Degree Of Approximation Of Fourier Series Of Functions In Besove Space By Riesz Means

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Abstract: The paper studies the degree of approximation of Fourier series of functions in Besove space by Riesz Means and this generalizes many known results.


Keywords: Besove space, Holders space, Lipschitz space, Riesz mean, Fourier series, modulus of smoothness.

I. Introduction

Let \( f \) be a \( 2\pi \) periodic function and let \( f \in L_p[0, 2\pi], p \geq 1 \). The fourier series of \( f \) at \( x \) is given by

\[
\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]

(1.1)

Let \( s_n(x) \) denote the \( n \)th partial sums of (1.1). We know ([6],p.50) that

\[
s_n(x) - f(x) = \frac{1}{\pi} \int_0^\pi \phi_k(u) D_n(u) du
\]

(1.2)

where \( \phi_k(u) = f(x + u) + f(x - u) - 2f(x) \)

(1.3)

\[
D_n(u) = \frac{1}{2} \sum_{k=0}^{n} \cos ku = \frac{\sin(k + \frac{1}{2})u}{2 \sin \frac{u}{2}}
\]

(1.4)

Riesz Means:

Let \( \sum a_n \) be an infinite series of partial sum \( \{s_n\} \). Let \( b_n \) be a sequence of numbers such that

\[
B_n = b_0 + b_1 + \ldots + b_n, B_n \neq 0 (n \geq 0)
\]

Then the transformation

\[
t_n = \frac{1}{B_n} \sum_{k=0}^{n} b_k s_k
\]

(1.5)

is called the Riesz means or \((R, b_n)\) means of the series \( \sum a_n \) or the sequence \( s_n \).

Modulus of Continuity:
Let \( A = R, R + [a,b] \subset R \) or \( T \) (which usually taken to be \( R \) with identification of points modulo \( 2\pi \)).

The modulus of continuity \( w(f, t) = w(t) \) of a function \( f \) on \( A \) can be defined as

\[
w(t) = w(f, t) = \sup_{[x-y] \in [t, x, y] \in A} |f(x) - f(y)|, t \geq 0.
\]

Modulus of Smoothness:
The \( k^{th} \) order modulus of smoothness [2] of a function \( f: A \rightarrow R \) is defined by

\[
w_k(f, t) = \sup_{0 < kh \leq t} \{ \sup_{x \in A} |\Delta_k^h(f, x)| : x, x + kh \in A \}, t \geq 0
\]

(1.6)

where
\[ \Delta^k_h(f,x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x + ih), k \in \mathbb{N}. \quad (1.7) \]

For \( k = 1 \), \( w_1(f,t) \) is called the modulus of continuity of \( f \). The function \( w \) is continuous at \( t = 0 \) if and only if \( f \) is uniformly continuous on \( A \), that is \( f \in \mathcal{C}(A) \). The \( k \)th order modulus of smoothness of \( f \in L_p(A), 0 < p < \infty \) or of \( f \in \mathcal{C}(A) \), if \( p = \infty \) is defined by

\[ w_k(f,t) = \sup_{0<h<\infty} \| \Delta^k_h(f,) \|_{p}, t \geq 0 \quad (1.8) \]

if \( p \geq 1, k = 1 \), then \( w_1(f,t) = w(f,t) \) is a modulus of continuity (or integral modulus of continuity). If \( p = \infty \), \( k = 1 \) and \( f \) is continuous then \( w_k(f, t) \) reduces to modulus of continuity \( w_1(f, t) \) or \( w(f, t) \).

**Lipschitz Space:**

If \( f \in \mathcal{C}(A) \) and

\[ w(f, t) = O(t^\alpha), 0 < \alpha \leq 1 \quad (1.9) \]

then we write \( f \in Lip \alpha \). If \( w(f, t) = O(t) \) as \( t \to 0^+ \) (in particular \( 1.9 \) holds for \( \alpha > 1 \)) then \( f \) reduces to a constant.

If \( f \in L_p(A), 0 < p < \infty \) and

\[ w(f, t) = O(t^\alpha), 0 < \alpha \leq 1 \quad (1.10) \]

then we write \( f \in Lip(\alpha, p), 0 < p < \infty, 0 < \alpha \leq 1 \).

The case \( \alpha > 1 \) is of no interest as the function reduces to a constant, whenever

\[ w(f, t) = O(t) \text{ as } t \to 0^+ \quad (1.11) \]

We note that if \( p = \infty \) and \( f \in \mathcal{C}(A) \), then \( Lip(\alpha, p) \) class reduces to \( Lip \alpha \) class.

