On Pseudo m – power Commutative near – rings

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Abstract: A near ring N is called weak commutative if \( xyz = zyx \) for every \( x,y,z \in N \). N is called pseudo commutative if \( xyz = zyx \) for every \( x,y \in N \). N is called quasi weak commutative if \( xyz = yxz \) for every \( x,y,z \in N \). N is called pseudo m – power commutative if \( x^m yz = zy^m x \) for every \( x,y,z \in N \). We obtain more results, generalising the results of [15].

I. Introduction

S.Uma, R.Balakrishnan and T.Tammizhchelvan[15] called a near ring N to be pseudo commutative if \( xyz = zyx \) for every \( x,y,z \in N \). G.GopalaKrishnamoorthy and S.Geetha[8] called a ring R to be \( m \)-power commutative if \( x^m y = y^m x \) for all \( x,y \in R \), where \( m \geq 1 \) is a fixed integer . They also called a ring R to be \( (m,n) \)-power commutative if \( x^m y^n = y^n x^m \) for all \( x,y \in R \), where \( m \geq 1 \) and \( n \geq 1 \) are fixed integers. We have defined a near ring to be pseudo \( m \)-power commutative if \( x^m yz = zy^m x \) for all \( x,y,z \in N \), where \( m \geq 1 \) is a fixed integer. In this paper we prove more general results on pseudo \( m \) – power commutative near rings, this generalising the results of [15].

II. Preliminaries

Throughout this paper, \( N \) denotes a right near ring with atleast two elements. For any non – empty subset \( A \) of \( N \), we denote \( A^{-\{0\}} \) as \( A^* \). The following definitions and results are needed for the development of this paper.

2.1 Definition
Let \( N \) be a near ring. An element \( a \in N \) is said to be idempotent if \( a^2 = a \). Nilpotent if there exists a positive integer \( k \) such that \( a^k = 0 \)

2.2 Lemma (Pilz [14])
Each near ring \( N \) is isomorphic to a sub direct product of subdirectly irreducible near rings

2.3 Definition
A near ring \( N \) is said to be zero symmetric if \( ab = 0 \) implies \( ba = 0 \), where \( a,b \in N \).

2.4 Lemma
If \( N \) is zero symmetric, then every left ideal \( A \) of \( N \) is a \( N \)-subgroup of \( N \)

2.5 Lemma
Let \( N \) be a regular near ring, \( a \in N \) and \( a = axa \), then
(i) \( ax \) and \( xa \) are idempotents and so the set of idempotent elements of \( N \) is non – empty.
(ii) \( axN = aN \) and \( Nxa = Na \)

2.6 Lemma
Let \( N \) be a regular near ring, \( a \in N \) and \( a = axa \), then
(i) \( ax \) and \( xa \) are idempotents and so the set of idempotent elements of \( N \) is non – empty.
(ii) \( axN = aN \) and \( Nxa = Na \)

2.7 Definition
A near ring \( N \) is said to be reduced if \( N \) has no non – zero nilpotent elements

2.8 Lemma [3]
Let \( N \) be a zero – symmetric reduced near – ring. For any \( a,b \in N \) and for any idempotent element \( e \in N \), \( abe = aeb \)
2.9 Lemma [5, 6]
A near ring $N$ is sub–directly irreducible if and only if the intersection of all non–zero ideals of $N$ is not zero.

2.10 Lemma [6]
Each simple near–ring is sub–directly irreducible.

2.11 Lemma [13]
An $N$–subgroup $A$ of $N$ is essential if $A \cap B = \{0\}$ where $B$ is any $N$ subgroup of $N$ implies $B = \{0\}$.

2.12 Definition
A near–ring $N$ is said to be an integral near–ring if $N$ has no non–zero divisors.

2.13 Lemma
Let $N$ be a near–ring such that for all $a \in N$, $a^2 = 0$ implies $a = 0$. Then $N$ has no non–zero nilpotent elements. That is, $N$ is reduced.

2.14 Definition
A near ring $N$ is said to satisfy intersection of factors property (I F P) if $ab = 0$ implies $anb = 0$ for all $n \in N$, where $a, b \in N$.

