IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 12, Issue 5 Ver. I (Sep. - Oct.2016), PP 92-96
www.iosrjournals.org

Cubic fuzzy H-ideals in BF-Algebras

B. Satyanarayana[1], Esraa Mohammed Wannas[2] and U. Bindu Madhavi[3]
[1,2]Department of Mathematics, Acharya Nagarjuna University, Nagarjuna Nagar-522 510, Andhra Pradesh, India.
[3]Department of Mathematics, Krishna University P.G. Centre, Nuzvid-521 201, Krishna (Dt.), Andhra Pradesh, India.

Abstract: *In this paper, we introduce the notion of cubic fuzzy H-ideals in BF-algebras and prove some interesting properties.*

I. Introduction

Zadeh [8] has introduced the concept of fuzzy subsets in 1965. This concept has been widely adopted and applied to many disciplines. Zhan and Tan [11] introduced the notion of fuzzy H-ideals in BCK-algebras and Satyanarayana et.al ([5], [6]) studied intuitionistic fuzzy H-ideals in BCK-algebras. Jun. et.al. [2] introduced the notion of cubic sets. In this paper we introduce the notion of cubic fuzzy H-ideals in BF-algebras and investigate some of its properties.

Definition 1.1([4], [7]). A BF-algebra is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

(i) $x * x = 0$,

(ii) $x * 0 = x$,

(iii) $0 * (x * y) = (y * x)$ for all $x, y \in X$.

Definition 1.2. A subset I of a BF-algebra $(X, *, 0)$ is called an ideal of X, if for any $x, y \in X$

(1). $0 \in I$

(2). $x * y$ and $y \in I \Rightarrow x \in y$

Definition 1.3. [11] A non-empty subset I of X is called an H-ideal of X if

(1). $0 \in I$

(2). $x * (y * z) \in I$ and $y \in I \Rightarrow x * z \in I$

Since $x * 0 = x$, It is clear that every H-ideal is an ideal.

Definition 1.4.[11] A fuzzy subset μ in a BF-algebra X is called a fuzzy H- ideal of X if

(i) $\mu(0) \geq \mu(x)$

(ii) $\mu(x * z) \geq \min\{\mu(x * (y * z)), \mu(y)\}$, for all $x, y, z \in X$.

Since $x * 0 = x$, It is clear that every fuzzy H-ideal is an fuzzy ideal.

The determination of maximum and minimum between two real numbers is very simple, but it is not simple for two intervals. Biswas [1] described a method to find max/sup and min/inf between two intervals and set of intervals. By an interval number $\tilde{a}$ on $[0,1]$, we mean an interval $\left[a^-, a^+\right]$, where $0 \leq a^- \leq a^+ \leq 1$. The set of all closed subintervals of $[0,1]$ is denoted by $D[0,1]$. The interval $[a, a]$ is identified with the number $a \in [0,1]$.

For an interval numbering $\tilde{a}_i = \left[a_i^-, b_i^+\right] \in D[0,1], i \in I$. We define

$$\inf \tilde{a}_i = \left[\min_{i \in I} a_i^-, \min_{i \in I} b_i^+\right], \quad \sup \tilde{a}_i = \left[\max_{i \in I} a_i^-, \max_{i \in I} b_i^+\right]$$

And put

(i) $\tilde{a}_1 \wedge \tilde{a}_2 = \min(\tilde{a}_1, \tilde{a}_2) = \min\left(\left[a_1^-, b_1^+\right], \left[a_2^-, b_2^+\right]\right)$

$$= \left[\min\left\{ a_1^-, a_2^+ \right\}, \min\left\{ b_1^-, b_2^+ \right\} \right]$$

(ii) $\tilde{a}_1 \vee \tilde{a}_2 = \max(\tilde{a}_1, \tilde{a}_2) = \max\left(\left[a_1^-, b_1^+ \right], \left[a_2^-, b_2^+ \right] \right)$

$$= \left[\max\left\{ a_1^-, a_2^- \right\}, \max\left\{ b_1^+, b_2^+ \right\} \right]$$