**Generalized Lipschitz Space:**

Let \( \alpha > 0 \) and suppose that \( k = [\alpha] + 1 \). For \( f \in L_p(A), 0 < p < \infty \), if

\[ w_k(f, t) = O(t^\alpha), t > 0 \quad (1.12) \]

then we write

\[ f \in Lip^+(\alpha, p), \alpha > 0, 0 < p \leq \infty \quad (1.13) \]

and say that \( f \) belongs to generalized Lipschitz space. The seminorm is then

\[ \| f \|_{Lip^+(\alpha, p), \alpha > 0} = \sup_{t > 0} (t^{-\alpha} w_k(f,t)). \]

It is known ([2], p-52) that the space \( Lip^+(\alpha, L_p) \) contains \( Lip(\alpha, L_p) \). For \( 0 < \alpha < 1 \) the spaces coincide, (for \( p = \infty \), it is necessary to replace \( L_\infty \) by \( \mathcal{C} \) of uniformly continuous function on \( A \)). For \( 0 < \alpha < 1 \) and \( p = 1 \) the space \( Lip^+(\alpha, L_p) \) coincide with \( Lip \alpha \).

For \( \alpha = 1, p = \infty \), we have

\[ Lip(1, \mathcal{C}) = Lip 1 \quad (1.14) \]

but

\[ Lip^+(1, \mathcal{C}) = \mathcal{Z} \quad (1.15) \]

is the Zygmund space [5] which is characterized by \( 1.12 \) with \( k = 2 \).
Holder ($H_\alpha$) Space:
For $0 < \alpha \leq 1$, let
\[ H_\alpha = \{ f \in C_{2\pi} : w(f, t) = O(t^\alpha) \} \] (1.16)
It is known [3] that $H_\alpha$ is a Banach Space with the norm $\| f \|_\alpha$ defined by
\[ \| f \|_\alpha = \| f \|_p + \sup_{t \geq 0} t^{-\alpha} w(t), \quad 0 < \alpha \leq 1 \] (1.17)
and
\[ H_{\alpha} \subseteq H_{\beta} \subseteq C_{2\pi}, \quad 0 < \beta \leq \alpha \leq 1 \] (1.18)

$H_{(\alpha, p)}$ Space:
For $0 < \alpha \leq 1$, let
\[ H_{(\alpha, p)} = \{ f \in L_p[0, 2\pi] : 0 < p \leq \infty, w(f, t)_p = O(t^\alpha) \} \] (1.19)
and introduce the norm $\| f \|_{(\alpha, p)}$ as follows
\[ \| f \|_{(\alpha, p)} = \| f \|_p + \sup_{t \geq 0} t^{-\alpha} w(f, t)_p, \quad 0 < \alpha \leq 1 \] (1.20)
It is known [1] that $H_{(\alpha, p)}$ is a Banach space for $p \geq 1$ and a complete $p$-normed space for $0 < p < 1$.
Also
\[ H_{(\alpha, p)} \subseteq H_{(\beta, p)} \subseteq L_p, 0 < \beta \leq \alpha \leq 1. \] (1.21)
Note that $H_{(\alpha, \infty)}$ is the space $H_\alpha$ defined above.

For study of degree of approximation problems the natural way to proceed to consider with some restrictions on some modulus of smoothness as prescribed in $H_\alpha$ and $H_{(\alpha, p)}$ spaces. As we have seen above only a constant function satisfies Lipschitz condition for $\alpha > 1$. However for generalized Lipschitz class there is no such restriction on $\alpha$. We required a finer scale of smoothness than is provided by Lipschitz class. For each $\alpha > 0$ Besove developed a remarkable technique for restricting modulus of smoothness by introducing a third parameter $q$ (in addition to $p$ on $\alpha$) and applying $\alpha \cdot q$ norms (rather than $\alpha, \infty$ norms) to the modulus of smoothness $w_k(f, t)_p$ of $f$.

