On Designs arising from Corona Product $H \circ K_3$

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Abstract: In this paper, we determine the partially balanced incomplete block designs and association scheme which are formed by the minimum dominating sets of the graphs $C_3 \circ K_3$, we determine the number of minimum dominating sets of graph $G = C_n \circ K_3$ and prove that the set of all minimum dominating sets of $G = C_n \circ K_3$ forms a partially balanced incomplete block design with two association scheme. Finally we generalize the results for the graph $H \circ K_3$.

Key Words: Minimum dominating sets, association schemes, PBIB designs.

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I. Introduction

By a graph, we mean a finite undirected graph without loops or multiple lines. For a graph $G = (V, E)$, let $V$ and $E$ respectively denote the vertex set and the edge set of graph $G$. For any vertex $u \in V$, $N(u) = \{v \in V : u \sim v \in E\}$ is called the open neighbourhood of $u$ in $V$, and the closed neighbourhood of $u$ in $G$ is $N[u] = N(v) \cup \{u\}$. The degree of $u$ in $G$, $\deg(u) = |N(u)|$. The open Neighborhood of a set of vertices $S$ in $G$ is $N(S) = \bigcup_{u \in S} N(u)$ and the closed neighbourhood of the set $S$ is $N[S] = N(S) \cup S$. A subset $D \subseteq V$ is called dominating set of $G = (V, E)$ if $N[D] = V$. The domination number $\gamma(G)$ of $G$ is the minimum cardinality of a dominating set.

A dominating set $D$ is called minimal dominating set if no proper subset $S \subseteq D$ is a dominating set.

The PBIBD with m-association scheme which are arising from dominating sets has been studied extensively by many for example see [8],[1]. In this paper, We study the PBIBD and the association scheme which can be obtained from the minimum dominating sets in $(C_n \circ K_3)$ graph. Finally we generalize the results for the graph $H \circ K_3$.

II. PBIBD arising from minimum dominating sets of $(C_n \circ K_3)$

Definition 1. Given $v$ objects a relation satisfying the following conditions is said to be an association scheme with $m$ classes:

(i) Any two objects are either first associates, or second associates….. or $m$th associates, the relation of association being symmetric.

(ii) Each object $a$ has $n_1$ ith associates, the number $n_1$ being independent of $a$.

(iii) If two objects $a$ and $b$ are $i$th associates, then the number of objects which are $j$th associates of $a$ and $k$th associates of $b$ is $p_{jk}^i$ and is independent of the pair of $i$th associates $a$ and $b$. Also $p_{jk}^i : jk \equiv p_{kj}$

If we have association scheme for the $v$ objects we can define a PBIBD as the following definition.

Definition 2. The PBIBD design is arrangement of $v$ objects into $b$ sets (called blocks) of size $k$ where $k < v$ such that

(i) Every object is contained in exactly $r$ blocks.

(ii) Each block contains $k$ distinct objects.

(iii) Any two objects which are $i$th associates occur together in exactly $\lambda_i$ blocks.
Theorem 3. From \((C_3 \circ K_3)\) we can get PBIBD with parameters 
\((v = 12, k = 3, r = 16, b = 64, \lambda_1 = 0, \lambda_2 = 4)\) and association scheme of 2-classes

With \(P_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\) and \(P_2 = \begin{bmatrix} 0 & 3 \\ 3 & 4 \end{bmatrix}\).

Proof. Let \(G = (V, E)\) be a corona graph \(C_3 \circ K_3\).

By labelling \(\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}\) as in Figure 1, we can define PBIBD as follows:

The point set is the vertices and the block set is the minimum dominating sets \(\{v_1, v_2, v_3\}, \{v_1, v_4, v_5\}, \{v_1, v_6, v_7\}, \{v_1, v_8, v_9\}, \{v_1, v_2, v_3, v_4\}, \{v_1, v_2, v_5, v_6\}, \{v_1, v_2, v_7, v_8\}, \{v_1, v_2, v_9, v_10\}, \{v_1, v_2, v_11, v_12\}\) and \(\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}\).

![Figure 1: C_3 \circ K_3](image)

We define the association scheme as follows, for any \(\alpha, \beta \in V(G)\), \(\alpha\) is first associate of \(\beta\) if \(\alpha\) and \(\beta\) appear in zero or three minimum dominating sets and \(\alpha\) is second associate of \(\beta\) otherwise, see Table 1.
Theorem 4. Let $G \simeq (C_n \square K_3)$. Then the number of minimum dominating sets of $G$ is $4^n$.

Proof. Let $G \simeq (C_n \square K_3)$. Then $\gamma(G) = n$. We need to find out all the sets of size $n$. For this, we have many possibilities:

**Case 1.** All the vertices of the minimum dominating set are from inside that is from $C_n$. Then there is only one minimum dominating set.

