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New Methods for the Construction of Spring Balance Weighing Designs

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Abstract: *If the light objects are weighed in groups rather than individually, and the weights of the individual objects are estimated, then the precision of the estimates of the weights of the objects increases quite considerably. In the present paper, two methods for the construction of spring balance weighing designs using Balanced incomplete block designs (BIBD) and Mutual orthogonal Latin squares (MOLS) are proposed and the methods are illustrated through suitable examples.*
Key words: *BIBD, MOLS, Spring Balance Designs*

I. Introduction

The problem of weighing light objects was introduced by Yates (1935) in furnishing an illustration of "independent factors" in complex experiments. The basic principles of theory of the design of efficient experiments for estimating the true unknown weights of ‘P’ objects by means of a specified number of weighings N (P≤N), and refers to the design of a certain class. To weigh the objects either chemical balance or spring balance device are used. In spring balance only one pan is available for placing the objects. So the elements of the design matrix assumes only the values 1 and 0 according as the corresponding object is weighed in the combination or not.

The results of N weighing operations determining the individual weights of P objects fit into the linear model

$$\underline{Y} = X\,\underline{\beta} + \underline{\varepsilon} \qquad (1.1)$$

Where $\underline{Y}$ is the Nx1 vector of the recorded weighings,
X is an NxP design matrix; i = 1, 2, ..., N; j = 1,2, ..., P;
$\underline{\beta}$ is the Px1 vector of true unknown weights of the objects
$\underline{\varepsilon}'$ is an Nx1 random vector of errors, with $E(\underline{\varepsilon}) = 0$, $E(\underline{\varepsilon}, \underline{\varepsilon}') = I\sigma^2$..
$X_{ij} = 1$ if the j^{th} object is included in the i^{th} weighing
= 0 if the j^{th} object is not included in the i^{th} weighing,

If X′X is non-singular, the least squares estimate of $\underline{\beta}$ is given by

$$\hat{\beta} = (X'X)^{-1} X'Y \qquad (1.2)$$

with covariance matrix $(X'X)^{-1}\sigma^2$. If X’X is singular take‘t’ additional weighings (t > 0) made with all the ‘P’ objects being placed in the pan.

Several authors like Banerjee (1948, 1949), Kulshrestha and Dey (1970), Meena (1980), Jacroux and Notz (1983), Cernaka and Kalulska (1986) studied the construction and estimation of weights of objects using spring balance weighing designs.

II. Optimality Criteria

Let **D** be a class of weighing designs for estimating the weights of ‘p’ objects. A design d∈ **D** is said to be optimal if $\phi[M(d)] \le \phi[M(d^*)]$ for any d*∈**D.** The function ϕ is said to be the criterion function**.** Then the following optimality criteria are defined for comparing designs belongs to **D**.

MV-optimality: A design d* belonging to **D** is said to be MV-optimal in **D** if $Var(\hat{\beta}_i)_{d^*} \le V(\hat{\beta}_i)_d$ for any other design d belonging to **D**.

A-Optimality: A design d* belonging to **D** is said to be A-optimal in **D** if $Trace\,(X'X)^{-1})_{d^*} \le Trace\,(X'X)^{-1})_d$ for any other design d belonging to **D**.

D-Optimality: A design d* belonging to **D** is said to be D-optimal in **D** if $|(X'X)^{-1}|_{d^*} \le |\,(X'X)^{-1}|_d$ for any other design d belonging to **D**.

Note: The efficiency of BIBD is E= λv / rk and the efficiency of spring balance weighing design based on variance was presented by Benarjee (1975) is $E= [(r-\lambda)\{r+\lambda(p-1)\}]/N[r+\lambda(p-2)]$ where p is the number of objects.

III. Construction Of Spring Balance Weighing Designs

In this section, an attempt is made to propose two new methods for the construction of spring balance weighing designs using balanced incomplete block designs and mutual orthogonal latin squares. The methods are illustrated through suitable examples. The optimality criteria for the respective designs are also studied.

3.1 Method-I: Consider a BIBD with parameters v, b, r, k and λ such that v > k + 1 whose incidence matrix is N_{bxv} and consider a matrix I_v where I_v is the identity matrix of order v. Derive the matrix N* of v(b-r) rows by adding the j^{th} row of I_v to the rows of N having 0(zero) in the j^{th} column (j = 1,2,..., v). Then the resulting N* constitutes a spring balance weighing design for weighing v objects in v(b-r) weighings.

The variances of estimated weights in case of spring balance design can be derived from the X'X matrix like as follows:

We have $\sum_{i=1}^{N} X_{ij}$ = v(r-λ) + (b-2r+λ); $\sum_{i=1}^{N} X_{ij} X_{ik}$ = vλ + 2(r-λ)-λk

The X′X matrix is

$$X'X = [(R-\lambda^*)Iv + \lambda^* Jv]$$

where R= v(r-λ)+ (b-2r+λ) and λ*= vλ+2(r-λ)-λk. Estimate the weight of objects using (1.2)

Remarks:

1. An incomplete weighing design can be obtained if v > k+1.
2. If v = k+1, the resulting design in singular.

The method is illustrated in the example 3.1.