2.15 Lemma [14]
A non–zero symmetric near–ring has intersection of factors property if and only if $(O:S)$ is an ideal for any subset $S$ of $N$.

2.16 Definition
(i) Let $N$ be a near–ring. An ideal $I$ of $N$ is called a prime ideal if for all ideals $A, B$ of $N$, $AB \subseteq I A \subseteq I$ or $B \subseteq I$.
(ii) $I$ is called a semi–prime ideal if for all ideals $A$ of $N$, $A^2 \subseteq I$ implies $A \subseteq I$.
(iii) $I$ is called a completely semi–prime ideal if for any $x \in N$, $x^2 \in I$ implies $x \in I$.
(iv) $I$ is called a completely prime ideal if for any $x, y \in N$, $xy \in I$ implies $x \in I$ or $y \in I$.
(v) $N$ is said to have strong intersection of factors property if for all ideals $I$ of $N$, $ab \in I$ implies $anb \in I$ for all $n \in N$.

2.17 Lemma
Let $N$ be a Pseudo Commutative near–ring. Then every idempotent element is central.

III. Main results

3.1 Lemma
Every pseudo $m$–power commutative (right) near–ring is zero symmetric.

Proof
Let $N$ be a pseudo $m$–power commutative near–ring. Then $x^{m}yz = zy^{m}x$ for all $x, y, z \in N$.

Now for all $a \in N$,
\[
a.0 = a.0^{m+1} = a.0^{m}.0 = 0^{m}.0a = 0a = 0
\]
This proves $N$ is zero symmetric.

3.2 Lemma
Every idempotent element in a pseudo $m$–power commutative near–ring is central.

Proof
Let $N$ be a pseudo $m$–power commutative near–ring and $e \in N$ be an idempotent element. Then it follows that $e^{k} = e$ for all $k \geq 2$.

Now for any $a \in N$,
\[
e a = e^{m+1}a = e^{m}.e.a
\]
\[
= a e^{m} e = ae^{m+1} = ae
\]
This proves $e$ is central.

3.3 Lemma
Homomorphic image of a pseudo $m$–power commutative near–ring is also a pseudo $m$–power commutative near–ring.

Proof
Let $N$ be a pseudo $m$–power commutative near–ring. Let $f : N \rightarrow M$ be an endomorphism of near–rings. For all $x, y, z \in N$,
\[
f(x)^{m}f(y)f(z) = f(x^{m}yz)
\]
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This proves M is pseudo m – power commutative.

3.4 Corollary
Let N be a pseudo m – power commutative near – ring. If I is an ideal of N, then N / I is also pseudo m – power commutative.

Proof
Since the canonical map \( \eta : N \rightarrow N/I \) is an endomorphism of near – rings, the corollary follows from the Lemma.

3.5 Theorem
Every pseudo m – power commutative near – ring N is isomorphic to a sub – directly product of sub – directly irreducible pseudo m – power commutative rings

Proof
By Lemma2.2, N is isomorphic to a subdirect product of subdirectly irreducible near – rings \( N_k \) and each \( N_k \) is a homomorphic image of N under the projection map \( \pi_k : N \rightarrow N_k \). The result follows from Lemma 3.4

3.6 Definition
Let N be a near – ring. N is said to be weak m – power commutative if \( ab^m c = ac^m b \) for all \( a, b, c \in N \)

3.7 Lemma
Any pseudo – m – power commutative near – ring with right identity is weak m – power commutative

Proof
Let N be a pseudo m – power commutative near – ring. Let \( a, b, c \in N \)
Now, \( ab^m c = (ab^m)c \)
\( = (a(b^m)c) \) (N is pseudo m – power commutative)
\( = (ac)(b^m) \)
\( ab^m c = ac^m b \)
This proves N is weak m – power commutative

3.8 Definition
Let N be a near – ring. N is said to be quasi – weak m – power commutative if \( x^m yz = y^m xz \) for all \( x, y, z \in N \)

3.9 Lemma
Any weak m – power commutative near – ring with left identity is quasi – weak m – power commutative

Proof
Let N be a weak m – power commutative near – ring. Let \( a, b, c \in N \)
Now \( a^m bc = e(a^m bc) \)
\( = (ea^m b)c \)
\( = (eb^m a)c \)
\( = b^m ac \)
This proves N is quasi weak m – power commutative.