Besove space:
Let $\alpha > 0$ be given and let $k = [\alpha] + 1$. For $0 < p, q \leq \infty$, the Besove space ([2], p-54) $B_q^\alpha(L_p)$ is defined as follows:
\[ B_q^\alpha(L_p) = \{ f \in L_p : f \in B_q^\alpha(L_p), \quad \| w_k(f, \cdot) \|_{(\alpha, q)} \text{ is finite} \} \]
where
\[ \| w_k(f, \cdot) \|_{(\alpha, q)} = \left\{ \begin{array}{ll}
\int_0^\infty (t^{-\alpha} w_k(f, t)_p)^q \frac{dt}{t} \frac{1}{q}, & 0 < q < \infty \\
\sup_{t \geq 0} t^{-\alpha} w_k(f, t)_p, & q = \infty.
\end{array} \right. \]
It is known ([2], p-55) that $\| w_k(f, \cdot) \|_{(\alpha, q)}$ is a seminorm if $1 \leq p, q \leq \infty$ and a quasi-seminorm in other cases.
The Besov norm for \( B^\alpha_q(L_p) \) is
\[
\| f \|_{B^\alpha_q(L_p)} = \| f \|_p + \| t^{-\alpha} w_k (f, t) \|_q \quad (1.22)
\]
It is known ([4], p.237) that for \( 2\pi \) -periodic function \( f \), the integral \( \int_0^\infty (t^{-\alpha} w_k (f, t))^{\frac{1}{q}} dt \) is replaced by \( \left( \int_0^\infty (t^{-\alpha} w_k (f, t_t))^{\frac{1}{q}} dt \right)^{\frac{1}{q}} \).

We know ([2], p.56, [4], p.236) the following inclusion relations.

- For fixed \( \alpha \) and \( p \)
  \( B^\alpha_q(L_p) \subset B^\alpha_{q_1}(L_{p_1}), q < q_1 \).

- For fixed \( p \) and \( q \)
  \( B^\alpha_q(L_p) \subset B^\beta_q(L_p), \beta < \alpha \).

- For fixed \( \alpha \) and \( q \)
  \( B^\alpha_q(L_p) \subset B^\alpha_q(L_{p_1}), p_1 < p \).

**Special Cases Of Besov Space:**

For \( q = \infty, B^\alpha_\infty(L_p), \alpha > 0, p \geq 1 \) is same as \( \text{Lip}^* (\alpha, L_p) \) the generalized Lipschitz space and the corresponding norm \( \| \cdot \|_{B^\alpha_\infty(L_p)} \) is given by
\[
\| f \|_{B^\alpha_\infty(L_p)} = \| f \|_p + \sup_{t > 0} t^{-\alpha} w_k (f, t), \quad (1.23)
\]
for every \( \alpha > 0 \) with \( k = [\alpha] + 1 \).

For the special case when \( 0 < \alpha < 1 \), \( B^\alpha_\infty(L_p) \) space reduces to \( H^{(\alpha,p)} \) space due to Das et al. [1] and the corresponding norm is given by
\[
\| f \|_{B^\alpha_\infty(L_p)} = \| f \|_{H^{(\alpha,p)}} = \| f \|_p + \sup_{t > 0} t^{-\alpha} w_k (f, t), 0 < \alpha < 1. \quad (1.24)
\]

For \( \alpha = 1 \), the norm is given by
\[
\| f \|_{B^1_\infty(L_p)} = \| f \|_p + \sup_{t > 0} t^{-1} w_k (f, t). \quad (1.25)
\]
Note that \( \| f \|_{B^1_\infty(L_p)} \) is not same as \( \| f \|_{L_1(L_p)} \) and the space \( B^1_\infty(L_p) \) includes the space \( H^{(1,p)} \), \( p \geq 1 \).

If we further specialize by taking \( p = \infty, B^\alpha_\infty, 0 < \alpha < 1 \), coincides with \( H^{(\alpha)} \) space due to Prossodorf [3] and the norm is given by
\[
\| f \|_{B^\alpha_\infty(L_\infty)} = \| f \|_{H^{(\alpha)}} = \| f \|_\infty + \sup_{t > 0} t^{-\alpha} w_k (f, t), 0 < \alpha < 1. \quad (1.26)
\]

For \( \alpha = 1, p = \infty \), the norm is given by
\[
\| f \|_{B^1_\infty(L_\infty)} = \| f \|_\infty + \sup_{t > 0} t^{-1} w_k (f, t), \alpha = 1 \quad (1.27)
\]
which is different from \( \| f \|_1 \) and \( B^1_\infty(L_\infty) \) includes the \( H^1 \) space.
II. Main Result

Theorem: Let $0 < \alpha < 2$ and $0 \leq \beta < \alpha$. If $f \in B^\alpha_q(L_p), p \geq 1$ and $1 < q \leq \infty$. Let $T_n(x)$ be the $(R, b_n)$ of Fourier series when $b_n \geq 0$ and non-increasing. Then

Case 1: $(1 < q < \infty)$

$$\|T_n(\cdot)\|_{\beta, q}^{\alpha, q} = O\left(\frac{1}{B_n} \left\{ \sum_{k=1}^{n} \left( \frac{B_{k+1}}{k^{\alpha-\beta/2}} \right)^q \right\}^{1/q} \right)$$

Case 2: $(q = \infty)$

$$\|T_n(\cdot)\|_{\beta, \infty}^{\alpha, \infty} = O\left(\frac{1}{B_n} \sum_{k=1}^{n} \frac{B_{k+1}}{k^{\alpha-\beta}} \right)$$

For the proof of the theorem we need the following Lemmas.

