) \quad ,\forall\, t\geq\tau.$$

iii) $A(\tau,\omega)$ Attracts every set in D, that is for every $\{B=B(\tau,\omega):\tau\in\mathbb{R}\,,\omega\in\Omega\}\in D$ ,

$$\lim_{t\to\infty} e^{-\beta t}\, d_H(\Phi\big(\mathrm{t},\tau-\mathrm{t},\theta_{-\mathrm{t}}\omega,\mathrm{B}(\tau-\mathrm{t},\theta_{-\mathrm{t}}\omega)\big),A(\tau,\omega))=0$$

Where $d_H$ is the hausdroff semi -metric given by

$$d_H(Y,Z)=\sup_{y\in Y}\inf_{z\in Z}\|y-z\|_X$$

For any $Y\in X$ and$Z\in X$ .

**proposition 2.8** Let D be a neighborhood-closed collection of $(\tau,\omega)$- parameterized families of nonempty subsets of X and $\Phi$ be a continuous cocycle on X over $\mathbb{R}$ and $(\Omega\,,\mathcal{F},\mathrm{P}\,,\{\theta_t\}_{t\in\mathbb{R}})$ is .then $\Phi$ has a pullback D-attractor A in D if and only if $\Phi$ is pullback D-asymptotically compact in X and $\Phi$ has a closed , $\mathcal{F}$ -measurable pullback D-absorbing set K in D. The unique pullback D-attractor $A=A(\tau,\omega)$ is given

$$A(\tau,\omega)=\bigcap_{r\geq0}\overline{\bigcup_{\tau\geq r}\Phi\big(t,\tau-t,\theta_{-\mathrm{t}}\omega,\mathrm{K}(\tau-\mathrm{t},\theta_{-\mathrm{t}}\omega)\big)}$$

$$\tau\in\mathbb{R},\omega\in\Omega$$

## 3. Cocycle for A Stochastic Damped Wave Equations

In this section ,we focus on the existence of a continuous cocycle for the stochastic wave equation on $\mathbb{H}^1\,(\mathbb{R}^3)\times\mathbb{L}^2\,(\mathbb{R}^3)$ for our purpose, we need first introduce a transformation $\xi(x,t)=u_t+\delta u$ , is a positive constant, then (1.1)-(1.2) can be rewritten in the form of the following equivalent system

$$\xi(x,t)=u_t+\delta u \tag{3.1}$$

$$\xi_t+(\sigma(u)-\delta)\xi(x,t)+\big(\lambda+\delta^2-\delta\sigma(u)\big)u+\mathrm{A}u+f(u)=g(x,t)+\varepsilon uo\frac{d\omega}{dt} \tag{3.2}$$

$$u(x,\tau)=u_0(x),\xi(x,\tau)=\xi_0(x)=u_1(x)+\delta u_0(x) \tag{3.3}$$

TO study the dynamical behavior of problem (3.1)-(3.3),we need to convert the stochastic system into deterministic one with a random parameter. We introduce an Ornstein-Uhlenbeck process driven by the Brownian motion, which satisfies the *It*ô differential equation

$$dz+\delta zdt=dw\,,\ \delta>0 \tag{3.4}$$

And the solution is given by

$$z(\theta_t\omega)=z(t,\omega)=-\delta\int_\infty^0 e^{\delta s}(\theta_t\omega)sds\ \ ,\ \in\mathbb{R}\ \ ,\ \omega\in\Omega \tag{3.5}$$

From [5],it is known that the random variable $|z(\omega)|$ is tempered and there is a invariant set $\overline{\Omega}\subseteq\Omega$ of full P measure such that $z(\theta_t\omega)=z(t,\omega)$is continuous in t for every $\omega\in\overline{\Omega}$ . For convenient we shall write $\overline{\Omega}$ as $\Omega$ .To define a cocycle for problem (3.1)-(3.3), let $v=\xi(x,t)-\varepsilon uz(\theta_t\omega)$ then (3.1)-(3.3) can be rewritten as the equivalent system with random coefficients but without white noise

$$u_t+\delta u=v+\varepsilon uz(\theta_t\omega) \tag{3.6}$$

$$\begin{aligned}v_t+(\sigma(u)-\delta)v+\big(\lambda+\delta^2-\delta\sigma(u)\big)u+\mathrm{A}u+f(u)\\=g(x,t)-\varepsilon(v+\sigma(u)u-2\delta u+\varepsilon uz(\theta_t\omega))z\theta_t\omega\end{aligned} \tag{3.7}$$

$$u(x,\tau)=u_0(x),v(x,\tau)=v_0(x)=u_1(x)+\delta u_0(x)-\varepsilon uz(\theta_t\omega) \tag{3.8}$$

Where $\mathrm{A}=-\Delta$ , $\in\mathbb{R}^3$ , $\tau\in\mathbb{R}$.

Let $E(\mathbb{R}^3)=H^1\,(\mathbb{R}^3)\times\mathbb{L}^2\,(\mathbb{R}^3)$ and endow it the standard norm

$$\|Y\|^2_{H^1\,(\mathbb{R}^3)\times\mathbb{L}^2\,(\mathbb{R}^3)}=(\|v\|^2+(\delta^2-\delta\alpha_3+\lambda)\|u\|^2+\|\nabla u\|^2)^{\frac{1}{2}} \tag{3.9}$$

For $y=(u,v)^{\intercal}\in E(\mathbb{R}^3)$ where$\intercal$ stands for the transformation.

We define new norm $\|Y\|^2_{E\,(\mathbb{R}^3)}$ by

$$\|Y\|^2_{E\,(\mathbb{R}^3)}=(\|v\|^2+(\lambda+\delta^2-\delta\sigma(u))\|u\|^2+\|\nabla u\|^2)^{\frac{1}{2}} \tag{3.10}$$

**Lemma 3.1** (see[30 ,35]) for the Ornstien-Uhlenbeck process $z(\theta_t\omega)$ in (3.4)-(3.5),we have

$$\lim_{t\to\pm\infty}\frac{|z(\theta_t\omega)|}{|t|}=0, \tag{3.11}$$

$$\lim_{t\to\pm\infty}\frac{1}{t}\int_{-t}^0 z(\theta_s\omega)ds=E[z(\theta_t\omega)]=0, \tag{3.12}$$

$$\lim_{t\to\pm\infty}\frac{1}{t}\int_{-t}^0 z(\theta_s\omega)ds=E[z(\theta_t\omega)]=\frac{1}{\sqrt{\pi\delta}}\,, \tag{3.13}$$

$$\lim_{t\to\pm\infty}\frac{1}{t}\int_{-t}^{0}|z(\theta_s\omega)|^2ds = E[|z(\theta_t\omega)|^2] = \frac{1}{2\delta}\,. \tag{3.14}$$

By (3.13)-(3.14), there exists $T_1 > 0$, such that for all $t \geq T_1(\omega)$,

$$\int_{-t}^{0} z(\theta_s\omega)ds < \frac{2}{\sqrt{\pi\delta}}t\,, \quad \int_{-t}^{0}|z(\theta_s\omega)|^2ds < \frac{1}{2\delta}\,t. \tag{3.15}$$

**Lemma 3.2** (see[25]) Put $\varphi(t,\tau,\omega,\varphi_0) = (u(t,\tau,\omega,u_0,v(t,\tau,\omega,v_0)))^{\intercal}$ ,where $\varphi_0 = (u_0,v_0)^{\intercal}$ , and let (1.3)-(1.7) hold. Then for every $\omega\in\Omega$ and $\varphi_0 \in E(\mathbb{R}^3)$ the problem (3.6)-(3.8) has a unique solution $(u(t,\tau-t,\omega,u_0,v(t,\tau-t,\omega,v_0))^{\intercal}$ which is continuous with respect to $(u_0,v_0)^{\intercal}$ in E for all $t$ >0. Hence the solution mapping

$$\Phi(t,\tau,\omega,\varphi_0) = \varphi(\tau,\tau,\omega,\varphi_0)\,. \tag{3.16}$$

Generates a continuous random dynamical system.

Introducing the homeomorphism $P(\theta_t\omega)(u,\xi)^{\intercal} = (u,\xi+\varepsilon uz(\theta_t\omega))^{\intercal}$ , $(u,\xi)^{\intercal}\in E$ with an inverse homeomorphism $P^{-1}(\theta_t\omega)(u,\xi)^{\intercal} = (u,\xi-\varepsilon uz(\theta_t\omega))^{\intercal}$ then the transformation.

$$\widetilde{\Phi}(t,\omega) = P(\theta_t\omega)\varphi(t,\omega)P^{-1}(t,\omega). \tag{3.17}$$

Also generate a random dynamical system associated with (3.1)-(3.3). Note that the two random dynamical systems are equivalent. By (3.17),it is easy to check that $\widetilde{\Phi}(t,\omega)$ has a random attractor provided $\Phi(t,\omega)$possesses a random attractor .Then we only need to consider the random dynamical system $\Phi(t,\omega)$

We also need the following condition on $g(x,t)\in L^2_{loc}(\mathbb{R},L^2(\mathbb{R}^3)$ there exists a positive constant $\sigma$ such that

$$\int_{-\infty}^{\tau}\|g(.,s)\|^2ds < \infty\ \forall\tau\in\mathbb{R}\ . \tag{3.18}$$

Which implies that?

$$\lim_{s\to\infty}\int_{-\infty}^{\tau}\int_{|x|\geq k}e^{\sigma s}|g(x,s)|^2dxds = 0\ \ \forall\tau\in\mathbb{R}\,. \tag{3.19}$$

The condition (3.18) does not require g(.,t)to be bounded in $L^2(\mathbb{R})$ when$|t|\to\infty$.

For any bounded nonempty subset B of E, denote by$\|B\| = sup_{\varphi\in B}\|\varphi\|_E$in the subsequent section B will be the following neighborhood-closed of E

$B = \{B(\tau,\omega){:}\,\tau\in R,\omega\in\Omega\}$:B is bounded and satisfy

$$\lim_{s\to\infty} e^{\sigma s}\|B(\tau+s,\theta_s\omega)\|_E^2 = 0\ \ . \tag{3.20}$$

## 4. Uniform Estimates Of Solutions

In this section, we drive uniform estimates on the solutions of the stochastic strongly damped wave equation (3.1)-(3.3) defined on $R^3$ when $t\to\infty$ with the purpose of proving the existence of a bounded random absorbing set and the asymptotic compactness of the random dynamical system associated with the equations. In particular, we will show that the tails of the solutions, i.e. solutions evaluated at large values of |*x*|, are uniformly small when the time is sufficiently large.