3.9 Definition
A near – ring N is said to be m – regular near – ring if for each \( a \in N \), where exists an element \( b \in N \) such that \( a = ab^m a \) where \( m \geq 1 \) is a fixed integer.

3.10 Lemma
Let N be a m – regular near – ring, \( a \in N \) and \( a = ab^m a \).
Then (i) \( ab^m, b^m a \) are idempotents
(ii) \( ab^m N = aN \) and \( Nb^m a = Na \)

Proof
(i) Let \( a \in N \). Since N is m – regular, there exists \( b \in N \) such that
\( a = ab^m a \) ...........(1)
Now \( (ab^m)^2 = (ab^m)(ab^m) \)
\( = (ab^m)a b^m \)
\( = ab^m \)
Similarly, \( (b^m a)^2 = (b^m a)(b^m a) = b^m (ab^m a) \)
\( = b^m a \)
Hence \( ab^m \) and \( b^m a \) are idempotents.
(ii) Let \( y \in ab^m N \)
\( \Rightarrow y = ab^m x \) for some \( x \in N \)
\[ \begin{align*}
\in aN \\
\Rightarrow ab^nN \subseteq aN \\
\text{Let } y \in aN \\
y = az \text{ for some } z \in N \\
= (ab^m a)z \\
= ab^m (az) \\
\in ab^nN \\
\text{That is, } aN \subseteq ab^nN \\
\text{Hence } ab^nN = aN \\
\text{Similarly it can be proved } N b^n a = N a
\end{align*} \]

### 3.11 Definition

Let \( N \) be a near-ring \( A \subseteq N \) then \( \sqrt{A} = \{ x \in N / x^k \in A \text{ for some } k \geq 1 \} \)

### 3.12 Theorem

Let \( N \) be a \( m \)-regular pseudo \( m \) power commutative near-ring.

\[ \text{Then } A = \sqrt{A} \text{ for every } N \text{-subgroup } A \text{ of } N. \]

**Proof**

Let \( A \) be an \( N \)-subgroup of \( N \).

Since \( N \) is \( m \)-regular for every \( a \in N \), there exists \( b \in N \) such that \( a = ab^m a \).

By Lemma 3.10(i), \( ab^m, b^m a \) are idempotents.

Since \( N \) is pseudo \( m \)-power commutative by Lemma 3.2, \( ab^m, b^m a \) are central.

Let \( a \in \sqrt{A} \). Then \( a^k \in A \) for some positive integer \( k \).

Now \( a = a b^m a = a (b^m a) \)
\[ a = b^m a = b^m (b^m a^2) a \]
\[ = b^{2m} a^3 \]
\[ = b^{3m} a^4 \]
\[ = b^{4m} a^5 \]
\[ \ldots \]

\[ \therefore a = b^{(k-1)m} a^k \in NA \subseteq A \text{ for all } k \geq 1 \]

Hence \( \sqrt{A} \subseteq A \)

Obviously \( A \subseteq \sqrt{A} \)

Hence \( A = \sqrt{A} \)

### 3.13 Theorem

Let \( N \) be a \( m \)-regular pseudo \( m \)-power commutative near-ring. Then

(i) \( N \) is reduced

(ii) \( N \) has IFP (A \( m \)-regular near-ring is said to have IFP if \( ab = 0 \) implies there exists \( n \in N \) such that \( an^m b = 0 \))

**Proof**

Let \( a \in N \) be such that \( a^2 = 0 \). By (i) of Theorem 3.12, \( a = b^m a^2 = b^m \cdot 0 = 0 \)

Hence \( N \) is reduced.

Let \( x, y \in N \) such that \( xy = 0 \)

Now \( (yx)^2 = (yx) (yx) = y (xy) x \)
\[ = y \cdot 0 \cdot x \]
\[ = y \cdot 0 \]
\[ = (yx)^2 = 0 \]

By (i) \( yx = 0 \)

That is, \( N \) is zero commutative.