(iii) $\tilde{a}_1 + \tilde{a}_2 = \left[a_1^- + a_2^- - a_1^- . a_2^-,\ b_1^+ + b_2^+ - b_1^+ . b_2^+ \right]$

(iv) $\tilde{a}_1 \leq \tilde{a}_2 \Leftrightarrow a_1^- \leq a_2^-$ and $b_1^+ \leq b_2^+$

(v) $\tilde{a}_1 = \tilde{a}_2 \Leftrightarrow a_1^- = a_2^-$ and $b_1^+ = b_2^+$,

(vi) $\tilde{a}_1 \leq \tilde{a}_2 \Leftrightarrow a_1^- \leq a_2^-$ and $b_1^+ \leq b_2^+$,

(vii) $mD = m[a_1^-, b_1^+] = [ma_1^-, mb_1^+]$, where $0 \leq m \leq 1$.

Obviously $(D[0,1], \leq, \vee, \wedge)$ forms a complete lattice with $[0, 0]$ as its least element and $[1,1]$ as its greatest element.

In [10], Zarandi and Borumand defined another type of fuzzy set called interval-valued fuzzy set (i-v FS). The membership value of an element of this set is not a single number, it is an interval and this interval is a sub-interval of the interval [0,1]. Let D [0, 1] be the set of a subintervals of the interval [0, 1].

The notion of interval-valued fuzzy set was first introduced by Zadeh as an extension of fuzzy set. An interval-valued fuzzy set is a fuzzy set whose membership function is many-valued and form an interval in the membership scale. This idea gives the simplest method to capture the impression of the membership grade for a fuzzy set.

Let X be a given nonempty set. An interval-valued fuzzy set (briefly, i-v fuzzy set) B on X is defined by $B = \{(x, [\mu_B^-(x), \mu_B^+(x)]) : x \in X\}$, Where $\mu_B^-(x)$ and $\mu_B^+(x)$ are fuzzy sets of X such that $\mu_B^-(x) \leq \mu_B^+(x)$ for all $x \in X$. Let $\tilde{\mu}_B(x) = [\mu_B^-(x), \mu_B^+(x)]$, then $B = \{(x, \tilde{\mu}_B(x)) : x \in X\}$, Where $\tilde{\mu}_B : X \to D[0,1]$.

Definition 1.5. Consider two elements $D_1, D_2 \in D[0,1]$. If $D_1 = [a_1^-, a_1^+]$ and $D_2 = [a_2^-, a_2^+]$, then $r\min(D_1, D_2) = [\min(a_1^-, a_2^-), \min(a_1^+, a_2^+)]$ which is denoted by $D_1 \wedge^r D_2$. Thus if $D_i = [a_i^-, a_i^+] \in D[0,1]$ for i=1,2,3,4….. then we define $r\sup_i(D_i) = [\sup(a_i^-), \sup(a_i^+)]$, i.e, $\vee_i^r D_i = [\vee_i a_i^-, \vee_i a_i^+]$. Now we call $D_1 \geq D_2$ if and only if $a_1^- \geq a_2^-$ and $a_1^+ \geq a_2^+$. Similarly, the relations $D_1 \leq D_2$ and $D_1 = D_2$ are defined.

Based on the (interval-valued fuzzy sets, Jun et al. [2] introduced the notion of cubic sets, and investigated several properties.

Definition 1.6. Let X be a non-empty set. A cubic set A in X is a Structure which is briefly denoted by $A = (\tilde{\mu}_A, \lambda_A)$ where $\tilde{\mu}_A = [\mu_A^-, \mu_A^+]$ is an interval valued fuzzy set in X and λ_A is fuzzy set in X.

Definition 1.7. Let $A = (\tilde{\mu}_A, \lambda_A)$ be cubic set in X, where X is BF subalgebra, then the set A is cubic BF subalgebra over the binary operation $*$ if it satisfies the following conditions

(i) $\tilde{\mu}_A(x * y) \geq r\min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$

(ii) $\lambda_A(x * y) \leq \max\{\lambda_A(x), \lambda_A(y)\}$.