We always assume that D is the collection of all tempered subsets of E from now on. Let $\delta\in(0,1)$ be small enough such that

$$\delta^2-\delta\alpha_2+\lambda>0\ ,3\alpha-\delta>0. \tag{4,1}$$

Throughout this section we assume that

$$|\varepsilon| = \frac{-2\sqrt{\delta}(\gamma_1\gamma_2+1)+\sqrt{4\delta(\gamma_1\gamma_2+1)^2-\pi\delta\gamma_1\sigma.}}{\gamma_1\sqrt{\delta}} \tag{4.2}$$

Where $\gamma_1 = 1+\frac{1}{\delta^2-\delta\alpha_2+\lambda}$ and $\gamma_2 = \delta+\frac{\alpha_2}{2}+(\alpha_2-\alpha)\delta$

The next lemma shows that $\Phi$ has a random absorbing set in D.

**Lemma 4.1** Assume that (1.3)-(1.7) hold and $g(x,t)\in L^2_{loc}(\mathbb{R},L^2(\mathbb{R}^3)$ satisfies (3.18) and (3.19). Then there exists $B(\tau-t,\theta_{-t}\omega)\in D$ such that $\{K(\omega)\}_{\omega\in\Omega}$ is a random absorbing set for $\Phi$ in D, that is, for any $B = \{B(\omega)\}_{\omega\in\Omega}$ and P-a.e$\omega\in\Omega$, there is $T = T(\tau,\omega,B) > 0$ such that

$$\Phi(t,\tau-t,\theta_{-t}\omega,B(\tau-t,\theta_{-t}\omega)) \subseteq K(\tau,\omega)\quad ,\forall\,t\geq T \tag{4.3}$$

**Proof** Taking inner product of the second equation of (3.7) with v in $L^2(\mathbb{R}^3)$,we find that

$$(v_t,v)+(\sigma(u)-\delta)(v,v)+\big(\lambda+\delta^2-\delta\sigma(u)\big)(u,v)+(Au,v)+(f(u),v)$$

$$= (g(x,t),v)-\varepsilon\Big(\big(v+\sigma(u)u-2\delta u+\varepsilon uz(\theta_t\omega)\big)z(\theta_t\omega),v\Big)\ \ . \tag{4.4}$$

By the equation (3.6), we substituting the above v into the second and third terms on the left-hand side of (4.4), we find that

$$v = \frac{du}{dt} + \delta u - \varepsilon u z(\theta_t\omega) \quad , \qquad (4.5)$$

$$(u, v) = \left(u, \frac{du}{dt} + \delta u - \varepsilon u z(\theta_t\omega)\right) \le \frac{1}{2}\frac{d}{dt}\|u\|^2 + \delta\|u\|^2 - |\varepsilon||z(\theta_t\omega)|\|u\|^2 \quad , \quad (4.6)$$

$$(Au, v) = \left(-\Delta u, \frac{du}{dt} + \delta u - \varepsilon u z(\theta_t\omega)\right) \le \left(\nabla u, \nabla\left(\frac{du}{dt} + \delta u - \varepsilon u z(\theta_t\omega)\right)\right)$$
$$\le \frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + \delta\|\nabla u\|^2 - |\varepsilon||z(\theta_t\omega)|\|\nabla u\|^2. \qquad (4.7)$$

From condition (1.4)-(1.6) we get

$$(f(u), v) = \left(f(u), \frac{du}{dt} + \delta u - \varepsilon u z(\theta_t\omega)\right)$$
$$\le \frac{1}{2}\frac{d}{dt}\int_{R^3} F(u)dx + \delta\beta_2\int_{R^3} F(u)dx - \beta_2|\varepsilon||z(\theta_t\omega)|\int_{R^3} F(u)dx + \int_{\mathbb{R}^3}\beta_1 dx - \delta\beta_2 + \beta_2|\varepsilon||z(\theta_t\omega)|\,. \quad (4.8)$$

Using the cauchy-schwartz inequality and the young inequality, we have

$$(g(x,t), v) \le \|g(x,t)\|\|v\| \le \frac{\|g(x,t)\|^2}{\alpha-\delta} + \frac{\alpha-\delta}{4}\|v\|^2\,; \qquad (4.9)$$

$$\varepsilon\big(\delta - \varepsilon z(\theta_t\omega)\big)z(\theta_t\omega))(u, v) \le \varepsilon\big(\delta - \varepsilon z(\theta_t\omega)\big)z(\theta_t\omega))\|u\|\|v\|$$
$$\le \left(\delta|\varepsilon||z(\theta_t\omega)| + \frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2\right)(\|v\|^2 + \|u\|^2) \quad ; \qquad (4,10)$$

$$\varepsilon(v, v)z(\theta_t\omega) = |\varepsilon||z(\theta_t\omega)|\|v\|^2\ . \qquad (4,11)$$

By substitute (4.6)-(4.11) into (4.4) and $\tilde{F} = \int_{\mathbb{R}^3} f(u)dx$ we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|v\|^2 + (\delta^2 - \delta\sigma(u) + \lambda)\|u\|^2 + \|\nabla u\|^2 + 2\tilde{F}(u)\right)$$
$$+\delta\left(\|v\|^2 + (\delta^2 - \delta\alpha_2 + \lambda)\|u\|^2 + \|\nabla u\|^2 + \beta_2\tilde{F}(u)\right)$$
$$-|\varepsilon||z(\theta_t\omega)|\left(\|v\|^2 + (\delta^2 - \delta\alpha_2 + \lambda)\|u\|^2 + \|\nabla u\|^2 + \beta_2\tilde{F}(u)\right)$$

$$\le \frac{\|g(x,t)\|^2}{\alpha-\delta} + \frac{\alpha-\delta}{4}\|v\|^2 + \left(\left(\delta - \frac{1}{2}\alpha_2\right)|\varepsilon||z(\theta_t\omega)| + \frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2\right)$$
$$(\|u\|^2 + \|v\|^2) - \beta_2\left(\int_{\mathbb{R}^3}\beta_1 dx + |\varepsilon|\,|z(\theta_t\omega)| - \delta\right)\ . \qquad (4.12)$$

$$\frac{1}{2}\frac{d}{dt}\left(\|v\|^2 + (\delta^2 - \delta\sigma(u) + \lambda)\|u\|^2 + \|\nabla u\|^2 + 2\tilde{F}(u)\right)$$
$$\le -\delta\left(\|v\|^2 + (\delta^2 - \delta\alpha_2 + \lambda)\|u\|^2 + \|\nabla u\|^2 + \beta_2\tilde{F}(u)\right)$$
$$+|\varepsilon||z(\theta_t\omega)|\left(\|v\|^2 + (\delta^2 - \delta\alpha_2 + \lambda)\|u\|^2 + \|\nabla u\|^2 + \beta_2\tilde{F}(u)\right)$$

$$+\frac{\|g(x,t)\|^2}{\alpha-\delta} + \frac{\alpha-\delta}{4}\|v\|^2 + \left(\left(\delta - \frac{1}{2}\alpha_2\right)|\varepsilon||z(\theta_t\omega)| + \frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2\right)$$
$$(\|u\|^2 + \|v\|^2) - \beta_2\left(\int_{\mathbb{R}^3}\beta_1 dx + |\varepsilon|\,|z(\theta_t\omega)| - \delta\right). \qquad (4.13)$$

Let

$$\sigma = \min\ \left\{\delta, \frac{\delta\beta_2}{2}\right\} \qquad (4.14)$$

$$\frac{1}{2}\frac{d}{dt}\left(\|v\|^2 + (\delta^2 - \delta\sigma(u) + \lambda)\|u\|^2 + \|\nabla u\|^2 + 2\tilde{F}(u)\right)$$

$$\leq \left[-\sigma + |\varepsilon||z(\theta_t\omega)| + \gamma_1\left(\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2 + \gamma_2|\varepsilon||z(\theta_t\omega)|\right)\right].$$
$$\left(\|v\|^2 + (\delta^2 - \delta\alpha_2 + \lambda)\|u\|^2 + \|\nabla u\|^2 + \beta_2\tilde{F}(u)\right)$$

$$+\frac{\|g(x,t)\|^2}{\alpha-\delta} - \beta_2\left(\int_{\mathbb{R}^3}\beta_1 dx + |\varepsilon|\,|z(\theta_t\omega)| - \delta\right). \qquad (4.15)$$

Let

$$\Gamma(t,\omega) = -\sigma + |\varepsilon||z(\theta_t\omega)| + \gamma_1\left(\tfrac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2 + \gamma_2|\varepsilon||z(\theta_t\omega)|\right) \qquad (4.16)$$

$$\frac{1}{2}\frac{d}{dt}\left(\|v\|^2 + (\delta^2 - \delta\sigma(u) + \lambda)\|u\|^2 + \|\nabla u\|^2 + 2\tilde{F}(u)\right)$$
$$\leq \Gamma(t,\omega)\left(\|v\|^2 + (\delta^2 - \delta\alpha_2 + \lambda)\|u\|^2 + \|\nabla u\|^2 + \beta_2\tilde{F}(u)\right)$$

$$+\frac{\|g(x,t)\|^2}{\alpha-\delta} - \beta_2\left(\int_{\mathbb{R}^3}\beta_1 dx + |\varepsilon|\,|z(\theta_t\omega)| - \delta\right). \qquad (4.17)$$

Recall the new norm $\|\varphi\|^2_{E(\mathbb{R}^3)}$ in (3.14); we obtain from (4.13)

$$\frac{1}{2}\frac{d}{dt}\left(\|\varphi\|_E^2 + \tilde{F}(u)\right) + \Gamma(t,\omega)\left(\|\varphi\|_E^2 + \tilde{F}(u)\right) + \frac{\|g(x,t)\|^2}{\alpha-\delta} - \beta_2\left(\int_{\mathbb{R}^3}\beta_1 dx + |\varepsilon|\,|z(\theta_t\omega)| - \delta\right). \qquad (4.18)$$

