Solution of aVariational inequality Problem for Accretive Operators in Banach Spaces

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Abstract: This paper introduces a two-step iterative process for finding a solution of a variational inequality problem for accretive operators in Banach spaces. The result obtained in this paper is motivated by the result given by Koji Aoyama et al [3]. Further, we consider the problem of finding a fixed point of a strictly pseudocontractive mapping in a Banach space.

Keywords: Accretive operators, sunny non-expansive retractions, Banach spaces, variational inequality problem.

I. Introduction

Let E be any smooth Banach space with || ||. Let E* denote the dual of E and < x, f > denote the value of f E at x E. Let C be a nonempty closed convex subset of E and let A be an accretive operator of C into E. The generalized variational inequality problem in Banach space is to find an element u E C such that < Au, J(v - u)> ≥ 0 ∀ v E C, where J is the duality mapping of E into E*.

Definition 1.1 A Banach space E is called uniformly convex iff for any ε, 0 < ε ≤ 2, the inequalities || x || ≤ 1, || y || ≤ 1 and || x - y || ≥ ε imply there exists a δ > 0 such that || x + y ||/2 ≤ 1 - δ.

Definition 1.2 Let E be any smooth Banach space. Then a function ρ_E: R+ → R+ is said to be modulus of smoothness of E if ρ_E(t) = sup \{|| x + y || + || x - y || - 1; || x || = 1, || y || = t \}.

Definition 1.3 A Banach space E is said to be uniformly smooth if

\[ \lim_{t \to 0} \rho_E(t) = 0 \]

Remark 1.4 Let q > 1. A Banach space E is said to be q-uniformly smooth if there exists a fixed constant c > 0 such that ρ_E(t) = ct^q for all t > 0. For more details, see [4, 11]. It is obvious that if E is q-uniformly smooth, then q ≤ 2 and E is uniformly smooth.

Definition 1.5 Let E be any mapping from E into E* satisfying J(x) = { f E* : < x, f > = || x ||^2 and || f || = || x || }. Then J is called the normalized duality mapping of E.

Definition 1.6 Let C be a non-empty subset of a Banach space E. A mapping T : C → C is called nonexpansive [10] if || Tx - Ty || ≤ || x - y || ∀ x, y E C. T is called η-strictly pseudo-contractive if there exists a constant η E (0, 1) such that

\[ < Tx - Ty, j(x - y) > \geq || x - y ||^2 - η || (1 - T)x - (1 - T)y ||^2 \] (1.1)

for every x, y E C and for some j(x - y) E J(x - y).

It is obvious that (1.1) is equivalent to

\[ < (1 - T)x - (1 - T)y, j(x - y) > \geq η || (1 - T)x - (1 - T)y ||^2 \] (1.2)

Definition 1.7 A Banach space E is said to be smooth if the limit

\[ \lim_{t \to 0} \frac{|| x + ty || - || x ||}{t} \]

exists for all x, y E U, where U = \{ x E E : || x || = 1 \}.

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Remark 1.8 It is known that \( J_q(x) = \|x\|^{q-2} J(x) \) for all \( x \in E \). If \( E \) is a Hilbert space, then \( J = I \). The normalized duality mapping \( J \) has the following properties:

1. If \( E \) is smooth, then \( J \) is single valued.
2. If \( E \) is strictly convex, then \( J \) is one-one and

\[ < x - y, x' - y' > > 0 \] for all \( (x, x'), (y, y') \in J \) with \( x \neq y \).
3. If \( E \) is reflexive, then \( J \) is surjective.
4. If \( E \) is uniformly smooth, then \( J \) is uniformly norm to norm continuous on each bounded subset of \( E \).
5. It is also known that \( q < y - x, j(x) \leq \|y\|^q - \|x\|^q \) for all \( x, y \in E \) and \( j \in J(x) \).