Proposition 1.8. If $A = (\tilde{\mu}_A, \lambda_A)$ is a cubic BF subalgebra in X, then for all $x \in X$, $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ and $\lambda_A(0) \leq \lambda_A(x)$. Thus $\tilde{\mu}_A(0)$ and $\lambda_A(0)$ are the upper bounds and lower bounds of $\tilde{\mu}_A(x)$ and $\lambda_A(x)$ respectively.

II. Cubic Fuzzy H-Ideals in BF-algebras

In this section, we apply the concept of cubic fuzzy set to H-ideal of BF-algebras and introduced the notions of cubic fuzzy H-ideals of BF-algebras and investigate some of its related properties.

Definition 2.1. Let $A=(X,\tilde{\mu}_{A,}\lambda_A)$ be cubic fuzzy set in X, where X is a BF algebra, then the set A is cubic fuzzy ideal over the binary operation $*$ it satisfies the following conditions:

(CF1) $\tilde{\mu}_A(0)\geq\tilde{\mu}_A(x)$ and $\lambda_A(0)\leq\lambda_A(x)$

(CF2) $\tilde{\mu}_A(x)\geq r\min\{\tilde{\mu}_A(x*y),\tilde{\mu}_A(y)\}$

(CF3) $\lambda_A(x)\leq\max\{\lambda_A(x*y),\lambda_A(y)\}$, for all $x,y\in X$.

Definition 2.2. A non empty sub set I of BF-algebra X is called an H-ideal of X, if

(i). $0\in I$

(ii). $x*(y*z)\in I$ and $y\in I\Rightarrow x*z\in I$

Definition 2.3. A cubic fuzzy set $A=(X,\tilde{\mu}_{A,}\lambda_A)$ in a BF-algebra X is called a cubic fuzzy H-ideal of X, if

(CFH 1) $\tilde{\mu}_A(0)\geq\tilde{\mu}_A(x)$ and $\lambda_A(0)\leq\lambda_A(x)$

(CFH 2) $\tilde{\mu}_A(x*z)\geq r\min\{\tilde{\mu}_A(x*(y*z)),\tilde{\mu}_A(y)\}$ and

(CFH 3) $\lambda_A(x*z)\leq\max\{\lambda_A(x*(y*z)),\lambda_A(y)\},\forall x,y,z\in X$.

Proposition 2.4. Every cubic fuzzy H-ideal $A=(X,\tilde{\mu}_{A,}\lambda_A)$ is a cubic fuzzy ideal.

Proof: By setting $z=0$, in (CFH 2) and (CFH 3) we get

$\tilde{\mu}_A(x)\geq r\min\{\tilde{\mu}_A(x*y),\tilde{\mu}_A(y)\}$ and $\lambda_A(x)\leq\max\{\lambda_A(x*y),\lambda_A(y)\}$, for all $x,y\in X$. Therefore $A=(X,\tilde{\mu}_{A,}\lambda_A)$ is a cubic fuzzy ideal of X.

Theorem 2.5. Let $A=(X,\tilde{\mu}_{A,}\lambda_A)$ be a cubic fuzzy H-ideal of X, if there is a sequence $\{x_n\}$ in X such that

(i) $\lim_{n\to\infty}\tilde{\mu}_A(x_n)=[1,1]$ then $\tilde{\mu}_A(0)=[1,1]$ and

(ii) $\lim_{n\to\infty}\lambda_A(x_n)=0$ then $\lambda_A(0)=0$.

Proof: Since $\tilde{\mu}_A(0)\geq\tilde{\mu}_A(x)$ for all $x\in X$.

Therefore, $\tilde{\mu}_A(0)\geq\tilde{\mu}_A(x_n)$ for every positive integer n.

Consider $[1,1]\geq\tilde{\mu}_A(0)\geq\lim_{n\to\infty}\tilde{\mu}_A(x_n)=[1,1]$

Hence $\tilde{\mu}_A(0)=[1,1]$. Since $\lambda_A(0)\leq\lambda_A(x)$ for all $x\in X$,

Thus $\lambda_A(0)\leq\lambda_A(x_n)$ for every positive integer n, Now $0\leq\lambda_A(0)\leq\lim_{n\to\infty}\lambda_A(x_n)=0$

Hence $\lambda_A(0)=0$.