Applying Gronwall's lemma over $\tau - t,\ \tau$ we find that, for all $t \geq 0$,

$$\|\varphi(\tau, \tau - t, \omega, \varphi_0)\|_E^2 + \tilde{F}(u(\tau,, \tau - t, \omega, u_0)) \leq e^{2\int_\tau^{\tau-t}\Gamma(r,\omega)dr}(\|\varphi_0\|_E^2 + \tilde{F}(u_0))$$

$$+\int_{\tau-t}^{\tau} e^{2\int_s^\tau \Gamma(r-\tau,\omega)dr}\left(\frac{\|g(x,t)\|^2}{\alpha-\delta} - \beta_2\left(\int_{\mathbb{R}^3}\beta_1 dx + |\varepsilon|\,|z(\theta_t\omega)| - \delta\right)\right)ds\ . \qquad (4.19)$$

By replacing $\omega$ *by* $\theta_{-\tau}\ \omega$ , we get from (4.19) such that for all $t \geq 0$,

$$\|\varphi(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_0)\|_E^2 + \tilde{F}(u(\tau,, \tau - t, \theta_{-\tau}\omega, u_0)) \leq e^{\int_\tau^{\tau-t}\Gamma(r,\omega)dr}(\|\varphi_0\|_E^2 + \tilde{F}(u_0))$$
$$+\int_{\tau-t}^{\tau} e^{2\int_s^\tau \Gamma(r-\tau,\omega)dr}\left(\frac{\|g(x,t)\|^2}{\alpha-\delta} - \beta_2\left(\int_{\mathbb{R}^3}\beta_1 dx + |\varepsilon|\,|z(\theta_{s-\tau}\omega)| - \delta\right)\right)ds. \qquad (4.20)$$

By (4.1)-(4.2) and Lemma 3.1 we can show that

$$e^{2\int_s^0 \Gamma(r,\omega)dr} \leq e^{2(\frac{\sigma}{2}s)} = e^s\ . \qquad (4.21)$$

For any $s$ since $|\varepsilon|\ |z(\theta_t\omega)|$ is tempered by (3.18),(4.2)and(4.21) it then follows from Lemma(3.1) $\varphi_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$. and the fact that $B(\omega)$ is tempered that note that (1.5) , (1.6) and due to $\varphi_0 = (u_0, v_0)^{\intercal} \in B(\tau - t, \theta_{-t}\omega)$. And $B \in D$,we get from (4.21) such that

$$\lim_{t\to-\infty} e^{-\sigma t}\,(\|\varphi_0\|_E^2 + \tilde{F}(u_0)) = 0 \qquad (4{,}22)$$

Therefore, there exists $T = T(\tau, \omega, B) > 0$such that $e^{-\sigma t}(\|\varphi_0\|_E^2 + \tilde{F}(u_0)) \geq 1$ for all $t \geq T$ thus the lemma follows from (3.19), (1.5)-(1.6) and (4.22) we can written the following results

$$\|\varphi(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_0)\|_E^2 + \tilde{F}\left(u(\tau,, \tau - t, \theta_{-\tau}\omega, u_0)\right) \leq$$
$$+\int_{\tau-t}^{\tau} e^{2\int_s^\tau \Gamma(r-\tau,\omega)dr}\left(\frac{\|g(x,t)\|^2}{\alpha-\delta} - \beta_2\left(\int_{\mathbb{R}^3}\beta_1 dx + |\varepsilon|\,|z(\theta_{s-\tau}\omega)| - \delta\right)\right)ds \ \leq \infty\,. \qquad (4.23)$$

Then we complete the proof. □

**Lemma 4.2** Assume that (1.3)-(1.7) hold and $g(x,t) \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^3)$ satisfies (3.18) and (3.19). Then there exists a random ball $\{K = K(\tau, \omega) : \tau \in \mathbb{R}\,, \omega \in \Omega\} \in B$ centered at 0 with random radius.

$$\varrho^2(\tau, \omega) = C\int_{\tau-t}^{\tau} e^{2\int_s^\tau \Gamma(r-\tau,\omega)dr}\,(\|g(x,t)\|^2 + 1 + |\varepsilon|\,|z(\theta_{s-\tau}\omega)|)ds \quad (4.24)$$

such that is a closed measurable D-pullback absorbing set for the continuous cocycle associated with problem (3.6)-(3.8)in D, that is ,for every $\tau \in \mathbb{R}\,, \omega \in \Omega$ and $\{B = B(\tau, \omega) : \tau \in \mathbb{R}\,, \omega \in \Omega\} \in D$ , there exists $T = T(\tau, \omega, B) > 0$, such that for all $t \geq T$

$$\Phi(t, \tau - t, \theta_{-t}\omega, B(\tau - t, \theta_{-t}\omega)) \subseteq K(\tau, \omega) \quad , \forall\, t \geq T$$

**Proof** This is an immediate consequence of(3.16)-(3.18 ),(4.14), (4.21) and Lemma4.1.□

We can choose a smooth function $\rho$ defined on $\mathbb{R}^+$ such that $0 \leq \rho(s) \leq 1$ for all $s \in \mathbb{R}^+$ and

$$\rho(s)=\begin{cases}0 & for\ 0\le s\le 1\\ 1 & for\ 1\le s\le 2\end{cases}$$

Then there exists a constant $\mu_1$ and $\mu_2$ such that $|\acute{\rho}(s)|\le\mu_1$ , $|\acute{\acute{\rho}}(s)|\le\mu_2$ for any $s\in\mathbb{R}^+$,given $\ge 1$ , denote by $H_k=\{x\in\mathbb{R}^3:|x|<k\}$ and $\mathbb{R}^3\setminus H_k$ the complement of $H_k$.to prove asymptotic compactness of the random dynamical system we prove the following lemma

**Lemma 4.3** Assume that (1.3)-(1.7) hold and $g(x,t)\in L^2_{loc}(\mathbb{R},L^2(\mathbb{R}^3))$ satisfies (3.18)and (3.19). Let $\{B=B(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in D$ and$\varphi_0(\omega)\in B(\tau,\omega)$. Then for every $\eta>0$ and P-a.e $\omega\in\Omega$,there exists $\tilde{T}=\tilde{T}(\tau,\omega,\mathrm{B},\eta)>0$ and $\tilde{k}=\tilde{k}(\tau,\omega,\eta)>1$ such that $\varphi(\tau,\tau-\mathrm{t},\theta_{-\mathrm{t}}\omega,\varphi_0(\omega))$ the solution of (3.6)-(3.8) satisfies, for all $t\ge\tilde{T}$ , $k\ge\tilde{k}$

$$\left\|\varphi(\tau,\tau-\mathrm{t},\theta_{-\tau}\omega,\varphi_0)\right\|^2_{E(\mathbb{R}^3\setminus H_k)}\le\eta\ . \qquad (4.26)$$

**Proof** Multiplying (3.7) by $\int_{\mathbb{R}^3}\rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\mathrm{v}$ in $L^2(\mathbb{R}^3)$,and integrating over $\mathbb{R}^3$ we find that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)|v|^2dx+(\sigma(u)-\delta)\int_{\mathbb{R}^3}\rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)|v|^2dx$$

$$+\left(\lambda+\delta^2-\delta\sigma(u)\right)\int_{\mathbb{R}^3}\rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)uvdx+\int_{\mathbb{R}^3}\rho(\mathrm{Au})\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)vdx+\int_{\mathbb{R}^3}\rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)f(u)vdx$$

$$=\int_{\mathbb{R}^3}\rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)g(x,t)vdx-\varepsilon\int_{\mathbb{R}^3}\rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\Big(\big(v+\sigma(u)u-2\delta u+\varepsilon uz(\theta_t\omega)\big)z(\theta_t\omega)\Big)vdx\ . \qquad (4.27)$$

Estimating the third left side of (4.27) we obtain

$$\int_{\mathbb{R}^3}\rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)uvdx=\int_{\mathbb{R}^3}\rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)u\left(\frac{du}{dt}+\delta u-\varepsilon uz(\theta_t\omega)\right)dx$$

$$\le\int_{\mathbb{R}^3}\rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\left(\frac{1}{2}\frac{d}{dt}|u|^2+\delta|u|^2-|\varepsilon||z(\theta_t\omega)||\nabla u|^2\right)dx\ . \qquad (4.28)$$

Estimating the fourth left side of (4.27) we obtain

$$\int_{\mathbb{R}^3}\rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)Auvdx=\int_{\mathbb{R}^3}\rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)(-\Delta u)\left(\frac{du}{dt}+\delta u-\varepsilon uz(\theta_t\omega)\right)dx$$

$$\le\int_{\mathbb{R}^3}\rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)(\nabla u)\nabla\left(\frac{du}{dt}+\delta u-\varepsilon uz(\theta_t\omega)\right)dx$$

$$\le\int_{\mathbb{R}^3}\nabla u\,\frac{2\mathrm{x}}{\mathrm{k}^2}\acute{\rho}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)vdx+\int_{\mathbb{R}^3}\nabla u\,\rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\nabla\left(\frac{du}{dt}+\delta u-\varepsilon uz(\theta_t\omega)\right)dx$$

$$\le\int_{k<|x|<2\sqrt{2}}\nabla u\,\frac{2\mathrm{x}}{\mathrm{k}^2}\mu_1\nabla uvdx+\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)|\nabla u|^2dx-\int_{\mathbb{R}^3}\rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)(-\delta+|\varepsilon||z(\theta_t\omega)|)|\nabla u|^2dx$$

$$\le\frac{\sqrt{2}}{k}\mu_1(\|\nabla u\|^2+\|\mathrm{v}\|^2)+\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)|\nabla u|^2dx-\int_{\mathbb{R}^3}\rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)(-\delta+|\varepsilon||z(\theta_t\omega)|)|\nabla u|^2dx\quad, \qquad (4.29)$$

Estimating the fifth left side of (4.27) by using the condition (1.3)-(1.6) we obtain

$$\int_{\mathbb{R}^3}\rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)f(u)vdx=\int_{\mathbb{R}^3}\rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)f(u)\left(\frac{du}{dt}+\delta u-\varepsilon uz(\theta_t\omega)\right)dx$$

$$\leq \frac{1}{2}\frac{d}{dt}\int_{R^3} \rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\tilde{F}(u)dx + \delta\beta_2\int_{R^3} \rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\tilde{F}(u)dx$$

$$-\beta_2|\varepsilon||z(\theta_t\omega)|\int_{R^3} \rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\tilde{F}(u)dx + \int_{\mathbb{R}^3} \rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)(\beta_1 - \delta\beta_2 + \beta_2|\varepsilon||z(\theta_t\omega)|\,)dx \quad , \quad (4.30)$$