Theorem 1.9 [3] Let \( E \) be a uniformly convex and \( 2 \)-uniformly smooth Banach space with best smooth constant \( K \) and \( C \) be a nonempty closed convex subset of \( E \). Let \( Q \) be a sunny nonexpansive retraction from \( E \) onto \( C \), \( \alpha > 0 \) and \( A \) be \( \alpha \)-inverse strongly accretive operator of \( C \) into \( E \). Let \( S(C, A) \neq \emptyset \) and the sequence \( \{x_n\} \) be generated by

\[ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Q_C(x_n - \lambda_n A x_n), \quad x_1 \in C, \quad n = 1, 2, 3, \ldots \]

where \( \{\alpha_n\} \) is a sequence of positive real numbers and \( \{\lambda_n\} \) is a sequence in \([0, 1]\) and \( \lambda_n \in [a, \alpha/K^2] \) for some \( a > 0 \) and let \( \alpha_n \in [b, c] \), where \( 0 < b < c < 1 \), then \( \{x_n\} \) converges weakly to some element \( z \in S(C, A) \).

After that for finding a common element of \( F(S) \cap VI(C, A) \), Nadezhkina and Takahashi [5] gave another result. They obtained the following weak convergence theorem:

Theorem 1.2 [5] Let \( C \) be a closed convex subset of a real Hilbert space \( H \). Let \( A \) be a monotone and \( k \)-Lipschitz continuous mapping from \( C \) to \( H \) and let \( S \) be a nonexpansive mapping of \( C \) into itself such that \( F(S) \cap VI(C, A) \neq \emptyset \). Let \( \{x_n\}, \{y_n\} \) be sequences generated by

\[ y_n = P_C(x_n - \lambda_n A y_n), \quad n \geq 0, \]

\[ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S_C(x_n - \lambda_n A x_n), \quad x_0 \in C \]

where \( \{\alpha_n\} \subset [a, b] \) for some \( a, b \in (0, 1/k) \) and \( \{\lambda_n\} \subset [c, d] \) for some \( c, d \in (0, 1) \). Then the sequences \( \{x_n\}, \{y_n\} \) generated by (1.3) converge weakly to some \( z \in F(S) \cap VI(C, A) \).

Motivated by above results, we provide the following iterative process for an accretive operator \( A \) in a Banach space \( E \),

\[ x_1 = x \in C, \]

\[ y_n = Q_C(x_n - \lambda_n A x_n), \quad n = 1, 2, 3, \ldots \]

where \( Q_C \) is a sunny nonexpansive retraction from \( E \) onto \( C \). Using this iterative process, we shall obtain a weak convergence theorem.

II. Preliminaries

Let \( D \) be a subset of \( C \) and \( Q \) be a mapping from \( C \) to \( D \). Then \( Q \) is said to be sunny if \( Q(Qx + t(x - Qx)) = Qx \), whenever \( Qx + t(x - Qx) \in C \) for \( x \in C \) and \( t \geq 0 \). A mapping \( Q : C \to C \) is called retraction if \( Q^2 = Q \). If \( Q \) is any retraction, then \( Qz = z \) for every \( z \in R(Q) \), where \( R(Q) \) is the range set of \( Q \). A subset \( D \) of \( C \) is called a sunny nonexpansive retraction of \( C \) if it exists a sunny nonexpansive retraction from \( C \) onto \( D \).

Now we collect some results.

Lemma 2.1 [7] Let \( C \) be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space \( E \) and \( T \) be a nonexpansive mapping of \( C \) into itself with \( F(T) \neq \emptyset \). Then the set \( F(T) \) is a sunny nonexpansive retraction of \( C \).

Lemma 2.2 [6, 8] Let \( C \) be a nonempty closed convex subset of a smooth Banach space \( E \) and let \( Q_C \) be a retraction of \( E \) onto \( C \). Then the following are equivalent

(i) \( Q_C \) is both sunny and nonexpansive.
(ii) \( < x - x_0, J(y - Q_C x) > \leq 0 \) for all \( x \in E, y \in C \).

Also it is well known that if \( E \) is a Hilbert space, then sunny nonexpansive retraction is coincident with metric projection.

Also \( Q_C \) satisfies

\[ x_0 = Q_C x \text{ iff } < x - x_0, J(y - x_0) > \leq 0 \text{ for all } y \in C. \]

Let \( E \) be a Banach space and let \( C \) be a nonempty closed convex subset of \( E \). An operator \( A \) of \( C \) into \( E \) is said to be accretive if there exists \( j(x - y) \in J(x - y) \) such that

\[ < Ax - Ay, j(x - y) > \geq 0 \text{ for all } x, y \in C. \]
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**Lemma 2.3** [3] Let C be a nonempty closed convex subset of a smooth Banach space E. Let Qc be a sunny nonexpansive retraction from E onto C and let A be an accretive operator of C into E. Then for all \( \lambda > 0 \),
\[
S(C, A) = \{ u \in C : J(\lambda u - \lambda A u) = u \}
\]
where
\[
S(C, A) = \{ u \in C : J(\lambda u - \lambda A u) = u \}
\]
An operator \( A : C \rightarrow E \) is said to be \( \alpha \)-inverse strongly accretive if
\[
\langle Ax - Ay, J(x - y) \rangle \geq \alpha \| Ax - Ay \|^2
\]
for all \( x, y \in C \).