Lemma 2.1 Let $1 \leq p \leq \infty$ and $0 < \alpha < 2$. If $f \in L_p([0, 2\pi])$, then for $0 < t, u \leq \pi$

i. $\|\phi(t, u)\|_p \leq 4w_k(f, t)_p$

ii. $\|\phi(t, u)\|_p \leq 4w_k(f, u)_p$

iii. $\|\phi(u)\|_p \leq 2w_k(f, u)_p$

Lemma 2.2 Let $0 < \alpha < 2$. Suppose that $0 \leq \beta < \alpha$. If $f \in B^\alpha_q(L_p), p \geq 1, 1 < q < \infty$, then

i. $\int_0^\pi |K_n(u)| \int_0^u \left\{ \int_0^\infty \left( \frac{\|\phi(t, u)\|_p}{t^{\beta/2}} \right)^q dt \right\}^{1/q} \frac{du}{t} = O(1) \int_0^\pi \left\{ \frac{\|u^{\alpha-\beta} | K_n(u)\|_q}{u^{\beta/2}} \right\}^{1/q} du$

ii. $\int_0^\pi |K_n(u)| \int_0^u \left\{ \int_0^\infty \left( \frac{\|\phi(t, u)\|_p}{t^{\beta/2}} \right)^q dt \right\}^{1/q} \frac{du}{t} = O(1) \int_0^\pi \left\{ \frac{\|u^{\alpha-\beta} | K_n(u)\|_q}{u^{\beta/2}} \right\}^{1/q} du$

Lemma 2.3 Let $0 < \alpha < 2$. Suppose that $0 \leq \beta < \alpha$. If $f \in B^\alpha_q(L_p), p \geq 1$ and $q = \infty$, then

$$\sup_{0 < t, u \leq \pi} t^{-\beta} \|\phi(t, u)\|_p = O(u^{\alpha-\beta})$$

Lemma 2.4 Let $b_n \geq 0$ and non-increasing and $K_n(u) = \frac{1}{B_n} \sum_{k=0}^{n} b_k D_k(u)$. then for $0 < u \leq \pi$ and

$$m = \left\lfloor \frac{\pi}{u} \right\rfloor$$

$$K_n(u) = \begin{cases} O(n) & u \leq \frac{\pi}{n} \\ O\left(\frac{B_m}{u^2B_n}\right) & u \geq \frac{\pi}{n} \end{cases}$$

Proof. We know

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\[ D_k(u) = \frac{1}{2} + \sum_{r=0}^{k} \cos ru = \frac{\sin(k + \frac{1}{2})u}{2 \sin \frac{u}{2}} \]  

(2.1)

\[ \Rightarrow |D_k(u)| = \frac{1}{2} + \sum_{r=0}^{k} |\cos ru| \]

\[ \leq \frac{1}{2} + \sum_{r=0}^{k} |\cos ru| \]

\[ \leq k + 1 \quad (2.2) \]

Now

\[ |K_n(u)| \leq \frac{1}{B_n} \sum_{k=0}^{n} b_k |D_k(u)| \]

\[ \leq \frac{1}{B_n} \sum_{k=0}^{n} b_k (k + 1) (by \ using \ (2.2)) \]

\[ = O \left( \frac{n + 1}{B_n} \sum_{k=0}^{n} b_k \right) \]

\[ = O(n) \]

Again

\[ K_n(u) = \frac{1}{B_n} \sum_{k=0}^{n} b_k \frac{\sin(k + \frac{1}{2})u}{2 \sin \frac{u}{2}} \]  

(by using (2.1))

\[ = \frac{1}{B_n} \sum_{k=0}^{n} b_k \left( \sum_{k=0}^{m} + \sum_{k=m+1}^{n} \right) \sin(k + \frac{1}{2})u \]

\[ \Rightarrow |K_n(u)| \leq \frac{1}{B_n} \left[ \sum_{k=0}^{m} |b_k| \sin(k + \frac{1}{2})u \right] \]

\[ + \left[ \sum_{k=m+1}^{n} |b_k| \sin(k + \frac{1}{2})u \right] \]

\[ = O \left( \frac{1}{uB_n} \right) \]  