Using the cauchy-schwartz inequality and the young inequality, we have

$$\int_{\mathbb{R}^3} \rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right) g(x,t)vdx \leq \int_{\mathbb{R}^3} \rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right) |g(x,t)||v|dx$$

$$\leq \frac{1}{\alpha-\delta}\int_{\mathbb{R}^3} \rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)|g(x,t)|^2dx + \frac{\alpha-\delta}{4}\int_{\mathbb{R}^3} \rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)|v|^2dx \quad , \quad (4.31)$$

$$\int_{\mathbb{R}^3} \rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\varepsilon\big(\delta - \varepsilon uz(\theta_t\omega)\big)z(\theta_t\omega))uvdx \leq \varepsilon\big(\delta - \varepsilon uz(\theta_t\omega)\big)z(\theta_t\omega))\int_{\mathbb{R}^3} \rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)|u||v|dx$$

$$\leq \int_{\mathbb{R}^3} \rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)(\delta|\varepsilon||z(\theta_t\omega)| + \frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2)(|v|^2 + |u|^2)\,dx \quad (4,32)$$

$$\int_{\mathbb{R}^3} \rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\varepsilon(v)vz(\theta_t\omega)dx = |\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^3} \rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)|v|^2\,dx. \quad (4,33)$$

By substitute (4.28)-(4.33) into (4.27) we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} \rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\Big(|v|^2 + (\delta^2 - \delta\sigma(u) + \lambda)|u|^2 + |\nabla u|^2 + 2\tilde{F}(u)\Big)dx$$

$$+\delta\int_{\mathbb{R}^3} \rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\Big(|v|^2 + (\delta^2 - \delta\alpha_2 + \lambda)|u|^2 + |\nabla u|^2 + \beta_2\tilde{F}(u)\Big)dx$$

$$-|\varepsilon||z(\theta_t\omega)|\int_{\mathbb{R}^3} \rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\Big(|v|^2 + (\delta^2 - \delta\alpha_2 + \lambda)|u|^2 + |\nabla u|^2 + \beta_2\tilde{F}(u)\Big)dx$$

$$\leq \int_{\mathbb{R}^3} \rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\frac{|g(x,t)|^2}{\alpha-\delta}\mathrm{dx} + \frac{\alpha-\delta}{4}\int_{\mathbb{R}^3} \rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)|v|^2\mathrm{dx} + \left(\left(\delta - \frac{1}{2}\alpha_2\right)|\varepsilon||z(\theta_t\omega)| + \frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2\right)$$

$$-\frac{\sqrt{2}}{k}\mu_1(\|\nabla u\|^2 + \|\mathrm{v}\|^2) + \int_{\mathbb{R}^3} \rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)(|\mathrm{u}|^2 + |\mathrm{v}|^2)dx - \beta_2(\beta_1 + |\varepsilon|\,|z(\theta_t\omega)| - \delta)dx\,. \quad (4.34)$$

Recall the new norm $\|\varphi\|^2_{E(\mathbb{R}^3)}$in (3.14), we obtain from (4.34)

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} \rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\Big(\|\varphi\|^2_{E(\mathbb{R}^3)} + 2\tilde{F}(u)\Big)dx$$

$$\leq \int_{\mathbb{R}^3} \rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\Big(\|\varphi\|^2_{E(\mathbb{R}^3)} + \tilde{F}(u)\Big)\Gamma(t,\omega)dx + \int_{\mathbb{R}^3} \rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\frac{|g(x,t)|^2}{\alpha-\delta}\mathrm{d}x$$

$$-\frac{\sqrt{2}}{k}\mu_1(\|\nabla u\|^2 + \|\mathrm{v}\|^2) - \beta_2\int_{\mathbb{R}^3} \rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)(\beta_1 + |\varepsilon|\,|z(\theta_t\omega)| - \delta)dx \quad (4.35)$$

Applying Gronwall's lemma over $\tau - t,\ \tau$ we find that, for all $t \geq 0$,

$$\int_{\mathbb{R}^3} \rho\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\big(\|\varphi(\tau,\tau-\mathrm{t},\omega,\varphi_0)\|^2_E + \tilde{F}(u(\tau,\tau-\mathrm{t},\omega,\mathrm{u}_0))\big)dx$$

$$\le e^{2\int_\tau^{\tau-t}\Gamma(r,\omega)dr}\int_{R^3}\rho\left(\frac{|x|^2}{k^2}\right)(\|\varphi_0\|_E^2+\tilde F(u_0))dx+\int_{\tau-t}^{\tau}e^{2\int_s^\tau\Gamma(r-\tau,\omega)dr}\int_{\mathbb{R}^3}\rho\left(\frac{|x|^2}{k^2}\right)\frac{|g(x,t)|^2}{\alpha-\delta}dxds$$
$$+\frac{\sqrt2}{k}\mu_1\int_{\tau-t}^{\tau}e^{2\int_s^\tau\Gamma(r-\tau,\omega)dr}\int_{R^3}\rho\left(\frac{|x|^2}{k^2}\right)(\|\nabla u(s,\tau-t,\omega,u_0)\|^2+\|u(s,\tau-t,\omega,u_0)\|^2)dxds$$
$$+\beta_2\int_{\tau-t}^{\tau}e^{2\int_s^\tau\Gamma(r-\tau,\omega)dr}\int_{\mathbb{R}^3}\rho\left(\frac{|x|^2}{k^2}\right)(\beta_1+|\varepsilon|\,|z(\theta_t\omega)|-\delta)dxds. \qquad (4.36)$$

By replacing $\omega$ *by* $\theta_{-\tau}\,\omega$ , we get from (4.36) such that for all$t\ge 0$,

$$\int_{R^3}\rho\left(\frac{|x|^2}{k^2}\right)(\|\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_0)\|_E^2+\tilde F(u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)))dx$$
$$\le e^{2\int_\tau^{\tau-t}\Gamma(r,\omega)dr}\int_{R^3}\rho\left(\frac{|x|^2}{k^2}\right)(\|\varphi_0\|_E^2+\tilde F(u_0))dx+\int_{\tau-t}^{\tau}e^{2\int_s^\tau\Gamma(r-\tau,\omega)dr}\int_{R^3}\rho\left(\frac{|x|^2}{k^2}\right)\frac{|g(x,t)|^2}{\alpha-\delta}dxds$$
$$+\frac{\sqrt2}{k}\mu_1\int_{\tau-t}^{\tau}e^{2\int_s^\tau\Gamma(r-\tau,\omega)dr}(\|\nabla u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2)ds$$
$$+\beta_2\int_{\tau-t}^{\tau}e^{2\int_s^\tau\Gamma(r-\tau,\omega)dr}\int_{\mathbb{R}^3}\rho\left(\frac{|x|^2}{k^2}\right)(\beta_1+|\varepsilon|\,|z(\theta_{s-\tau}\omega)|-\delta)dxds. \qquad (4.37)$$

Follows the procedures in the proof of lemma 4.1, it is similar to (4.21) and (4.22) that by (4.2), we estimate the term of right hand side of (4.37). For any initial data $\varphi_0=(u_0,v_0)^\intercal\in B(\tau-t,\theta_{-t}\omega)$. And $B\in D$ and (1.5) we have

$$\lim_{t\to\infty}e^{-t\sigma}\int_{R^3}\rho\left(\frac{|x|^2}{k^2}\right)(\|\varphi_0\|_E^2+\tilde F(u_0))dx\ \le\eta\ . \qquad (4{,}38)$$

Then there exists $\tilde T_1=\tilde T_1(\tau,\omega,B,\eta)>0$ such that $t\ge\tilde T_1$

For the second and last terms on the right hand side of (4.37) , there exists $\tilde k_1=\tilde k_1(\tau,\omega,\eta)>1$ such that for all $k\ge\tilde k_1$ by lemma 4.1 ,(4.21),lemma3.1and (3.18) there are $\tilde T_2=\tilde T_2(\tau,\omega,B,\eta)>0$ and $\tilde k_1=\tilde k_1(\tau,\omega,\eta)>1$such that for all $t\ge\tilde T_2$ and $k\ge\tilde k_1$

$$C\int_{\tau-t}^{\tau}e^{2\int_s^\tau\Gamma(r-\tau,\omega)dr}\int_{R^3}\rho\left(\frac{|x|^2}{k^2}\right)(|g(x,t)|^2+(1+|\varepsilon|\,|z(\theta_{s-\tau}\omega)|-\delta))dxds\le\ \eta\ . \qquad (4.39)$$

Next we estimate the third term on the right hand side of (4.37) by lemma4.1there are $\tilde T_3=\tilde T_3(\tau,\omega,B,\eta)>0$ and $\tilde k_2=\tilde k_2(\tau,\omega,\eta)>1$ such that for all $t\ge\tilde T_3$ and $k\ge\tilde k_2$

$$\frac{\sqrt2}{k}\mu_1\int_{\tau-t}^{\tau}e^{2\int_s^\tau\Gamma(r-\tau,\omega)dr}(\|\nabla u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2+\|u(s,\tau-t,\theta_{-\tau}\omega,u_0)\|^2)ds\le\ \eta\,. \qquad (4.40)$$

Letting

$$\tilde T=\max(\tilde T_1,\tilde T_2,\tilde T_3)$$
$$\tilde K=\max(\tilde k_1,\tilde k_2)\ . \qquad (4{,}41)$$

Then combining with (4.38),(4.39)and (4.40) ,we have for all $t>\tilde T$ and $k>\tilde K$

$$\int_{R^3}\rho\left(\frac{|x|^2}{k^2}\right)\left(\|\varphi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_0)\|_E^2+\tilde F\big(u(\tau,\tau-t,\theta_{-\tau}\omega,u_0)\big)\right)dx\le 3\eta \qquad (4.42)$$

Which implies (4.26) we get?

$$\|\Phi(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_0)\|_{E(R^3\setminus H_k)}^2\ \le c\eta$$