It is obvious from above equation that
\[
\| Ax - Ay \| \leq \frac{1}{\alpha} \| x - y \|.
\]

**Lemma 2.4** [3] Let C be a nonempty closed convex subset of a 2-uniformly smooth Banach space E. Let \( \alpha > 0 \) and let \( A : C \rightarrow E \) be an \( \alpha \)-inverse strongly accretive operator. If \( 0 < \lambda \leq \frac{\alpha}{K^2} \), then \( I - \lambda A \) is a nonexpansive mapping of C into E, where K is the 2-uniformly smoothness constant of E.

**Lemma 2.5** [9] Let C be a nonempty closed convex subset of a uniformly convex Banach space with a Frechet differentiable norm. Let \( \{ T_1, T_2, \ldots \} \) be a sequence of nonexpansive mappings of C into itself with
\[
\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset
\]
for all \( n \geq 1 \). Then the set
\[
\bigcap_{n=1}^{\infty} \bigcap_{m=n}^{\infty} F(T_m)
\]
consists of almost one point, where \( \overline{cD} \) D is the closure of the convex hull of D.

**Lemma 2.6** [2] Let q be a given real number with \( 1 < q \leq 2 \) and let E be a q-uniformly smooth Banach space. Then,
\[
\| x + y \|^q \leq \| x \|^q + q \langle x, y \rangle + 2 \| Kx \|^2,
\]
for all \( x, y \in E \), where \( J_q \) is the generalized duality mapping of E and K is the q-uniformly smoothness constant of E.

**Theorem 2.7** [1] Let D be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let T be a nonexpansive mapping of D into itself. If \( \{ u_n \} \) is a sequence of D such that \( u_n \rightarrow u_0 \) and let
\[
\lim_{n \to \infty} \| u_n - T u_n \| = 0,
\]
then \( u_0 \) is a fixed point of T.

### III. Main Result

In this section, we shall prove our main result.

**Theorem 3.1** Let E be a uniformly convex and 2-uniformly smooth Banach space with best smooth constant K and C be a nonempty closed convex subset of E. Let Qc be a sunny nonexpansive retraction from E onto C, \( \alpha > 0 \) and let \( A : C \rightarrow E \) be an \( \alpha \)-inverse strongly accretive operator of C into E. Let \( S(C, A) \neq \emptyset \) and the sequence \( \{ x_n \} \) be generated by
\[
y_n = Qc(y_n - \lambda_n A x_n),
\]
\[
x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Qc(y_n - \lambda_n A y_n), \quad x_n \in C, \quad n = 1, 2, 3, \ldots
\]
where \( \{ \alpha_n \} \) is a sequence of positive real numbers satisfying \( \lambda_n \leq \alpha \) and \( \lambda_n \in [a, \alpha/K^2] \) for some \( a > 0 \) and let \( \alpha_n \in [b, c] \), where \( 0 < b < c < 1 \), then \( \{ x_n \} \) converges weakly to some element z of \( S(C, A) \).

**Proof.** Let \( z_n = Qc(y_n - \lambda_n A y_n) \) for \( n = 1, 2, \ldots \). Let \( u \in S(C, A) \). Now,
\[
\| y_n - u \| \leq \| Qc(x_n - \lambda_n A x_n) - Qc(u - \lambda_n A u) \| (3.2)
\]
Also,
\[
\| z_n - u \| \leq \| Qc(y_n - \lambda_n A y_n) - Qc(u - \lambda_n A u) \| (3.3)
\]
Now, for every \( n = 1, 2, \ldots \),
\[
\| x_n - u \| \leq \| x_n - u \| + (1 - \alpha_n) \| z_n - u \|
\]
\[
\leq \alpha_n \| x_n - u \| + (1 - \alpha_n) \| z_n - u \|
\]
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Using (3.2) and (3.3), \( \|x_{n+1} - u\| \leq \|x_n - u\| \) (3.4)

(3.4) shows that \{ \|x_n - u\| \} is non-increasing sequence.

