Prime Graph vs. Zero Divisor Graph

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Abstract: In this paper we consider prime graph of $R$ (denoted by $\mathcal{P}_G(R)$) of an associative ring $R$ (introduced by Satyanarayana, Syam Prasad and Nagaraju \textsuperscript{[22]}). We also consider zero divisor graph of a finite associative ring $R$ (denoted by $\mathcal{Z}_D(G(R))$). It is proved that every prime graph is a subgraph of the zero divisor graph but the converse need not be true. An example of a ring for which $\mathcal{P}_G(R) \neq \mathcal{Z}_D(G(R))$ was presented.

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I. Introduction

Let $G = (V, E)$ be a graph consist of a finite non-empty set $V$ of vertices and finite set $E$ of edges such that each edge $e_k$ is identified as an unordered pair of vertices $\{v_i, v_j\}$, where $i, j \in V$ are called end points of $e_k$. The edge $e_k$ is also denoted by either $v_i v_j$ or $v_j v_i$. We also write $G(V, E)$ for the graph. Vertex set and edge set of $G$ are also denoted by $V(G)$ and $E(G)$, respectively. An edge associated with a vertex pair $\{v_i, v_j\}$ is called a self-loop. The number of edges associated with the vertex is the degree of the vertex, and $\delta(v)$ denotes the degree of the vertex $v$. If there is more than one edge associated with a given pair of vertices, then these edges are called parallel edges or multiple edges. A graph that does not have self-loop or parallel edges is called a simple graph. We consider simple graphs only. For an associative ring $R$, prime graph of $R$ (denoted by $\mathcal{P}_G(R)$) was introduced in Satyanarayana, Syam Prasad and Nagaraju \textsuperscript{[22]}. For a commutative ring $R$, the notion of ‘zero divisor graph’ is given in Beck \textsuperscript{[1988]}. In this paper, we consider the associative rings (need not be commutative) and provided some examples on the zero divisor graphs of $\mathbb{Z}_n$ where $n$ is a positive integer.

1.2 Definitions:

(i) A graph $G(V, E)$ is said to be a **star graph** if there exists a fixed vertex $v$ (called the center of the star graph) such that $E = \{vu / u \in V$ and $u \neq v\}$. A star graph is said to be an **$n$-star graph** if the number of vertices of the graph is $n$.

(ii) In a graph $G$, a subset $S$ of $V(G)$ is said to be a **dominating set** if every vertex not in $S$ has a neighbour in $S$. The **domination number**, denoted by $\gamma(G)$ is defined as min $\{|S| / S$ is a dominating set in $G\}$.

(iii) In a connected graph, a closed walk running through every vertex of $G$ exactly once (except the starting vertex at which the walk terminates) is called as **Hamiltonian circuit**. A graph containing a Hamiltonian circuit is called **Hamiltonian graph**.

1.3 Theorem: (Th. 13.8, page 361, [18]) A given connected graph $G$ is an Eulerian graph if and only if all the vertices of $G$ are of even degree.

For other preliminary results and notations we use \textsuperscript{[18],[20] or [21]}

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II. Prime Graph of a Ring

2.1 Definition: (Satyanarayana, Syam Prasad and Nagaraju [22]) Let \( R \) be an associative ring. A graph \( G(V, E) \) is said to be a prime graph of \( R \) (denoted by \( \text{PG}(R) \)) if \( V = R \) and \( E = \{ xy / xRy = 0 \text{ or } yRx = 0, \text{ and } x \neq y \} \).

For convenience of the reader we included the following example.

2.2 Example (Example 9.4.2 of Satyanarayana and Syam Prasad [20]): Consider \( \mathbb{Z}_n \), the ring of integers modulo \( n \).