Then we complete the proof. □

**High-Mode Estimates**.

Now we drive uniform estimates on the high-mode parts of solution in bounded domains $H_{2k} = \{x \in \mathbb{R}^3 : |x| < 2k\}$ these estimates will also be used to establish pullback asymptotic compactness ,denote $q(s)$=1-$p(s)$ ,where $p(s)$ is the cut-off function defined by (4.25) given positive integer r, we define two new variable $\tilde{u}$ and $\tilde{v}$ by

$$\tilde{u}(\mathrm{t}, \tau, \omega, \tilde{\mathrm{u}}_0) = \mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)(\mathrm{t}, \tau, \omega, \mathrm{u}_0)$$

$$\tilde{v}(\mathrm{t}, \tau, \omega, \tilde{\mathrm{v}}_0) = \mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)(\mathrm{t}, \tau, \omega, \mathrm{v}_0) \qquad (4,43)$$

Then $\tilde{\varphi}(\mathrm{t}, \tau, \omega, \tilde{\varphi}_0) = \left(\tilde{u}(\mathrm{t}, \tau, \omega, \tilde{\mathrm{u}}_0), \tilde{v}(\mathrm{t}, \tau, \omega, \tilde{\mathrm{v}}_0)\right)^{\mathrm{T}}$ is the solution of problem (3.6)-(3.7) on the bounded domain $H_{2k}$ ,where $\tilde{\varphi}_0 = \mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\varphi_0 \in E(H_{2k})$

Multiplying (3.7) by $\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)$ ,we obtain

$$\tilde{v} = \left(\frac{d\tilde{u}}{dt} + \delta\tilde{u} - \varepsilon\tilde{u}z(\theta_t\omega)\right) \qquad (4.44)$$

$$\tilde{v}_t + (\sigma(u) - \delta)\tilde{v} + \left(\lambda + \delta^2 - \delta\sigma(u)\right)\tilde{u} + A\tilde{u} + \mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)f(u)$$

$$= \mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)g(x,t) - \varepsilon\left(\tilde{v} + (\sigma(u) - 2\delta)\tilde{u} + \varepsilon\tilde{u}z(\theta_t\omega)\right)z(\theta_t\omega) + uA\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right) - 2\nabla u\nabla\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right) \qquad (4.45)$$

$\tilde{u} = \tilde{v} = 0$ For $|x| = 2k$. Consider the eigenvalue problem

$$A\tilde{u} = \tilde{u} \text{ In } H_{2k} \text{ with } \tilde{u} = 0 \text{ on } \partial H_{2k} \qquad (4.46)$$

The problem has a family of eigenfunctions $\{e_i\}_{i\in N}$ with the eigenvalue$\{\lambda_i\}_{i\in N}$ ,$\lambda_1 \leq, \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots, \lambda_i \to +\infty (i \to \infty)$ such that $\{e_i\}_{i\in N}$ is an orthonormal basis of $L^2(H_{2k})$ given n, let

$$X_n = \{e_1 \ldots\ldots e_n\} \text{ And } \rho_n : L^2(H_{2k}) \to X_n$$

Be the projection operator.

**Lemma 4.4** Assume that (1.3)-(1.7) hold and $g(x,t) \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^3))$ satisfies (3.18)-(3.19).Let $\{B = \mathcal{B}\tau,\omega : \tau \in \mathbb{R}$ ,$\omega \in \Omega \in \mathcal{D}$ and $\varphi 0(\omega) \in \mathcal{B}\tau,\omega$. Then for every $\eta > 0$ and P-a.e $\omega \in \Omega$,there exists $\mathcal{T} = \mathcal{T}\omega$,B,$\eta > 0$and $\tilde{k} = \tilde{k}(\omega, \eta) > 1$ and $N = N(\omega, \eta) > 0$ such that $\varphi(\mathrm{t}, \omega, \varphi_0(\omega))$ the solution of (3.5)-(3.7) with satisfies, for all $t \geq \tilde{T}, k \geq \tilde{k}$ *and* $n \geq N$

$$\|(1 - \mathrm{p_n})\,\tilde{\varphi}(\tau, \tau - \mathrm{t}, \theta_{-\tau}\omega, \varphi_0)\|^2_{E(H_{2k})} \leq \eta \qquad (4.47)$$

**Proof** Let $\tilde{u}_{n,1} = \mathrm{p_n}\tilde{u}$ , $\tilde{v}_{n,1} = \mathrm{p_n}\tilde{v}$ and $\tilde{u}_{n,2} = (1 - \mathrm{p_n})\tilde{u}$ , $\tilde{v}_{n,2} = (1 - \mathrm{p_n})\,\tilde{v}$ applying $(1 - \mathrm{p_n})$ to equation (4.44)

$$\tilde{v}_{n,2} = \left(\frac{d}{dt}\tilde{u}_{n,2} + \delta\tilde{u}_{n,2} - \varepsilon\tilde{u}_{n,2}z(\theta_t\omega)\right). \qquad (4.48)$$

Then applying $(1 - \mathrm{p_n})$ to equation (4.45) and taking the inner product of resulting equation with $\tilde{v}_{n,2}$ in $L^2(H_{2k})$ we have

$$\left(\tilde{v}_{n,2_t}, \tilde{v}_{n,2}\right) + (\sigma(u) - \delta)\left(\tilde{v}_{n,2}, \tilde{v}_{n,2}\right) + \left(\lambda + \delta^2 - \delta\sigma(u)\right)\left(\tilde{u}_{n,2}, \tilde{v}_{n,2}\right) + \left(A\tilde{u}, \tilde{v}_{n,2}\right)$$
$$+(1 - \mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\left(f(u), \tilde{v}_{n,2}\right) = (1 - \mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\left(g(x,t), \tilde{v}_{n,2}\right)$$

$$-\varepsilon\left(\left(\tilde{v}_{n,2} + (\sigma(u) - 2\delta)\tilde{u} + \varepsilon\tilde{u}z(\theta_t\omega)\right)z(\theta_t\omega), \tilde{v}_{n,2}\right) + \left(uA\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right) - 2\nabla u\nabla\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right), \tilde{v}_{n,2}\right). \qquad (4.49)$$

Estimating the third left side of (4.49) we obtain

$$\left(\tilde{u}_{n,2}, \tilde{v}_{n,2}\right) = \left(\tilde{u}_{n,2}, \frac{d}{dt}\tilde{u}_{n,2} + \delta\tilde{u}_{n,2} - \varepsilon\tilde{u}_{n,2}z(\theta_t\omega)\right)$$
$$\leq \frac{1}{2}\frac{d}{dt}\left\|\tilde{u}_{n,2}\right\|^2 + \delta\left\|\tilde{u}_{n,2}\right\|^2 - |\varepsilon||z(\theta_t\omega)|\left\|\tilde{u}_{n,2}\right\|^2 \leq \frac{1}{2}\frac{d}{dt}\left\|\tilde{u}_{n,2}\right\|^2 - (-\delta + |\varepsilon||z(\theta_t\omega)|)\left\|\tilde{u}_{n,2}\right\|^2. \qquad (4.50)$$

Estimating the fourth left side of (4.49) we obtain

$$\left(A\tilde{u}_{n,2},\tilde{v}_{n,2}\right)=\left(-\Delta\tilde{u}_{n,2},\tilde{v}_{n,2}\right)=\left(-\Delta\tilde{u}_{n,2},\frac{d}{dt}\tilde{u}_{n,2}+\delta\tilde{u}_{n,2}-\varepsilon\tilde{u}_{n,2}z(\theta_t\omega)\right)$$

$$\leq (\nabla\tilde{u}_{n,2},\nabla\left(\frac{d}{dt}\tilde{u}_{n,2}+\delta\tilde{u}_{n,2}-\varepsilon\tilde{u}_{n,2}z(\theta_t\omega)\right)\leq (\nabla\tilde{u}_{n,2},\nabla\left(\frac{d}{dt}\tilde{u}_{n,2}-\left(-\delta+\varepsilon z(\theta_t\omega)\right)\tilde{u}_{n,2}\right)$$

$$\leq\frac{1}{2}\frac{d}{dt}\left\|\nabla\tilde{u}_{n,2}\right\|^2-(-\delta-\ |\varepsilon||z(\theta_t\omega)|)\left\|\nabla\tilde{u}_{n,2}\right\|^2. \qquad (4.51)$$

Estimating the fifth left side of (4.49) by using the condition (1.3)-(1.6) we obtain

$$\left((1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)f(u),\tilde{v}_{n,2}\right)=\left((1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)f(u),\frac{d}{dt}\tilde{u}_{n,2}+\delta\tilde{u}_{n,2}-\varepsilon\tilde{u}_{n,2}z(\theta_t\omega)\right)$$

$$\leq\frac{d}{dt}\left((1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)f(u),\tilde{u}_{n,2}\right)-\left(-\delta+\varepsilon z(\theta_t\omega)\right)\left((1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)f(u),\tilde{u}_{n,2}\right)$$

$$-\left((1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\acute{f}_u(u)\tilde{u}_{n,2_t},\tilde{u}_{n,2}\right)\ . \qquad (4.52)$$

For the first term on the right- hand- side of (4.49)by using the cauchy-schwartz inequality and the young inequality ,we have

$$\left((1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)g(x,t),\tilde{v}_{n,2}\right)\leq\left\|(1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)g(x,t)\right\|\left\|\tilde{v}_{n,2}\right\|$$

$$\leq\frac{\left\|(1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)g(x,t)\right\|^2}{\alpha-\delta}+\frac{\alpha-\delta}{4}\left\|\tilde{v}_{n,2}\right\|^2. \qquad (4.53)$$

For the second term on the right- hand- side of (4.49)by using the cauchy-schwartz inequality and the young inequality ,we have

$$\varepsilon\left(\left(\sigma(u)-\delta+\varepsilon uz(\theta_t\omega)\right)z(\theta_t\omega)\right)(\tilde{u}_{n,2},\tilde{v}_{n,2})\leq\varepsilon\left(\left(\sigma(u)-\delta+\varepsilon uz(\theta_t\omega)\right)z(\theta_t\omega)\right)\left\|\tilde{u}_{n,2}\right\|\left\|\tilde{v}_{n,2}\right\|$$

$$\leq\left((\delta+\frac{1}{2}\alpha_2)|\varepsilon||z(\theta_t\omega)|+\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2\right)(\left\|\tilde{u}_{n,2}\right\|^2+\left\|\tilde{v}_{n,2}\right\|^2) \qquad (4{,}54)$$

$$\varepsilon\left(\tilde{v}_{n,2},\tilde{v}_{n,2}\right)z(\theta_t\omega)=|\varepsilon||z(\theta_t\omega)|\left\|\tilde{v}_{n,2}\right\|^2. \qquad (4{,}55)$$