**Case 2.** The vertices of minimum dominating set i.e. not from the vertices of $C_n$. The number of ways to select minimum dominating sets of size $n$ from outside is $3n$.

**Case 3.** We select some vertices of minimum dominating sets from inside and some from outside. So we start by selecting one vertex from inside and $(n - 1)$

<table>
<thead>
<tr>
<th>Elements</th>
<th>First Associates</th>
<th>Second Associates</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1$</td>
<td>$v_1^1, v_1^{11}$</td>
<td>$v_2, v_2^{11}$</td>
</tr>
<tr>
<td>$v_2$</td>
<td>$v_1, v_1^{11}, v_1^{11}$</td>
<td>$v_2, v_2^{11}$</td>
</tr>
<tr>
<td>$v_3$</td>
<td>$v_1, v_1^{11}, v_1^{11}$</td>
<td>$v_2, v_2^{11}$</td>
</tr>
<tr>
<td>$v_4$</td>
<td>$v_1, v_1^{11}, v_1^{11}$</td>
<td>$v_2, v_2^{11}$</td>
</tr>
</tbody>
</table>

Table 1:

vertices from outside. There are $\binom{n}{1} 3^{n-1}$ ways. Similarly 2 vertex from inside

$(n - 2)$ vertices from outside. There are $\binom{n}{2} 3^{n-1}$ ways. By continuing in same way till $(n - 1)$ vertices from inside and one from outside, there are $\binom{n}{n-1} 3$ ways.

Hence the total number of minimum dominating sets is:

$$3 \cdot \binom{n}{1} 3^{n-1} + \binom{n}{2} 3^{n-2} + \ldots + \binom{n}{n-1} 3 + 1$$

$$= \sum_{i=0}^{n-1} \binom{n}{i} 3^{n-i}$$

$$= 4^n$$

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Theorem 5. Let \( G \cong (C_n \circ K_3) \). Any two vertices in \( G \) either belong to zero minimum dominating set or \( 4^{n-2} \) minimum dominating sets.

Proof. By labeling the vertices of the graph \( G \) as \( \{v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n\} \)

where \( \{v_1, v_2, \ldots, v_n\} \) are the vertices of \( C_n \) and \( \{v'_1, v'_2, \ldots, v'_n\} \)
are the vertices of the copies \( K_3 \).

Suppose \( A = \{v_1, v_2, \ldots, v_n\} \) and \( B = \{v'_1, v'_2, \ldots, v'_n\} \).

Let \( u, v \) be any two vertices, we have the following cases:

Case 1. \( u \) and \( v \) belong to \( A \) then there are \( 4^{n-2} \) minimum dominating sets containing \( u \) and \( v \).

Case 2. \( u \) and \( v \) belong to \( B \) then there are \( 4^{n-2} \) ways to select minimum dominating sets containing \( u \) and \( v \).

Case 3. Let \( u \in A \) and \( v \in B \), we have two subcases:

Case (i). Let \( u \) and \( v \) in the same triangle then there does not exist any minimum dominating sets containing \( u \) and \( v \).

Case (ii). If \( u \) and \( v \) are from the different triangle then there are \( 4^{n-2} \) ways to select minimum dominating sets.

Theorem 6. Let \( G \cong (C_n \circ K_3) \). Then every vertex \( v \in V(G) \) contained in \( 4^{n-1} \) minimum dominating sets.

Proof. Let \( G \cong (C_n \circ K_3) \). The vertices of \( G \) can be partitioned into \( n \) sets, each set containing 3 vertex as the triangles \( \Delta_1, \Delta_2, \ldots, \Delta_n \). Let \( v \in V(G) \) be any vertex such that \( v \in \Delta_i \) for some \( 1 \leq i \leq n \). Any minimum dominating set containing \( v \) will contain \((n-1)\) vertices from the other triangle \( \Delta_j \) where \( i = j \). But it is not allowed to take two vertex from the same triangle so we need to take one vertex from each triangle.

Hence the ways to select \( n-1 \) vertices from the \( \Delta_j \) triangles \( i = j \) is \( 4^{n-1} \).

Finally, we can generalize Theorem 6 as following.

Theorem 7. For any graph \( G \cong (H \circ K_3) \), there is PBIBD and association scheme associate with \( G \) as the following parameters,

\( (v=4n, \; k=n, r=4^{n-1}, \; b=4^n, \; \lambda_1 = 0, \; \lambda_2 = 4^{n-2}) \) and

\[
P_1 = \begin{bmatrix} p_{11} & 0 \\ p_{21} & 4(n-1) \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 3 & 4(n-2) \end{bmatrix}.
\]

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