Example 3.1*:* Consider a BIBD with parameters v = 4, b = 6, r = 3, k = 2 and λ= 1 whose incidence matrix is N and consider an identity matrix I_4 for the same number of treatments. Then by adding the elements, of i^{th} row of I_4 to those rows of N, which contain zero in the j^{th} column. The resulting spring balance weighing design for four objects in 12 weighings is given below.

Incidence matrix of BIBD N_{bxv}	Identity matrix Iv	Spring Balance design X
1 1 0 0	1 0 0 0	1 0 1 1
0 0 1 1	0 1 0 0	1 1 0 1
1 0 1 0	0 0 1 0	1 1 1 0
0 1 0 1	0 0 0 1	0 1 1 1
1 0 0 1		1 1 1 0
0 1 1 0		1 1 0 1
		1 1 1 0
		0 1 1 1
		1 0 1 1
		1 1 0 1
		1 0 1 1
		0 1 1 1

We have $\sum_{i=1}^{N} X_{ij}^{2}$ = 9; $\sum_{i=1}^{N} X_{ij} X_{ik}$ = 6

The optimality criteria of the design are

Variance	Covariance	A–Optimality	D–Optimality
(7/27) σ^2	**(-2/27) σ^2**	**28/27**	**3^{-6}**

Remark:

1. An incomplete weighing design can be obtained if v > k + 1.
2. If v = k + 1, the resulting design in singular.

3.2 M*ethod*-II: Consider a complete set of n-1 mutual orthogonal latin squares (MOLS) of order 'n', where 'n' is prime or prime power. Construct an array of order n(n-1) x n by arranging the n-1 MOLS together such that each row consisting of 'n' elements P_0, P_1, ... , P_{n-1}. Replace 'f' of P_i's by 1's and the remaining (n-f) P_i's by 0's in each row. The resulting matrix consisting of 'f' ones and (n-f) zeros in each row provides a design matrix of spring balance weighing design for weighing n objects in n(n-1) weighings.

From the above design, we get

$$\sum_{i=1}^{N} X_{ij} = f(n-1); \qquad \sum_{i=1}^{N} X_{ij} X_{ik} = f(f-1)$$

The X′X matrix is

$$X'X = [(r-\lambda)I_v + \lambda\, J_v]$$

where $r = f(n-1)$ and $\lambda = f(f-1)$.

The variance of estimated weight in this case can be obtained as

$$V(\hat{\beta}_i) = [\, \{(p-2)f + 1\} \,/\, \{(p-f)\,(p-1)\, f^2\} \,]\, \sigma^2$$

$$Cov(\hat{\beta}_i\ \hat{\beta}_j) = [\, \{ -(f-1) \} \,/\, \{ (p-f)\,(p-1)\, f^2 \} \,]\, \sigma^2$$

If $X'X$ is singular we take t additional weighings ($t>0$) made with all the p objects being placed in the pan. In this case the variance and covariance becomes

$$V(\hat{\beta}_i) = [\, \{(p-2)f^2 + f + (p-1)t\} \,/\, \{\ [f(p-f)]\, [pt + (p-1)f^2]\ \} \,]\, \sigma^2$$

$$Cov(\hat{\beta}_i\ \hat{\beta}_j) = [\, -\{ f(f-1) + t \} \,/\, \{\ [f(p-f)]\, [pt + (p-1)f^2]\ \} \,]\, \sigma^2$$

Remarks:

1. When ‘n’ is even all the n-1 MOLS are to be considered.
2. When ‘n’ is odd, either all n-1 MOLS or a set of n-1/2 MOLS such that ${}^{n}C_2$ pairs of the ‘n’ elements occur exactly once in any two columned sub matrix of the array of order n(n-1)/2 x n can be considered.

The method is illustrated in the example 3.2

Example 3.2: Consider the set of 4 MOLS of order 5. Construct a 10x5 design, by selecting two MOLS such that all pairs of elements occur exactly once in any two columned sub matrix of the array of order 10x5. Then form 10 blocks, replacing P_0, P_1, and P_2 by 1 and P_3, P_4, by 0, the resulting design provides a spring balance weighing design for weighing 5-objects in 10 weighings.

$$\begin{pmatrix} P_0 & P_1 & P_3 & P_4 & P_2 \\ P_1 & P_2 & P_4 & P_0 & P_3 \\ P_2 & P_3 & P_4 & P_0 & P_3 \\ P_3 & P_4 & P_1 & P_2 & P_0 \\ P_4 & P_0 & P_3 & P_3 & P_1 \\ P_0 & P_3 & P_4 & P_2 & P_1 \\ P_1 & P_4 & P_0 & P_3 & P_2 \\ P_2 & P_0 & P_1 & P_4 & P_3 \\ P_3 & P_1 & P_2 & P_0 & P_4 \\ P_4 & P_2 & P_3 & P_1 & P_0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Optimality’s of Spring Balance Design

Variance	Covariance	A–Optimality	D–Optimality
$(5/18)\sigma^2$	**$(-1/18)\sigma^2$**	**25/18**	**½ (3^6)**

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