For the last term the right- hand- side of (4.49) we obtain

$$\left(uA\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)-2\nabla u\nabla\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right),\tilde{v}_{n,2}\right)=\left(-u\left(\frac{4x^2}{k^4}\acute{\acute{q}}\ \left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)+\frac{2}{\mathrm{k}^2}\acute{\mathrm{q}}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\right)-\frac{4x}{k^2}\nabla\mathrm{u}\acute{\mathrm{q}}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\ ,\tilde{v}_{n,2}\right)$$

$$\leq\int_{k<|x|<2\sqrt{2}}\ \left(-u\left(\frac{4x^2}{k^4}\acute{\acute{q}}\ \left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)+\frac{2}{\mathrm{k}^2}\acute{\mathrm{q}}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\right)-\frac{4x}{k^2}\nabla\mathrm{u}\acute{\mathrm{q}}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\ ,\tilde{v}_{n,2}\right)$$

$$=\left(-u\left(\frac{8}{k^2}\mu_2+\frac{2}{\mathrm{k}^2}\mu_1\right)-\frac{4\sqrt{2}}{k}\ \ \mu_1\nabla\mathrm{u}\ \ ,\tilde{v}_{n,2}\right)\leq\left(\frac{8\mu_2+2\mu_1}{k^2}\right)\|u\|\left\|\tilde{v}_{n,2}\right\|+\frac{4\sqrt{2}}{k}\ \ \mu_1\|\nabla u\|\left\|\tilde{v}_{n,2}\right\|$$

$$\leq\frac{1}{\alpha-\delta}\left(\left(\frac{8\mu_2+2\mu_1}{k^2}\right)^2\|u\|^2+\frac{32{\mu_1}^2}{k^2}\ \ \|\nabla u\|^2\right)+\frac{\alpha-\delta}{4}\left\|\tilde{v}_{n,2}\right\|^2$$

$$=\frac{1}{\alpha-\delta}\left(\frac{(8\mu_2+2\mu_1)^2}{k^4}\|u\|^2+\frac{32{\mu_1}^2}{k^2}\ \ \|\nabla u\|^2\right)+\frac{\alpha-\delta}{4}\left\|\tilde{v}_{n,2}\right\|^2. \qquad (4.56)$$

By substitute (4.50)-(4.55) into (4.49) we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\left\|\tilde{v}_{n,2}\right\|^2+(\delta^2-\delta\alpha_3+\lambda)\left\|\tilde{u}_{n,2}\right\|^2+\left\|\nabla\tilde{u}_{n,2}\right\|^2+2\left((1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)f(u),\tilde{u}_{n,2}\right)\right)$$

$$-(-\delta+|\varepsilon||z(\theta_t\omega)|)\left(\left\|\tilde{v}_{n,2}\right\|^2+(\delta^2-\delta\alpha_2+\lambda)\left\|\tilde{u}_{n,2}\right\|^2+\left\|\nabla\tilde{u}_{n,2}\right\|^2+\beta_2\left((1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)f(u),\tilde{u}_{n,2}\right)\right)$$

$$\le \frac{\left\|(1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)g(x,t)\right\|^2}{\alpha-\delta}+\frac{2\delta-\alpha}{2}\left\|\tilde{v}_{n,2}\right\|^2+\left((1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\acute{f}_u(u)\tilde{u}_{n,2_t},\tilde{u}_{n,2}\right)$$
$$+\frac{1}{\alpha-\delta}\left(\frac{(8\mu_2+2\mu_1)^2}{k^4}\|u\|^2+\frac{32\mu_1{}^2}{k^2}\ \ \|\nabla u\|^2\right)$$
$$+\left(\frac{2\delta+\alpha_2+2(\alpha_2-\alpha)\delta}{2}|\varepsilon||z(\theta_t\omega)|+\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2\right)\left(\left\|\tilde{u}_{n,2}\right\|^2+\left\|\tilde{v}_{n,2}\right\|^2\right). \qquad (4.57)$$

It follows that

$$\frac{1}{2}\frac{d}{dt}\left(\left\|\tilde{v}_{n,2}\right\|^2+(\delta^2-\delta\alpha_3+\lambda)\left\|\tilde{u}_{n,2}\right\|^2+\left\|\nabla\tilde{u}_{n,2}\right\|^2+2\left((1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)f(u),\tilde{u}_{n,2}\right)\right)$$
$$\le 2\left(-\delta+|\varepsilon||z(\theta_t\omega)|+\gamma_1\left(\frac{2\delta+\alpha_2+2(\alpha_2-\alpha)\delta}{2}|\varepsilon||z(\theta_t\omega)|+\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2\right)\right).$$
$$\left(\left\|\tilde{v}_{n,2}\right\|^2+(\delta^2-\delta\alpha_2+\lambda)\left\|\tilde{u}_{n,2}\right\|^2+\left\|\nabla\tilde{u}_{n,2}\right\|^2+\beta_2\left((1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)f(u),\tilde{u}_{n,2}\right)\right)$$
$$\frac{1}{\alpha-\delta}\left(\frac{(8\mu_2+2\mu_1)^2}{k^4}\|u\|^2+\frac{32\mu_1{}^2}{k^2}\ \ \|\nabla u\|^2\right)\ +\frac{\left\|(1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)g(x,t)\right\|^2}{\alpha-\delta}$$
$$+2\left(\delta-|\varepsilon||z(\theta_t\omega)|-\gamma_1\left(\frac{2\delta+\alpha_2+2(\alpha_2-\alpha)\delta}{2}|\varepsilon||z(\theta_t\omega)|+\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2\right)\right)$$
$$\left(\left(1-\mathrm{p_n}\right)\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\mathrm{f}(u),\tilde{u}_{n,2}\right)\ +\left(\left(1-\mathrm{p_n}\right)\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\acute{f}_u(u)\tilde{u}_{n,2_t},\tilde{u}_{n,2}\right). \quad (4.58)$$

Then (4.16) using we can written

$$\frac{1}{2}\frac{d}{dt}\left(\left\|\tilde{v}_{n,2}\right\|^2+(\delta^2-\delta\alpha_3+\lambda)\left\|\tilde{u}_{n,2}\right\|^2+\left\|\nabla\tilde{u}_{n,2}\right\|^2+2\left((1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)f(u),\tilde{u}_{n,2}\right)\right)$$
$$\le (2\Gamma(t,\omega)+2(-\delta+\sigma))\left[\left\|\tilde{v}_{n,2}\right\|^2+\left(\delta^2-\alpha_2\delta+\lambda\right)\left\|\tilde{u}_{n,2}\right\|^2+\left\|\nabla\tilde{u}_{n,2}\right\|^2+2\left((1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\mathrm{f}(u),\tilde{u}_{n,2}\right)\right]$$
$$\frac{1}{\alpha-\delta}\left(\frac{(8\mu_2+2\mu_1)^2}{k^4}\|u\|^2+\frac{32\mu_1{}^2}{k^2}\ \ \|\nabla u\|^2\right)\ +\frac{\left\|(1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)g(x,t)\right\|^2}{\alpha-\delta}$$
$$+2\left(\delta-|\varepsilon||z(\theta_t\omega)|-\gamma_1\left(\frac{2\delta+\alpha_2+2(\alpha_2-\alpha)\delta}{2}|\varepsilon||z(\theta_t\omega)|+\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2\right)\right)$$
$$\left((1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\mathrm{f}(u),\tilde{u}_{n,2}\right)\ +\left((1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\acute{f}_u(u)\tilde{u}_{n,2_t},\tilde{u}_{n,2}\right). \qquad (4.59)$$

Recalling the new norm $\|.\|_E$ in (3.10),we have

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\left\|\widetilde{\varphi}_{\mathrm{n},2}\right\|_{\mathrm{E}}^2+2\left((1-\mathrm{p_n})q\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)f(u),\tilde{u}_{n,2}\right)\right)$$

$$\leq \left(2\Gamma(t,\omega)+2(-\delta+\sigma)\right)\left(\left\|\tilde{\varphi}_{n,2}\right\|_E^2+2\left((1-p_n)q\left(\frac{|x|^2}{k^2}\right)f(u),\tilde{u}_{n,2}\right)\right)$$

$$\frac{1}{\alpha-\delta}\left(\frac{(8\mu_2+2\mu_1)^2}{k^4}\|u\|^2+\frac{32\mu_1{}^2}{k^2}\ \|\nabla u\|^2\right)\ +\frac{\left\|(1-p_n)q\left(\frac{|x|^2}{k^2}\right)g(x,t)\right\|^2}{\alpha-\delta}$$
$$+2\left(\delta-|\varepsilon||z(\theta_t\omega)|-\gamma_1\left(\frac{2\delta+\alpha_2+2(\alpha_2-\alpha)\delta}{2}|\varepsilon||z(\theta_t\omega)|+\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2\right)\right)$$
$$\left((1-p_n)q\left(\frac{|x|^2}{k^2}\right)f(u),\tilde{u}_{n,2}\right)\ +\left((1-p_n)q\left(\frac{|x|^2}{k^2}\right)\acute{f}_u(u)\tilde{u}_{n,2_t},\tilde{u}_{n,2}\right)\ . \qquad (4.60)$$

Using condition (1.4)and young inequality, we have

$$4\left(-\frac{1}{2}|\varepsilon||z(\theta_t\omega)|-\frac{1}{2}\gamma_1\left(\frac{2\delta+\alpha_2+2(\alpha_2-\alpha)\delta}{2}|\varepsilon||z(\theta_t\omega)|+\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2\right)\right)$$
$$.\left(\left(1-p_n\right)q\left(\frac{|x|^2}{k^2}\right)f(u),\tilde{u}_{n,2}\right)$$

$$\leq\left(-\frac{1}{2}|\varepsilon||z(\theta_t\omega)|-\frac{1}{2}\gamma_1\left(\frac{2\delta+\alpha_2+2(\alpha_2-\alpha)\delta}{2}|\varepsilon||z(\theta_t\omega)|+\frac{1}{2}\varepsilon^2|z(\theta_t\omega)|^2\right)\right)$$
$$\left(\left(1-p_n\right)q\left(\frac{|x|^2}{k^2}\right)(k\|u\|.\left\|\tilde{u}_{n,2}\right\|)\right)$$
$$\leq\mu_3(1+|z(\theta_t\omega)|^2)(\|u\|.\left\|\tilde{u}_{n,2}\right\|)\leq\mu_3\lambda_{n+1}^{\frac{-1}{2}}(1+|z(\theta_t\omega)|^2)(\|u\|.\left\|\tilde{u}_{n,2}\right\|)$$
$$\leq\frac{1}{2}\left\|\nabla\tilde{u}_{n,2}\right\|^2+\mu_3\lambda_{n+1}^{-1}(1+|z(\theta_t\omega)|^4)\|u\|^2. \qquad (4.61)$$