(2.3)

\[ |\sum_{k=0}^{m} b_k \sin(k + \frac{1}{2})u| \leq \sum_{k=0}^{m} b_k |\sin(k + \frac{1}{2})u| \]

\[ \leq \sum_{k=0}^{m} b_k . \]

Again,

\[ |\sum_{k=m+1}^{n} b_k \sin(k + \frac{1}{2})u| \leq b_m \max_{m \leq n} \sum_{k=m}^{m} \sin(k + \frac{1}{2})u \]  

(as \ \ b_k \ is \ monotonically \ decreasing)

\[ = b_m \cdot \frac{1}{u} \]

From equation (2.3), we have
Degree Of Approximation Of Fourier Series Of Functions In Besove Space By Riesz Means

Let $T_n(x)$ be the Riesz transformation of the Fourier series $s_n(x)$, that is by (1.5)

$$T_n(x) = \frac{1}{B_n} \sum_{k=0}^{n} b_k s_k(x)$$

Case 1: Let $T_n(x)$ be the Riesz transformation of the Fourier series $s_n(x)$, that is by (1.5)

$$T_n(x) = \frac{1}{B_n} \sum_{k=0}^{n} b_k s_k(x)$$

We have for $p \geq 1$ and $0 \leq \beta < \alpha < 2$, by use of Besove norm defined in (1.24)

$$\|T_p(\cdot)\|_{B^{\beta}_{q}(L^p)} \leq \|T_p(\cdot)\|_{p} + \|w_k(T_p,\cdot)\|_{\beta, q}$$

Applying Lemma 2.1(iii) in equation (2.7), we have

$$\|T_n(\cdot)\|_{p} \leq \frac{1}{\pi} \int_{0}^{\beta} |\phi(u)| \|K_n(u)\|_{p} du$$

$$\leq \frac{1}{\pi} \int_{0}^{\beta} 2w_k(f,u)_{p} |K_n(u)|_{p} du$$

$$= \frac{2}{\pi} \int_{0}^{\beta} |K_n(u)| w_k(f,u)_{p} du$$

Applying Hölder’s inequality, we have

$$\|T_n(\cdot)\|_{p} \leq \frac{2}{\pi} \left\{ \int_{0}^{\beta} \left| K_n(u) \right| u^{\frac{1}{q}-\frac{1}{p}} \right\}^{\frac{q}{q-1}} \left\{ \int_{0}^{\beta} \left( \frac{w_k(f,u)_{p}}{u^{\frac{1}{q}}-\frac{1}{q}} \right)^{q} \right\}^{\frac{1}{q}}$$

By definition of Besove space, we have

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\[ \| T_n(\cdot) \|_p \leq O(1) \left\{ \int_0^{\pi} \left| K_n(u) \right| u^{\frac{\alpha - 1}{q}} \frac{q}{q-1} \, du \right\}^{-\frac{1}{q}} \]

\[ = O(1) \left\{ \int_0^{\pi} \left| K_n(u) \right| u^{\frac{\alpha - 1}{q}} \frac{q}{q-1} \, du \right\}^{-\frac{1}{q}} + \int_0^{\pi} \left| K_n(u) \right| u^{\frac{\alpha - 1}{q}} \frac{q}{q-1} \, du \right\}^{-\frac{1}{q}} \]

\[ = O(1) \left\{ \int_0^{\pi} \left| K_n(u) \right| u^{\frac{\alpha - 1}{q}} \frac{q}{q-1} \, du \right\}^{-\frac{1}{q}} + \int_0^{\pi} \left| K_n(u) \right| u^{\frac{\alpha - 1}{q}} \frac{q}{q-1} \, du \right\}^{-\frac{1}{q}} \]

By using Lemma 2.4 in I of (2.8), we have

\[ I = \left\{ \int_0^{\pi} \left| K_n(u) \right| u^{\frac{\alpha - 1}{q}} \frac{q}{q-1} \, du \right\}^{-\frac{1}{q}} \]

\[ = O(n) \left\{ \int_0^{\pi} \left| K_n(u) \right| u^{\frac{\alpha - 1}{q}} \frac{q}{q-1} \, du \right\}^{-\frac{1}{q}} \]

\[ = O(n) \left\{ \int_0^{\pi} \left| K_n(u) \right| u^{\frac{\alpha - 1}{q}} \frac{q}{q-1} \, du \right\}^{-\frac{1}{q}} \]