Theorem 2.6. A cubic fuzzy set $A=(X,\tilde{\mu}_{A,}\lambda_A)$ in X is a cubic fuzzy H-ideal of X if and only if μ_A^-, μ_A^+ and λ_A are fuzzy H-ideals of X.

Proof: Let μ_A^-, μ_A^+ and λ_A are fuzzy H-ideals of X and $x,y,z\in X$.

Then by definition $\mu_A^-(0)\geq\mu_A^-(x),\mu_A^+(0)\geq\mu_A^+(x)$,

$\mu_A^-(x*z)\geq\min\{\mu_A^-(x*(y*z)),\mu_A^-(y)\}$,

$\mu_A^+(x*z)\geq\min\{\mu_A^+(x*(y*z)),\mu_A^+(y)\}$,

$\lambda_A(x*z)\leq\max\{\lambda_A(x*(y*z)),\lambda_A(y)\}$

Now $\tilde{\mu}_A(x*z)=[\mu_A^-(x*z),\mu_A^+(x*z)]$

$\geq[\min\{\mu_A^-(x*(y*z)),\mu_A^-(y)\},\min\{\mu_A^+(x*(y*z)),\mu_A^+(y)\}]$

$=r\min\{[\mu_A^-(x*(y*z)),\mu_A^+(x*(y*z))],[\mu_A^-(y),\mu_A^+(y)]\}$

$=r\min\{\tilde{\mu}_A(x*(y*z)),\tilde{\mu}_A(y)\}$

Therefore A is cubic fuzzy H-ideal of X.

Conversely assume that $A = (X, \tilde{\mu}_A, \lambda_A)$ is cubic fuzzy H-ideal of X. For any $x,y,z \in X$,

$$[\mu_A^-(x*z), \mu_A^+(x*z)] = \tilde{\mu}_A(x*z)$$
$$\geq r\min\{\tilde{\mu}_A(x*(y*z)), \tilde{\mu}_A(y)\}$$
$$= r\min\{[\mu_A^-(x*(y*z)), \mu_A^+(x*(y*z))], [\mu_A^-(y), \mu_A^+(y)]\}$$
$$= [\min\{\mu_A^-(x*(y*z)), \mu_A^-(y)\}, \min\{\mu_A^+(x*(y*z)), \mu_A^+(y)\}]$$

Thus

$$\mu_A^-(x*z) \geq \min\{\mu_A^-(x*(y*z)), \mu_A^-(y)\},$$
$$\mu_A^+(x*z) \geq \min\{\mu_A^+(x*(y*z)), \mu_A^+(y)\},$$
$$\lambda_A(x*z) \leq \max\{\lambda_A(x*(y*z)), \lambda_A(y)\}.$$

Hence μ_A^-, μ_A^+ and λ_A are fuzzy H-ideals of X.

Theorem 2.7. Let $A = (X, \tilde{\mu}_A, \lambda_A)$ be a cubic fuzzy H-ideal of BF-algebra X and let $n \in N$ (the set of natural numbers). Then

(i) $\tilde{\mu}_A(\prod^n x*x) \geq \tilde{\mu}_A(x)$, for any odd number n.

(ii) $\lambda_A(\prod^n x*x) \leq \lambda_A(x)$, for any odd number n

(iii) $\tilde{\mu}_A(\prod^n x*x) = \tilde{\mu}_A(x)$, for any even number n.

(iv) $\lambda_A(\prod^n x*x) = \lambda_A(x)$, for any even number n.

Theorem 2.8. If $A = (X, \tilde{\mu}_A, \lambda_A)$ be a cubic fuzzy H-ideal of BF-algebra X, then the non empty upper s-level cut $U(\tilde{\mu}_A; \tilde{s})$ and non-empty lower t-level cut $L(\lambda_A; t)$ are H-ideals of X, for any $\tilde{s} \in D[0,1]$ and $t \in [0,1]$.

Proof: proof is straight forward.