Using condition (1.4)and young inequality, we have

$$2\delta((1-p_n)q\left(\frac{|x|^2}{k^2}\right)f(u),\tilde{u}_{n,2})\leq 2\delta(1-p_n)q\left(\frac{|x|^2}{k^2}\right)(k\|u\|.\left\|\tilde{u}_{n,2}\right\|)\leq\mu_4\|u\|.\left\|\tilde{u}_{n,2}\right\|$$
$$\leq\mu_4\lambda_{n+1}^{\frac{-1}{2}}\left(\|u\|.\left\|\nabla\tilde{u}_{n,2}\right\|\right)\leq\frac{1}{2}\left\|\nabla\tilde{u}_{n,2}\right\|^2+\mu_4\lambda_{n+1}^{-1}\|u\|^2; \qquad (4.62)$$

Using condition (1.7) we have

$$2\delta((1-p_n)q\left(\frac{|x|^2}{k^2}\right)(f_u^{'}(u)u_t,\tilde{u}_{n,2})\leq\mu_5(k\|u_t\|.\left\|\tilde{u}_{n,2}\right\|)\leq\mu_5\lambda_{n+1}^{\frac{-1}{2}}\left(\|u_t\|.\left\|\nabla\tilde{u}_{n,2}\right\|\right)$$
$$\leq\frac{1}{2}\left\|\nabla\tilde{u}_{n,2}\right\|^2+\mu_5\lambda_{n+1}^{-1}\|u_t\|^2. \qquad (4.63)$$

Substituting (4.61)-(4.63) from (4.60) we have

$$\frac{d}{dt}\left(\left\|\tilde{\varphi}_{n,2}\right\|_E^2+\left(2(1-p_n)q\left(\frac{|x|^2}{k^2}\right)f(u),\tilde{u}_{n,2}\right)\right)$$
$$\leq 2\Gamma(t,\omega)\left(\left\|\tilde{\varphi}_{n,2}\right\|_E^2+2(1-p_n)q\left(\frac{|x|^2}{k^2}\right)f(u),\tilde{u}_{n,2}\right)+c\left(\left\|\tilde{\varphi}_{n,2}\right\|_E^2+\left(2(1-p_n)q\left(\frac{|x|^2}{k^2}\right)f(u),\tilde{u}_{n,2}\right)\right)$$
$$+c\left(\left\|(1-p_n)q\left(\frac{|x|^2}{k^2}\right)g(x,t)\right\|^2+\frac{c}{k^4}\|u\|^2+\frac{c}{k^2}\|\nabla u\|^2\right)$$
$$+c\lambda_{n+1}^{-1}\left(\|u_t\|^2+\|u\|_{H^1}^2+|z(\theta_t\omega)|^4\right); \qquad (4.64)$$

Note that $\lambda_n\to+\infty$ when $(n\to+\infty)$ there exists $\tilde{N}_1=\tilde{N}_1(\eta)>1$ and $\tilde{k}_1=\tilde{k}_1(\eta)>0$ such that for all $\geq\tilde{N}_1$ $,k\geq\tilde{k}_1$

$$\frac{d}{dt}\left(\left\|\tilde{\varphi}_{n,2}\right\|_E^2+\left(2(1-p_n)q\left(\frac{|x|^2}{k^2}\right)f(u),\tilde{u}_{n,2}\right)\right)$$
$$\leq 2\Gamma(t,\omega)\left(\left\|\tilde{\varphi}_{n,2}\right\|_E^2+\left(2(1-p_n)q\left(\frac{|x|^2}{k^2}\right)f(u),\tilde{u}_{n,2}\right)\right)+c\left(\left\|\tilde{\varphi}_{n,2}\right\|_E^2+\left(2(1-p_n)q\left(\frac{|x|^2}{k^2}\right)f(u),\tilde{u}_{n,2}\right)\right)$$

$$+c\left(\left\|(1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)g(x,t)\right\|^2+\frac{1}{k^4}\|\mathrm{u}\|^2+\frac{1}{k^2}\|\nabla \mathrm{u}\|^2\right)$$
$$+c\eta\left(\|\mathrm{u_t}\|^2+\|u\|_{H^1}^2+|z(\theta_t\omega)|^4\right)\ ; \qquad (4.65)$$

Applying Gronwall's lemma over $\tau-t,\tau$ we find that, for all $t\geq 0$,

$$\left\|\widetilde{\varphi}_{n,2}(\tau,\tau-\mathrm{t},\omega,\widetilde{\varphi}_0)\right\|_E^2+2\left((1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\mathrm{f}\big(u(\tau,\tau-\mathrm{t},\omega,\mathrm{u_0})\big),\tilde{u}_{n,2}(\tau,\tau-\mathrm{t},\omega,\tilde{\mathrm{u}}_0)\right)$$
$$\leq e^{2\int_\tau^{\tau-t}\Gamma(r,\omega)dr}\left(\|\widetilde{\varphi}_0\|_E^2+2\left((1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\mathrm{f}(\mathrm{u_0}),\tilde{u}_0\right)\right)+C\int_{\tau-t}^{\tau}e^{2\int_s^\tau\Gamma(r-\tau,\omega)dr}\left\|\widetilde{\varphi}_{n,2}(\tau,\tau-\mathrm{t},\omega,\widetilde{\varphi}_0)\right\|_E^2ds$$
$$+2\mathrm{C}\int_{\tau-t}^{\tau}e^{2\int_s^\tau\Gamma(r-\tau,\omega)dr}\left((1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\mathrm{f}\big(u(\tau,\tau-\mathrm{t},\omega,\mathrm{u_0})\big),\tilde{u}_{n,2}(\tau,\tau-\mathrm{t},\omega,\tilde{\mathrm{u}}_0)\right)ds$$
$$+c\int_{\tau-t}^{\tau}e^{2\int_s^\tau\Gamma(r-\tau,\omega)dr}\left\|(1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)g(x,t)\right\|^2ds+\frac{c}{k^4}\int_{\tau-t}^{\tau}e^{2\int_s^\tau\Gamma(r-\tau,\omega)dr}\,(\|\mathrm{u}(\mathrm{s},\tau-\mathrm{t},\omega,\tilde{u}_0)\|^2ds$$
$$+\frac{c}{k^2}\int_{\tau-t}^{\tau}e^{2\int_s^\tau\Gamma(r-\tau,\omega)dr}\,\|\nabla\mathrm{u}(\mathrm{s},\tau-\mathrm{t},\omega,\tilde{u}_0)\|^2)ds$$
$$+C\eta\int_{\tau-t}^{\tau}e^{2\int_s^\tau\Gamma(r-\tau,\omega)dr}\,(\|\mathrm{u_t}(\mathrm{s},\tau-\mathrm{t},\omega,\tilde{u}_0)\|^2+\|u(s,\tau-t\,,\omega,u_0)\|_{H^1}^2+|z(\theta_t\omega)|^4)ds.\quad (4.66)$$

*By replacing* $\omega$ *by* $\theta_{-\tau}\omega$ *, we get from* (4.66) *such that for all* $t\geq 0$

$$\left\|\widetilde{\varphi}_{n,2}(\tau,\tau-\mathrm{t},\theta_{-\tau}\omega,\widetilde{\varphi}_0)\right\|_E^2+\left(2(1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\mathrm{f}\big(u(\tau,\tau-\mathrm{t},\theta_{-\tau}\omega,\mathrm{u_0})\big),\tilde{u}_{n,2}(\tau,\tau-\mathrm{t},\theta_{-\tau}\omega,\tilde{\mathrm{u}}_0)\right)$$
$$\leq e^{2\int_\tau^{\tau-t}\Gamma(r,\omega)dr}\left(\|\widetilde{\varphi}_0\|_E^2+\left(2(1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\mathrm{f}(\mathrm{u_0}),\tilde{u}_0\right)\right)$$
$$+c\int_{\tau-t}^{\tau}e^{2\int_s^\tau\Gamma(r-\tau,\omega)dr}\left\|\widetilde{\varphi}_{n,2}(\mathrm{s},\tau-\mathrm{t},\theta_{-\tau}\omega,\widetilde{\varphi}_0)\right\|_E^2ds$$
$$+c\int_{\tau-t}^{\tau}e^{2\int_s^\tau\Gamma(r-\tau,\omega)dr}\left(2(1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\mathrm{f}\big(u(s,\tau-\mathrm{t},\theta_{-\tau}\omega,\mathrm{u_0})\big),\tilde{u}_{n,2}(s,\tau-\mathrm{t},\theta_{-\tau}\omega,\tilde{\mathrm{u}}_0)\right)ds$$
$$+c\int_{\tau-t}^{\tau}e^{2\int_s^\tau\Gamma(r-\tau,\omega)dr}\left\|(1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)g(x,t)\right\|_{\mathrm{E}}^2ds+\frac{c}{k^4}\int_{\tau-t}^{\tau}e^{2\int_s^\tau\Gamma(r-\tau,\omega)dr}\,(\|\mathrm{u}(\mathrm{s},\tau-\mathrm{t},\theta_{-\tau}\omega,\tilde{u}_0)\|^2ds$$
$$+\frac{c}{k^2}\int_{\tau-t}^{\tau}e^{2\int_s^\tau\Gamma(r-\tau,\omega)dr}\,\|\nabla\mathrm{u}(\mathrm{s},\tau-\mathrm{t},\theta_{-\tau},\omega,\tilde{u}_0)\|^2)ds$$
$$+c\eta\int_{\tau-t}^{\tau}e^{2\int_s^\tau\Gamma(r-\tau,\omega)dr}\,\|\mathrm{u_t}(\mathrm{s},\tau-\mathrm{t}\,,\theta_{-\tau}\omega,\tilde{u}_0)\|^2+\|\mathrm{u}(\mathrm{s},\tau-\mathrm{t},\theta_{-\tau}\omega,\tilde{u}_0)\|_{H^1}^2+|z(\theta_t\omega)|^4)ds.\quad (4.67)$$