Applying Lemma 2.4 in J of (2.8), we have

\[ J = \left\{ \int_0^{\pi} \left| K_n(u) \right| u^{\frac{\alpha - 1}{q}} \frac{q}{q-1} \, du \right\}^{-\frac{1}{q}} \]

\[ = O\left( \frac{1}{B_n} \right) \left\{ \int_0^{\pi} \left| K_n(u) \right| u^{\frac{\alpha - 1}{q}} \frac{q}{q-1} \, du \right\}^{-\frac{1}{q}} \]

\[ = O\left( \frac{1}{B_n} \right) \left\{ \int_0^{\pi} \left| K_n(u) \right| u^{\frac{\alpha - 1}{q}} \frac{q}{q-1} \, du \right\}^{-\frac{1}{q}} \]

\[ = O\left( \frac{1}{B_n} \right) \left\{ \int_0^{\pi} \left| K_n(u) \right| u^{\frac{\alpha - 1}{q}} \frac{q}{q-1} \, du \right\}^{-\frac{1}{q}} \]

\[ = O\left( \frac{1}{B_n} \right) \left\{ \int_0^{\pi} \left| K_n(u) \right| u^{\frac{\alpha - 1}{q}} \frac{q}{q-1} \, du \right\}^{-\frac{1}{q}} \]

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\[ = O\left( \frac{1}{B_n} \right) \left\{ \sum_{k=1}^{n} \frac{k^q}{k^{q-1}} \int_{0}^{\frac{k}{q}} u^{\frac{q}{q-1}(a-2)} \, du \right\} \]  
\[ = O\left( \frac{1}{B_n} \right) \left\{ \sum_{k=1}^{n} \frac{1}{k^{q-1}} \left( \frac{q}{k^{a-2}} \right) \right\} \]  
\[ = O\left( \frac{1}{B_n} \right) \left\{ \sum_{k=1}^{n} \left( B_{k+1} \right)^{\frac{q}{q-1}} \right\} \]  
(2.10)

Substitute the value of (2.9) and (2.10) in (2.8), we have

\[ \|T_n(\cdot)\|_p = O\left( \frac{1}{B_n} \right) \left\{ \sum_{k=1}^{n} \left( B_{k+1} \right)^{\frac{q}{q-1}} \right\} \]  
(2.11)

By using Besove space, we have

\[ \|w_k(T_n, \cdot)\|_{\beta,q} = \left\{ \int_{0}^{\pi} \left( t^{-\beta} \|w_k(T_n, t)\|_p \right)^q \frac{dt}{t} \right\}^{1/q} \]  
\[ = \int_{0}^{\pi} \left( \frac{w_k(T_n, t)}{t^\beta} \right)^q \frac{dt}{t} \]

From definition of \( w_k(T_n, t)_p \), we have

\[ w_k(T_n, t)_p = \|T_n(\cdot, t)\|_p \leq \pi \left\{ \int_{0}^{\pi} \left( \|T_n(\cdot, t)\|_p \right)^q \frac{dt}{t^\beta} \right\}^{1/q} \]  
\[ = \left\{ \int_{0}^{\pi} \left( \int_{0}^{\pi} |T_n(x,t)|^p \frac{dx}{t^\beta} \right)^{\frac{q}{p}} \frac{dt}{t^{\beta+1}} \right\}^{1/q} \]  
\[ = \left\{ \int_{0}^{\pi} \left( \int_{0}^{\pi} \phi(x,t,u)K_n(u)du \right)^{\frac{q}{p}} \frac{dt}{t^{\beta+1}} \right\}^{1/q} \]

By repeated application of generalized Minkowski’s inequality, we have

\[ \|w_k(T_n, \cdot)\|_{\beta,q} \leq \frac{1}{\pi} \left\{ \int_{0}^{\pi} \left( \int_{0}^{\pi} \left( \int_{0}^{\pi} |\phi(x,t,u)|^p |K_n(u)|^p \, dx \right)^{\frac{q}{p}} \frac{dt}{t^{\beta+1}} \right\}^{1/q} \]
\[
I' = \left\{ \int_{0}^{\pi} \left( \left\| K_{n}(u) \right\| u^{a-\beta} \right)^{q/3} du \right\}^{1 - \frac{1}{q}}
\]

\[
= \left\{ \left( \int_{0}^{\frac{\pi}{n}} + \int_{\frac{\pi}{n}}^{\pi} \right) \left( K_{n}(u) \right) u^{a-\beta} \right\}^{q/3} \right\}^{1 - \frac{1}{q}}
\]

\[
\leq \left\{ \int_{0}^{\frac{\pi}{n}} \left( K_{n}(u) \right) u^{a-\beta} \right\}^{q/3} du \right\}^{1 - \frac{1}{q}} + \left\{ \int_{\frac{\pi}{n}}^{\pi} \left( K_{n}(u) \right) u^{a-\beta} \right\}^{q/3} du \right\}^{1 - \frac{1}{q}}
\]

\[
= I_{1}' + I_{2}', \quad \text{(say)} \quad (2.13)
\]