Corollary 2.9. Let $A = (X, \tilde{\mu}_A, \lambda_A)$ be cubic fuzzy set. If A is a cubic fuzzy H-ideal of BF-algebra X then the sets $J = \{x \in X / \tilde{\mu}_A(x) = \tilde{\mu}_A(0)\}$ and $K = \{x \in X / \lambda_A(x) = \lambda_A(0)\}$ are H-ideals of X.

Proof: Since $0 \in X$, $\tilde{\mu}_A(0) = \tilde{\mu}_A(0)$ and $\lambda_A(0) = \lambda_A(0)$ implies $0 \in J$ and $0 \in K$, So $J \neq \phi$ and $K \neq \phi$. Let $x*(y*z) \in J$, $y \in J \Rightarrow \tilde{\mu}_A(x*(y*z)) = \tilde{\mu}_A(0)$ and $\tilde{\mu}_A(y) = \tilde{\mu}_A(0)$.

Since

$\tilde{\mu}_A(x*z) \geq r\min\{\tilde{\mu}_A(x*(y*z)), \tilde{\mu}_A(y)\} = r\min\{\tilde{\mu}_A(0), \tilde{\mu}_A(0)\} = \tilde{\mu}_A(0) \Rightarrow \tilde{\mu}_A(x*z) \geq \tilde{\mu}_A(0)$ but $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x*z)$. It follows that $x*z \in J$, for all $x,y,z \in X$. Thus J is H-ideal of X.

Let $x*(y*z) \in K$, $y \in K \Rightarrow \lambda_A(x*(y*z)) = \lambda_A(0)$ and $\lambda_A(y) = \lambda_A(0)$.

Since $\lambda_A(x*z) \leq \max\{\lambda_A(x*(y*z)), \lambda_A(y)\} = \max\{t,t\} = t$ but $\lambda_A(0) \leq \lambda_A(x*z)$.

Therefore $x*z \in K$, for all $x,y,z \in X$. Thus K is H-ideal of X.

Theorem 2.10. Let $A = (X, \tilde{\mu}_A, \lambda_A)$ be a cubic fuzzy ideal of BF-algebra X. If $\tilde{\mu}_A(x*y) \geq \tilde{\mu}_A(x)$ and $\lambda_A(x*y) \leq \lambda_A(x)$ for any $x,y \in X$, then $A = (X, \tilde{\mu}_A, \lambda_A)$ be a cubic fuzzy H-ideal of BF-algebra X.

Proof: Let $A = (X, \tilde{\mu}_A, \lambda_A)$ be a cubic fuzzy ideal of BF-algebra X. If $\tilde{\mu}_A(x*y) \geq \tilde{\mu}_A(x)$ and $\lambda_A(x*y) \leq \lambda_A(x)$ for any $x,y \in X$. We have by hypothesis

$$r\min\{\tilde{\mu}_A(x*(y*z)), \tilde{\mu}_A(y)\} \leq r\min\{\tilde{\mu}_A((x*z)*(y*z)), \tilde{\mu}_A(y*z)\}$$

$\leq \tilde{\mu}_A(y*z)$

$\tilde{\mu}_A(y*z) \geq \text{rmin}\{\tilde{\mu}_A(x*(y*z)), \tilde{\mu}_A(y)\}$, and

$\max\{\lambda_A(x*(y*z)), \lambda_A(y)\} \geq \max\{\lambda_A((x*z)*(y*z)), \lambda_A(y*z)\}$

$\geq \lambda_A(y*z)$

$\lambda_A(y*z) \leq \max\{\lambda_A(x*(y*z)), \lambda_A(y)\}$.

Hence $A = (X, \tilde{\mu}_A, \lambda_A)$ is a cubic fuzzy H-ideal of BF-algebra X.

Definition 2.11. Let f be a mapping from a set X in to a set Y. Let $B = (\tilde{\mu}_B, \lambda_B)$ be cubic fuzzy set in Y. Then the inverse image of B is defined as $f^{-1}(B) = \{(x, f^{-1}(\tilde{\mu}_B), f^{-1}(\lambda_B)) / x \in X\}$ with the membership function and non-membership function respectively given by $f^{-1}(\tilde{\mu}_B)(x) = \tilde{\mu}_B(f(x))$ and $f^{-1}(\lambda_B)(x) = \lambda_B(f(x))$. It can be shown that $f^{-1}(B)$ is cubic fuzzy set.