Now for any \( n \in N \), \( (xn^m y)^2 = xn^m y \cdot xn^m y \)
\[ = xn^m (y x) n^m y \]
By (i) \( x^n y = 0 \)

### Theorem 3.14

Let \( N \) be a \( m \)-regular pseudo \( m \)-power commutative near-ring. Then every \( N \) subgroup is an ideal.

**Proof**

Let \( a \in N \). Since \( N \) is \( m \)-regular, there exists \( b \in N \) such that \( a = ab^m a \).

By Lemma 3.10(i) \( bma \) is idempotent

Let \( b^m a = e \)

Then \( Ne = Nb^m a = Na \) (by Lemma 3.10 (ii))

Let \( S = \{ n-ne \in N / n \in N \} \)

Claim: \( (O : S) = \{ y \in N / sy = 0 \} \forall s \in S = Ne \)

Now \( n-ne = ne-ne^2 = ne-ne = 0 \) \( \forall neN \)

Since \( N \) has IFP, we have

\( (n-ne)Ne = 0 \)

Hence \( Ne \subseteq (O : S) \) ..........(1)

Let \( y \in (O : S) \). Then \( sy = 0 \) \( \forall s \in S \) ..........(2)

Now \( N \) is \( m \)-regular. \( y = x^n y \) for some \( x \in N \)

Since \( xy^n - (yx^n)e \) is an ideal, by (2) we get

\( (xy^n - (yx^n)e) y = 0 \)

That is, \( xy^n y - x^n ey = 0 \)

\( y - y(x^n ey) = 0 \) ..........(3)

Since \( N \) is zero symmetric reduced ring by Lemma 2.8, \( x^n ey = x^n ye \)

So, (3) becomes \( y - y(x^n ye) = 0 \)

\( y = y(\text{ye}) = 0 \)

Hence \( y = ye \) is an ideal of \( N \).

Thus \( M \) becomes an ideal of \( N \).

### Theorem 3.15

Let \( N \) be a \( m \)-regular pseudo \( m \)-power commutative near-ring. Then (i) \( N = Na = Na^2 = aN = aNa \) for all \( a \in N \)

(ii) Any ideal of \( N \) is completely semi prime

**Proof**

Since \( N \) is \( m \)-regular, for every \( a \in N \), there exists \( b \in N \) such that \( a = ab^m a \)

Then \( a = ab^m a = (ab)^m a = abm(a) = b^m a \) (by Lemma 3.10 (i))

Also \( a = ab^m a = (b^m a) a = b^m a^2 \in Na^2 \)

Hence \( N \subseteq Na^2 \) ..........(1)

Now \( Na \subseteq N \subseteq Na^2 = (Na)a \subseteq Na \subseteq N \)

So, \( Na = Na^2 = N \) ..........(2)

We shall now prove that \( Na^2 = aN \)

Let \( x \in Na^2 \).

Then \( x = na^2 \) for some \( n \in N \)

\[ = n(a^m a^2) a \]

\[ = nb^m a^2 \]

\[ = (a^m bn) a^2 \text{ (pseudo } m \text{- power commutative)} \]

\[ = a(a^m bna) aN \]
That is, \( N\alpha^2 \subseteq a\mathbb{N} \) ...................(3)

Let \( y \in a\mathbb{N} \).
Then \( y = an \) for some \( n \in \mathbb{N} \)
\( = b^m\alpha(a^n\beta)^m \)
\( = b^m\alpha(a^n\beta)^m \)
\( = b^m(a^n\beta^m) \) (pseudo \( m \)-power commutative)
\( = (b^m\alpha)b^m \alpha^2 \in Na^2 \)

So \( a\mathbb{N} \subseteq Na^2 \) .......................(4)

(3) and (4) gives \( Na^2 = a\mathbb{N} \) ..................(5)

Next we shall prove that \( a\mathbb{N} = aN \mathbb{A} \)
Let \( x \in a\mathbb{N} \).
Then \( x = an \) for some \( n \in \mathbb{N} \)
\( = (ab)^m(a^n) \)
\( = a(b^m(a^n) \alpha a(\alpha N)) \subseteq a\mathbb{N} \)