So, there exists \( \lim_{n \to \infty} \|x_n - u\| \) and hence \{ \( x_n \) \} is a bounded sequence. (3.2) and (3.3) shows that \{ \( y_n \), \( \{ Ax_n \} \) and \{ \( x_n \) \} are also bounded.

Next, we shall show that \( \lim_{n \to \infty} \|x_n - y_n\| = 0 \). Conversely, let \( \lim_{n \to \infty} \|x_n - y_n\| \neq 0 \). Then there exists \( \epsilon > 0 \) and a subsequence \{ \( x_{n_i} - y_{n_i} \) \} of \{ \( x_n - y_n \) \} such that \( \|x_{n_i} - y_{n_i}\| \geq \epsilon \) for each \( i = 1, 2, \ldots \). Since \( E \) is uniformly convex, so the function \( \| \cdot \| \) is uniformly convex on bounded convex subset \( B(0, \|x_1 - u\|) \), where \( B(0, \|x_1 - u\|) = \{ x \in E : \|x\| \leq \|x_1 - u\| \} \).

So, for any \( \epsilon \), there exists \( \delta > 0 \) such that \( \|x - y\| \geq \epsilon \) implies
\[
\|x + (1- \lambda)y\|^2 \\
\leq \lambda \|x\|^2 + (1- \lambda)\|y\|^2 - \lambda(1- \lambda)\delta,
\]

where \( x, y \in B(0, \|x_1 - u\|), \lambda \in (0, 1) \). So for \( i = 1, 2, \ldots \),
\[
\|x_{n_{i+1}} - u\|^2 = \|\alpha_{n_i}(x_{n_i} - u) + (1 - \alpha_{n_i})(z_{n_i} - u)\|^2 \\
\leq \alpha_{n_i}\|x_{n_i} - u\|^2 + (1 - \alpha_{n_i})\| y_{n_i} - u\|^2 - \alpha_{n_i}(1 - \alpha_{n_i})\delta \\
\leq \alpha_{n_i}\|x_{n_i} - u\|^2 + (1 - \alpha_{n_i})\| x_{n_i} - u\|^2 - \alpha_{n_i}(1 - \alpha_{n_i})\delta \\
\leq \| x_{n_i} - u\|^2 - \alpha_{n_i}(1 - \alpha_{n_i})\delta
\]

Therefore,
\[
0 < (1 - c)\delta \leq \alpha_{n_i}(1 - \alpha_{n_i})\delta \leq \|x_{n_i} - u\|^2 - \|x_{n_{i+1}} - u\|^2
\]

Since right hand side of inequality (3.5) converges to 0, so we get a contradiction.

Hence, \( \lim_{n \to \infty} \|x_n - y_n\| = 0 \) (3.6)

Now, since \( \{x_n\} \) is bounded, so there exists a subsequence \{ \( x_{n_j} \) \} of \( \{x_n\} \) that weakly converges to \( z \). Also \( \lambda_{n_i} \in [a, a/K^2] \), so \( \{ \lambda_{n_i} \} \) is bounded. Hence, there exists a subsequence \{ \( \lambda_{n_{i_j}} \) \} of \{ \( \lambda_{n_i} \) \} that weakly converges to \( \lambda_0 \in [a, a/K^2] \). Without loss of generality assume that \( \lambda_{n_i} \to \lambda_0 \). Since \( Q_c \) is nonexpansive, so
\[
y_{n_i} = Q_c(x_{n_i} - \lambda_{n_i}Ax_{n_i}) \text{ implies that}
\]
\[
\|Q_c(x_{n_i} - \lambda_{n_i}Ax_{n_i}) - x_n\| \\
\leq \|Q_c(x_{n_i} - \lambda_{n_i}Ax_{n_i}) - y_{n_i}\| + \|y_{n_i} - x_n\| \\
= \|Q_c(x_{n_i} - \lambda_{n_i}Ax_{n_i}) - Q_c(x_{n_i} - \lambda_{n_i}Ax_{n_i})\| + \|y_{n_i} - x_n\| \\
\leq |\lambda_0 - \lambda_{n_i}| \|Ax_{n_i}\| + \|y_{n_i} - x_n\| \\
\leq M |\lambda_0 - \lambda_{n_i}| + \|y_{n_i} - x_n\|
\]

where \( M = \sup \{ \|Ax_{n_i}\| : n = 1, 2, 3, \ldots \} \). Equation (3.6), (3.7) and convergence of \{ \( \lambda_{n_i} \) \} implies that
\[
\lim_{n \to \infty} \| Q_c(I - \lambda_{n_0}A)x_{n_i} - x_n \| = 0
\]