Next we estimate each term on the right hand side of (4.67) for the first term by using condition (1.4) ,(4.21)we find that there exists$\varphi_0\in B\big(\tau-t_n,\theta_{-t_n}\omega\big),B\in\tilde{T}_1=\tilde{T}_1(\tau,B,\omega,\eta)>0$ and $\widetilde{\mathbb{k}}_1=\widetilde{\mathbb{k}}_1(\tau,\omega,\eta)\geq 1$ such that $t\geq\tilde{T}_1,k\geq\widetilde{\mathbb{k}}_1$

$$e^{2\int_\tau^{\tau-t}\Gamma(r,\omega)dr}\left(\|\widetilde{\varphi}_0\|_E^2+2\left((1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\mathrm{f}(\mathrm{u_0}),\tilde{u}_0\right)\right)\leq\eta\ ; \qquad (4.68)$$

For the fourth term of right hand side, condition (3.18) and (4.21) since $g\in L^2(\mathbb{R}^3)$, there is $\widetilde{N}_1=\widetilde{N}_1(\tau,\omega,\eta)>0$ such that for all $n\geq\widetilde{N}_1$

$$c\int_{\tau-t}^{\tau} e^{2\int_s^\tau \Gamma(r-\tau,\omega)dr}\left\|(1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)g(x,s)\right\|_{\mathrm{E}}^2 ds \le \eta \ ; \qquad (4.69)$$

For the fifth terms on the right hand side of (4.67) ,by lemma 4.1 there exists $\tilde{T}_2=\tilde{T}_2(\tau,B,\omega,\eta)>0$ and $\tilde{\mathbb{k}}_2=\tilde{\mathbb{k}}_2(\tau,\omega,\eta)\ge 1$ such that for all $t\ge\tilde{T}_2\ and\ k\ge\tilde{\mathbb{k}}_2$

$$\int_{\tau-t}^{\tau} e^{2\int_s^\tau \Gamma(r-\tau,\omega)dr}\left(\frac{c}{k^4}\|\mathrm{u(s,\tau-t,\theta_{-\tau}\omega,u_0)}\|^2+\frac{c}{k^2}\|\nabla\mathrm{u}(\mathrm{s,\tau-t,\theta_{-\tau}\omega},\tilde{u}_0)\|^2\right)ds \le \ \eta \quad (4.70)$$

For the second, third and last terms on the right hand side of (4.67), by lemma 4.1 ,(4.21) , lemma3.1 ,(4.5) and (3.15) there exists $\tilde{T}_3=\tilde{T}_3(\tau,B,\omega,\eta)>0$ such that for all $t\ge\tilde{T}_3$ we have

$$c\int_{\tau-t}^{\tau} e^{2\int_s^\tau \Gamma(r-\tau,\omega)dr}\left\|\tilde{\varphi}_{n,2}(\mathrm{s,\tau-t,\theta_{-\tau}\omega},\tilde{\varphi}_0)\right\|_E^2 ds$$

$$c\int_{\tau-t}^{\tau} e^{2\int_s^\tau \Gamma(r-\tau,\omega)dr}\left(2(1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\mathrm{f}\big(u(s,\tau-\mathrm{t},\theta_{-\tau}\omega,\mathrm{u_0})\big),\tilde{u}_{n,2}(s,\tau-\mathrm{t},\theta_{-\tau}\omega,\tilde{\mathrm{u}}_0)\right)ds$$

$$c\eta\int_{\tau-t}^{\tau} e^{2\int_s^\tau \Gamma(r-\tau,\omega)dr}\|\mathrm{u_t(s,\tau-t\,,\theta_{-\tau}\omega},\tilde{u}_0)\|^2+\|\mathrm{u(s,\tau-t,\theta_{-\tau}\omega},\tilde{u}_0)\|_{H^1}^2+|z(\theta_t\omega)|^4)ds \quad \le\eta\,.(4.71)$$

Letting

$$\tilde{\mathrm{T}}=max\{\tilde{\mathrm{T}}_1,\tilde{\mathrm{T}}_2,\tilde{\mathrm{T}}_3\}$$
$$\tilde{K}=max\{\tilde{\mathrm{K}}_1,\tilde{\mathrm{K}}_2\} \qquad (4.72)$$

Then combining with (4.68)-(4.71),we have for all $t>\tilde{T}$ and $n>\tilde{N}$

$$\|\,\tilde{\varphi}_{\mathrm{n},2}(\tau,\tau-\mathrm{t},\theta_{-\mathrm{t}}\omega,\tilde{\varphi}_0)\|_E^2+2\left((1-\mathrm{p_n})\mathrm{q}\left(\frac{|\mathrm{x}|^2}{\mathrm{k}^2}\right)\mathrm{f}(\mathrm{u_0},\tilde{\mathrm{u}}_0)\right)\le 4\eta \qquad (4.73)$$

This implies (4.47) we get

$$\|(1-P_n)\tilde{\varphi}(t,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega))\|_{E(\mathbb{H}_{2k})}^2\le 4\eta. \qquad (4.74).$$

Then we complete the proof. □

## 5. Random Attractors

In this section, we prove the existence of a D-random attractor for the random dynamical system  associated with the stochastic wave equation(3.1)-(3.2)on $\mathbb{R}^3$.It follows from lemma 4.1 that $\Phi$ has a closed random absorbing set in D,which along with the D-pullback asymptotic compactness will imply the existence of a unique D-random attractor. the D-pullback asymptotic compactness of $\phi$ is given below and will be proved by using the uniform estimates on the tails of solutions.

**Lemma 5.1**.we assume that(1.3)-(1.7) hold and $g(x,t)\in L^2_{loc}(\mathbb{R},L^2(\mathbb{R}^3)$ satisfies (3.18)and (3.19). Then the solution of problem (3.6)-(3.8)or random dynamical system $\phi$ is D-pullback asymptotically compact in $E(\mathbb{R}^3)$ ;that is , for every $\tau\in\mathbb{R}$ , $\omega\in\Omega$ and $B=\{B(\tau,\omega):\tau\in R,\omega\in\Omega\}\in D,\omega\in\Omega$ ,the sequence $\{\varphi(\tau,\tau-t_n,\theta_{-\tau}\omega,\varphi_{0,n})\}$has a convergent subsequence in $E(\mathbb{R}^3)$provided $t_n\to\infty$ and $\varphi_{0,n}\in B(\tau-t_n,\theta_{-t_n}\omega)$.

**Proof** Let $t_n\to\infty$, $B\in D$ and $\varphi_{0,n}\in B(\tau-t_n,\theta_{-t_n}\omega)$ .Then by Lemma4.1, for P-a.e $\omega\in\Omega$ ,we have that $\{\varphi(\tau,\tau-t_n,\theta_{-\tau}\omega,\varphi_{0,n})\}$ is bounded in $E(\mathbb{R}^3)$ ;that is,for every $\tau\in\mathbb{R},\omega\in\Omega$ ,there exists $M_1=M_1(\tau,\omega,B)>0$ such that for all $m\ge M_1$

$$\left\|\varphi(\tau,\tau-t_n,\theta_{-\tau}\omega,\varphi_{0,n})\right\|_{E(\mathbb{R}^3)}^2\le\varrho^2(\tau,\omega) \qquad (5.1)$$

In addition, it follows from Lemma4.3 that there exists $k_1=k_1(\tau,\omega,\eta)>0$ and $\widetilde{M_2}=\tilde{M}_2(\tau,B,\omega,\eta)>0$ such that for all $m\ge M_2$

$$\left\|\varphi(\tau,\tau-t_n,\theta_{-\tau}\omega,\varphi_{0,n})\right\|_{E(\mathbb{R}^3\setminus H_{k_1})}^2\le\eta\,. \qquad (5.2)$$

Next, by using Lemma4.4,there are$N=N(\tau,\omega,\eta)>0$, $k_2=k_2(\tau,\omega,\eta)>k_1$and $\widetilde{M_3}=\tilde{M}_3(\tau,B,\omega,\eta)>0$ such that for all $m\ge M_3$

$$(1-P_n)\left\|\tilde{\varphi}\big(\tau,\tau-t_n,\theta_{-\tau}\omega,\varphi_{0,n}\big)\right\|_{E(H_{2k_2})}^2\le\eta\,. \qquad (5.3)$$

using (4.43) and (5.1) ,we find that $\{P_n\tilde{\varphi}\big(\tau,\tau-t_n,\theta_{-\tau}\omega,\varphi_{0,n}\big)\}$ is bounded in finite-dimensional space $P_nE\big(H_{2k_2}\big)$ ,which associates with (5.3) implies that $\{\tilde{\varphi}\big(\tau,\tau-t_n,\theta_{-\tau}\omega,\varphi_{0,n}\big)\}$ is precompact in $H_0^1\big(H_{2k_2}\big)\times L^2\big(H_{2k_2}\big)$.

Note that $q\left(\frac{|x|^2}{k^2}\right) = 1$ for $\{x \in \mathbb{R}^3 : |x| \le k_2\}$.recalling (4.43) ,we find that $\{\varphi(\tau, \tau - t_n, \theta_{-\tau}\omega, \varphi_{0,n})\}$ is precompact in $E(H_{k_2})$ , which along with (5.2) show that the precompactness of this sequence in $E(\mathbb{R}^3)$ this completes the proof.□

The main result of this section can now be state as follows.

**Theorem5.2**. we assume that(1.3)-(1.7) hold and $g(x,t) \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^3)$ satisfies (3.18)and (3.19). Then the continuous cocycle Φ associated with problem (3.6)-(3.8)or random dynamical system Φ has a unique D-pullback attractor $\mathrm{A} = \{\mathrm{A}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$ in $\mathbb{R}^3$.

**Proof**. Notice that the continuous cocycle Φ has a closed random absorbing set $\{\mathrm{A}(\omega)\}_{\omega\in\Omega}$ in D by Lemma4.2 .on the other hand,by (3.16) and Lemma5.1 ,the continuous cocycle Φ is D-pullback asymptotically compact in $\mathbb{R}^3$. Hence the existence of a unique D- random attractor for Φ follows from proposition 2.1 immediately. □

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