So, \( a\mathbb{N} \subseteq a\mathbb{N} \) ..........................(6)

Obviously \( a\mathbb{N} \subseteq a\mathbb{N} \) ..........................(6)

Hence \( a\mathbb{N} = Na \) ..............................(7)

From (2), (5) and (7) we get
\( N = Na = Na^2 = a\mathbb{N} = a\mathbb{N} \)

Let \( I \) be any ideal of \( N \) and \( a^2 \in I \)
Now \( a = a^2 b^m \in I \)
That is, \( a^2 \in I \) implies \( a \in I \)
Hence \( I \) is Completely semi – prime.

3.16 Definition
A near – ring \( N \) is said to have the property \( P_4 \) if for all ideals \( I \) of \( N \),
\( xy \in I \) implies \( yx \in I \), where \( x,y \in N \)

3.17 Theorem
Every \( m \)-regular pseudo \( m \)-power Commutative near – ring satisfies the property \( P_4 \)

Proof
Let \( N \) be a \( m \)-regular pseudo \( m \)-power Commutative near – ring and \( I \) be an ideal of \( N \). Let \( a,b \in N \) such that \( ab \in I \)
Then \( (ba)^2 = (ba)(ba) \)
\( = b(ab)a \)
\( \in N I N \subseteq I \)

That is, \( (ba)^2 \in I \)
By Theorem 3.15 (ii), \( ba \in I \)
Thus \( N \) satisfies the property \( P_4 \)

3.18 Theorem
Let \( N \) be a \( m \)-regular pseudo \( m \)-power Commutative near – ring. Then (i) For every ideal \( I \) of \( N \), \( (I:S) \) is an ideal of \( N \), where \( S \) is any subset of \( N \)
(ii) For every ideal \( I \) of \( N \), \( x_1,x_2,x_3,...,x_n \in I \) if \( x_1,x_2,x_3,...,x_n \in I \), then \( <x_1>, <x_2>, <x_3>,..., <x_n> \subseteq I \).

Proof
Let \( I \) be an ideal of \( N \) and \( S \) be any subset of \( N \).
By Lemma 2.5, \( (I:S) = \{ n \in N/ ns \subseteq I \} \) is a left ideal of \( N \).
If \( a \in I \), then \( aS \subseteq I \). So, as \( I \) is any ideal of \( S \).
Then by Theorem 3.16, \( aS \subseteq I \). Then for any \( n \in N \), \( (sa)n \in I \).
That is, \( s(an) \in I \). By Theorem 3.17, \( (an)s \in I \). So an \( \in I \) for any \( n \in I \).
Hence \( (I:S) \) is a right ideal. Consequently \( (I:S) \) is an ideal. This completes the proof 3.17 (i).

Let \( x_1,x_2,x_3,...,x_n \in I \)
\( \Rightarrow x_1 \in (I : x_2,x_3,...,x_n) \)
\( \Rightarrow <x_1> \subseteq (I : x_2,x_3,...,x_n) \)
\( \Rightarrow <x_1> \subseteq (I : x_3,...,x_n) \subseteq I \)
3.19 Theorem
Let N be a m – regular pseudo m – power Commutative near – ring.

Then (i) N has strong IFP
(ii) N is a semi – prime near –ring

Proof
Let I be an ideal of N such that ab ∈ I, where a,b ∈ N. By Lemma 3.1, N is zero symmetric NI ⊆ I.

By Theorem 3.15 aN = Na.

Hence an = ma^n for some m,n ∈ N
Then any n ∈ N, anb = ma^n b

That is, N has strong IFP

Let M be an N – subgroup of N. Then by Theorem 3.14, M is an ideal of such that I^2 ⊆ M.

Since N is zero symmetric, NI ⊆ I.

If a ∈ I, then a = ab^ma ∈ I(NI) ⊆ I^2 ⊆ M.

So, any N – subgroup M of N is a semi – prime ideal. In particular {0} is semi – prime ideal and hence N is a semi – prime near – ring.

3.20 Note
When m = 1, all the results of [15] are obtained.

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