Also, \( Q_c(I - \lambda_{0}A) \) is nonexpansive, so (3.8), lemma 2.3 and theorem 2.7 implies \( z \in F(Q_c(I - \lambda_{0}A)) = S(C, A) \).

Lastly, we shall prove that \( \{x_n\} \) is convergent to some element of \( S(C, A) \). Let \( T_n = \alpha_{n_1}I + (1 - \alpha_{n_1})Q_c(I - \lambda_{n_0}A) \), for \( n = 1, 2, \ldots \) (3.9)

Then, \( x_{n+1} = T_nx_{n+1} \ldots \ldots \ldots \ldots T_1x \) and \( z \in \bigcap_{n=1}^{\infty} \text{co} \{x_m : m \geq n\} \). Also from lemma 2.4, \( T_n \) is nonexpansive mapping of \( C \) into itself. And from lemma 2.3, we have,
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\[ \bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{n=1}^{\infty} F(Q_c(I - \lambda_n A)) = S(C, A). \]

Using theorem (2.5), we obtain

\[ \bigcap_{n=1}^{\infty} \{x_m : m \geq n\} \bigcap S(C, A) = \{z\} \quad (3.10) \]

Hence, the sequence \( \{x_n\} \) is weakly convergent to some element of \( S(C, A) \).

IV. Application

Using our main result, we shall prove a result for strongly accretive operator.

Let \( C \) be a subset of a smooth Banach space \( E \). Let \( \alpha > 0 \). An operator \( A \) of \( C \) into \( E \) is said to be \( \alpha \)-strongly accretive if

\[ < Ax - Ay, J(x - y) > \geq \alpha \| x - y \|^2 \quad \text{for all } x, y \in C. \]

Let \( \beta > 0 \). An operator \( A \) of \( C \) into \( E \) is said to be \( \beta \)-Lipschitz continuous if

\[ \| Ax - Ay \| \leq \beta \| x - y \|, \quad \text{for all } x, y \in C. \]

**Theorem 4.1** Let \( E \) be a uniformly convex and 2-uniformly smooth Banach space with best smooth constant \( K \) and \( C \) be a nonempty closed convex subset of \( E \). Let \( Q_C \) be a sunny nonexpansive retraction from \( E \) onto \( C \), \( \alpha > 0 \), \( \beta > 0 \) and \( A \) be \( \alpha \)-strongly accretive operator and \( \beta \)-Lipschitz continuous operator of \( C \) into \( E \). Let \( S(C, A) \neq \emptyset \) and the sequence \( \{x_n\} \) be generated by

\[ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Q_C(y_n - \lambda_n A y_n), \quad x_n \in C, \quad n = 1, 2, 3, \ldots \]

where \( \{\lambda_n\} \) is a sequence of positive real numbers satisfying \( 0 \leq \lambda_n \leq 1 \) and \( \lambda_n \in [a, \alpha/K^2] \) for some \( a > 0 \) and let \( \alpha_n \in [b, c] \), where \( 0 < b < c < 1 \), then \( \{x_n\} \) converges weakly to a unique element \( z \) of \( S(C, A) \).

**Proof.** Since \( A \) is an \( \alpha \)-strongly accretive and \( \beta \)-Lipschitz continuous operator of \( C \) into \( E \), we have

\[ < Ax - Ay, J(x - y) > \geq \alpha \| x - y \|^2 \geq \frac{\alpha}{\beta^2} \| Ax - Ay \|^2, \quad \text{for all } x, y \in C. \]

So \( A \) is \( \frac{\alpha}{\beta^2} \)-inverse strongly accretive. Since \( A \) is strongly accretive and \( S(C, A) \neq \emptyset \), so the set \( S(C, A) \) consists of one point \( z \). Using theorem 3.1, \( \{x_n\} \) converges weakly to a unique element \( z \) of \( S(C, A) \).

**References**