Theorem 2.12. Let $f : X \to Y$ be a homomorphism of BF-algebras. If $B = (\tilde{\mu}_B, \lambda_B)$ is a cubic fuzzy H-ideal of Y, then the pre image $f^{-1}(B) = \{(x, f^{-1}(\tilde{\mu}_B), f^{-1}(\lambda_B)) / x \in X\}$ of B under f is a cubic fuzzy H-ideal of X.

Proof: Assume that $B = (\tilde{\mu}_B, \lambda_B)$ is a cubic fuzzy H-ideal of Y. Let $x, y, z \in X \Rightarrow f(x), f(y), f(z) \in Y$

Consider $f^{-1}(\tilde{\mu}_B)(0) = \tilde{\mu}_B(f(0)) \geq \tilde{\mu}_B(f(x)) = f^{-1}(\tilde{\mu}_B)(x)$ and

$f^{-1}(\lambda_B)(0) = \lambda_B(f(0)) \leq \lambda_B(f(x)) = f^{-1}(\lambda_B)(x)$.

Thus $f^{-1}(\tilde{\mu}_B)(x*z) = \tilde{\mu}_B(f(x*z) \geq \text{rmin}\{\tilde{\mu}_B(f(x*(y*z)), \tilde{\mu}_B(f(y))\}$

$= \text{rmin}\{f^{-1}(\tilde{\mu}_B)(x*(y*z)), f^{-1}(\tilde{\mu}_B)(y)\}$

And $f^{-1}(\lambda_B)(x*z) = \lambda_B(f(x*z) \leq \max\{\lambda_B(f(x*(y*z)), \lambda_B(f(y))\}$

$= \max\{f^{-1}(\lambda_B)(x*(y*z)), f^{-1}(\lambda_B)(y)\}$

Therefore $f^{-1}(B) = \{(x, f^{-1}(\tilde{\mu}_B), f^{-1}(\lambda_B)) / x \in X\}$ is a cubic fuzzy H-ideal of Y.

References

[1]. Biswas. R., Rosenfeld's fuzzy subgroups with interval valued membership function, Fuzzy Sets and Systems 1 (63) (1994), 87-90.
[2]. Jun. Y.B., Kim. C.S., and Yang. K.O., Cubic sets, Ann. Fuzzy Math. Inform, 4(1) (2012) 83- 98.
[3]. Jun. Y.B., Kim. C.S., and Kang. M.S., Cubic sub algebras and ideals of BCK/BCI algebras, Far East Journal of Mathematical Sciences 44(2),(2010),239-250.
[4]. Neggers. J., Ahn. S.S and Kim. H.S., On BF-algebra, International Journal of Math. Sci. 27 (2001), 749-757.
[5]. Satyanarayana B., Bindu Madhavi U. and Durga Prasad R., On intuitionistic fuzzy H-ideals in BCK-algebras, International Journal of Algebra, 4, (15)(2010), 743-749.
[6]. Satyanarayana B., Bindu Madhavi U. and Durga Prasad R., On foldness of intuitionistic fuzzy H-ideals in BCK-algebras, International Mathematical Form, 5(45)(2010), 2205-2211.
[7]. Satyanarayana. B.,Ramesh. D., and Pragathi Kumar. Y., Interval valued intuitionistic Fuzzy Homomorphism of BF-algebras, Mathematical Theory and Modeling 3 (2013), 161-165.
[8]. Zadeh. L. A., Fuzzy sets, Inform. Control 8, 338-353(1965).
[9]. Zadeh. L. A., The concept of a linguistic variable and its application to approximate reasoning. I, Information Sci. And Control 8 (1975) Z199-249.
[10]. Zarandi. A and Borumand Saeid. A., Interval-valued fuzzy BF-algebras, Int. J. Contemp. Math. Sciences, 4 (17) (2009) 803-814.
[11]. Zhan. J and Tan. Z., Fuzzy H-ideals in BCK-algebras, Southeast Asian Bulletin of Mathematics 29 (2005) 1165-1173