Applying Lemma 2.4 in \( I_{1}' \) and \( I_{2}' \), we have

\[
I_{1}' = \left\{ \int_{0}^{\pi} \left( K_{n}(u) \right) u^{a-\beta} \right\}^{q/3} du \right\}^{1 - \frac{1}{q}}
\]

\[
= O(n) \left\{ \int_{0}^{\pi} u^{a-\beta} \left( \frac{q}{q-1} \right)^{1/q} du \right\}
\]
\[ I_2' = \left\{ \sum_{k=1}^{\infty} \left( B_{k+1} u^{\alpha-\beta-2} \right)^{\frac{q}{q-1}} \right\}^{1-\frac{1}{q}} \]

Let \( h(u) = (B_{k+1} u^{\alpha-\beta-2})^{\frac{q}{q-1}} \) and \( H(u) \) is a primitive of \( h(u) \), then

\[ \int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}} (B_{k+1} u^{\alpha-\beta-2})^{\frac{q}{q-1}} du = \int_{\frac{\pi}{k+1}}^{\frac{\pi}{k}} h(u) du \]

\[ = H\left(\frac{\pi}{k}\right) - H\left(\frac{\pi}{k+1}\right) \]

\[ = \frac{\pi}{k} - \frac{\pi}{k+1} h(c) \quad \text{for some} \quad \frac{\pi}{k+1} < c < \frac{\pi}{k} \]

\[ = O\left(1\right) \left( \frac{1}{k^2} \right)^{\frac{q}{q-1}} \]

\[ = O\left( \frac{B_{k+1}}{k^{\alpha-\beta-\frac{2}{q}}} \right)^{\frac{q}{q-1}} \]

\[ I_2' = O\left( \frac{1}{B_n} \right) \left\{ \sum_{k=1}^{\infty} \left( \frac{B_{k+1}}{k^{\alpha-\beta-\frac{2}{q}}} \right)^{\frac{q}{q-1}} \right\}^{1-\frac{1}{q}} \quad (2.15) \]

From (2.13), (2.14) and (2.15), we have
\[ I' = O\left( \frac{1}{B_n} \right) \left\{ \sum_{n=1}^{\infty} \left( \frac{B_{n+1} - \frac{1}{q}}{k} \right)^{\frac{1}{q}} \right\}^{\frac{1}{q}} \quad (2.16) \]

\[ J' = \left\{ \int_0^1 \left( K_n(u) \left| u^{\alpha - \frac{1}{q}} \right| \right)^{\frac{1}{q}} du \right\}^{\frac{1}{q}} \]

\[ = \left\{ \left( \sum_{n=1}^{n} \left( K_n(u) \left| u^{\alpha - \frac{1}{q}} \right| \right)^{\frac{1}{q}} du \right) \right\}^{\frac{1}{q}} \]

\[ \leq \left\{ \left( \sum_{n=1}^{n} \left( K_n(u) \left| u^{\alpha - \frac{1}{q}} \right| \right)^{\frac{1}{q}} du \right) \right\}^{\frac{1}{q}} + \left\{ \int_0^1 \left( K_n(u) \left| u^{\alpha - \frac{1}{q}} \right| \right)^{\frac{1}{q}} du \right\}^{\frac{1}{q}} \]

\[ = (J_1^1 + J_1^2) \quad (say) \quad (2.17) \]

Applying Lemma 2.4 in \( J_1^1 \) and \( J_1^2 \), we have

\[ J_1^1 = \left\{ \int_0^1 \left( K_n(u) \left| u^{\alpha - \frac{1}{q}} \right| \right)^{\frac{1}{q}} du \right\}^{\frac{1}{q}} \]

\[ = O(n) \left\{ \int_0^1 u^{\frac{1}{q} - \frac{1}{2} n} du \right\}^{\frac{1}{q}} \]

\[ = O(n) \left\{ \int_0^1 u^{\frac{1}{q} - \frac{1}{2} n} du \right\}^{\frac{1}{q}} \]

\[ = O\left( \frac{1}{n^{\alpha - \beta}} \right) \quad (2.18) \]

\[ J_1^2 = \left\{ \int_0^1 \left( K_n(u) \left| u^{\alpha - \frac{1}{q}} \right| \right)^{\frac{1}{q}} du \right\}^{\frac{1}{q}} \]

\[ = O\left( \frac{1}{B_n} \right) \left\{ \int_0^1 \left( B_n u^{\alpha - \frac{1}{q}} \right)^{\frac{1}{q}} du \right\}^{\frac{1}{q}} \]
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\[ = O\left(\frac{1}{B_n}\right) \left\{ \sum_{k=1}^{n} \left( \frac{B_{k+1}}{k} \right)^{\frac{q}{q-1}} \right\}^{1 - \frac{1}{q}} \]

From (2.17), (2.18), (2.19), we have

\[ J_1 = O\left(\frac{1}{B_n}\right) \left\{ \sum_{k=1}^{n} \left( \frac{B_{k+1}}{k} \right)^{\frac{q}{q-1}} \right\}^{1 - \frac{1}{q}} \]

From (2.12), (2.16) and (2.20), we have

\[ \| w_k (T_n) \|_{\beta,q} = O\left(\frac{1}{B_n}\right) \left\{ \sum_{k=1}^{n} \left( \frac{B_{k+1}}{k} \right)^{\frac{q}{q-1}} \right\}^{1 - \frac{1}{q}} \]

From (2.21), (2.11) and (2.7), for \( p \geq 1, 1 < q < \infty \) and \( 0 \leq \beta < \alpha < 2 \), we have

\[ \| T_n (\cdot) \|_{B_q^\beta (L_p)} = \| T_n (\cdot) \|_{p} + \| w_k (T_n, \cdot) \|_{\beta,q} \]

\[ = O\left(\frac{1}{B_n}\right) \left\{ \sum_{k=1}^{n} \left( \frac{B_{k+1}}{k} \right)^{\frac{q}{q-1}} \right\}^{1 - \frac{1}{q}} \]

Case 2: Now, we consider the case \( q = \infty \)

\[ \| T_n (\cdot) \|_{B_q^\beta (L_p)} = \| T_n (\cdot) \|_{p} + \| w_k (T_n, \cdot) \|_{\beta,\infty} \]

We know \( T_n(x) = \frac{1}{\pi} \int_0^\infty \phi_n(u)K_n(u)du \)

Applying Lemma 2.1(iii), we have

\[ \| T_n (\cdot) \|_{p} \leq \frac{1}{\pi} \int_0^\infty \| \phi_n (u) \|_{p} K_n(u)du \]

\[ \leq \frac{2}{\pi} \int_0^\infty K_n(u) \left| w_k (f, u) \right|_p du \]

\[ = O(1) \int_0^\infty K_n(u) |u|^\alpha du \quad (by \ the \ hypothesis) \]
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\[ I^{\Pi} = \int_{0}^{\pi} |K_n(u)| u^d du = O(n^{1/2}) \] (by lemma 2.4) \hfill (2.5)

\[ J^{\Pi} = \int_{n}^{\infty} |K_n(u)| u^d du \]

From (2.24), (2.25) and (2.26), we have

\[ \|T_n(\cdot)\|_p = O\left(\frac{1}{B_n} \sum_{k=1}^{n} \frac{B_{k+1}}{k^\alpha} \right) \] \hfill (2.7)

Again,

\[ \|w_n(T_n, \cdot)\|_{\beta, \alpha} = O(1) \left[ \int_{0}^{\pi} u^{\alpha-\beta} |K_n(u)| du + \sum_{n}^{\infty} u^{\alpha-\beta} |K_n(u)| du \right] \]

Using Lemma 2.4 in \( I^{\Pi} \) and \( J^{\Pi} \), we have

\[ I^{\Pi} = \int_{0}^{\pi} |K_n(u)| u^{\alpha-\beta} du \]

\[ J^{\Pi} = \int_{n}^{\infty} u^{\alpha-\beta} |K_n(u)| du \]

From (2.30) and (2.31), we have

\[ \|w_n(T_n, \cdot)\|_{\beta, \alpha} = O\left(\frac{1}{B_n} \sum_{k=1}^{n} \frac{B_{k+1}}{k^{\alpha-\beta}} \right) \] \hfill (2.31)
From (2.23), (2.27) and (2.31), we have

\[
\| T_n (\cdot) \|_{\mathcal{B}^\alpha_p (t_p)} = O \left( \frac{1}{B_n} \sum_{k=1}^{n} \frac{B_{k+1}}{k^{\alpha - \beta}} \right)
\]  

(2.32)

Hence the Theorem follows from (2.22) and (2.32).

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**References**