(i) Let us construct the graph \( \text{PG}(R) \), where \( R = \mathbb{Z}_3 \). We know that \( R = \mathbb{Z}_3 = \{0, 1, 2\} \). So \( V(\text{PG}(R)) = \{0, 1, 2\} \). Since \( 0R1 = 0, 0R2 = 0 \) there exists an edge between 0 and 1, and also an edge between 0 and 2. There are no other edges, as there are no two non-zero elements \( x, y \in R \) with \( xRy = 0 \). So \( E(\text{PG}(R)) = \{01, 02\} \). Now \( \text{PG}(R) \) is given in Figure 2.2 (i).

![Fig. 2.2 (i)](image)

(ii) Let us construct the graph \( \text{PG}(R) \), where \( R = \mathbb{Z}_4 \). We know that \( R = \mathbb{Z}_4 = \{0, 1, 2, 3\} \). So \( V(\text{PG}(R)) = \{0, 1, 2, 3\} \). Since \( 0R1 = 0, 0R2 = 0, 0R3 = 0 \), we have that \( 01, 02, 03 \in E(\text{PG}(R)) \). There are no other edges, as there are no two distinct non-zero elements \( x, y \in R \) such that \( xRy = 0 \). So \( E(\text{PG}(R)) = \{01, 02, 03\} \). Now \( \text{PG}(R) \) is given in Figure 2.2 (ii).

![Fig. 2.2 (ii)](image)

III. Zero Divisor Graph of an Associative Ring

In this section, we wish to study zero divisor graph of an associative ring

3.1 Definition: (Vasantha kandasamy and Florentin Smarandache [23]) A graph \( G = (V, E) \) is said to be a zero divisor graph of a commutative ring \( R \) if \( V = R \) and \( E = \{ xy / x \neq y, x, y \in R, x \neq 0 \neq y, xy = 0 \} \cup \{ \overline{x0} / \overline{0x} / 0 \neq x \in R \} \) where \( \overline{xy} \) denotes an edge between \( x, y \in V \).

This definition ‘zero divisor graph’ is same as that of Beck [1988].

3.2 Notation: (i) We denote zero divisor graph of ring \( R \) by \( \text{ZDG}(R) \)

(ii) In the graph \( \text{ZDG}(R) \), we have that \( V(\text{ZDG}(R)) = R \) and \( E(\text{ZDG}(R)) = \{ xy / x \neq y, x, y \in R, x \neq 0 \neq y, xy = 0 \} \cup \{ \overline{x0} / \overline{0x} / 0 \neq x \in R \} \)

3.3 Example: (Vasantha kandasamy and Florentin Smarandache [23])

Consider \( Z_n \), the ring of integers modulo \( n \).

Consider \( \text{ZDG}(R) \) with \( R = \mathbb{Z}_{10} \). We know that \( R = \mathbb{Z}_{10} = \{0,1,2,3,\ldots,9\} \),
So \( V(ZDG(R)) = \{0,1,2,3,4,5,6,7,8,9\} \). Since \( 5.8 = 5.4 = 5.6 = 0 \) (mod 10), there exist edges between the vertices 5 and 8; 5 and 4; also between 5 and 6. Since ‘0’ is adjacent to all the elements in R, we get \( \{01,02,03,04,05,06,07,08,09\} \in E(ZDG(R)) \).

Therefore, \( E(ZDG(R)) = \{01,02,03,04,05,06,07,08,09,25,58,54,56\} \).

Now ZDG(R) given by the figure 3.3.

3.4 Observations: (i) ZDG(\( Z_{10} \)) contains a 10-star graph as its subgraph; (ii) The domination number is 1; (iii) Since 02, 25, 50 form a triangle, we conclude that the graph cannot be a bipartite graph; (iv) ZDG(\( Z_{10} \)) is not an Eulerian graph (by using the Th. 13.8, p 361 of [18]); and (v) Since ZDG(\( Z_{10} \)) contains pendant vertices, it contains no Hamiltonian circuit.

The following definition is an extension of the concept “zero divisor graph” to Associative rings.

3.5 Definition: Consider an associative ring \( R \) (need not be commutative) with identity 1. The zero divisor graph (in notation, ZDG(R)) is defined as \( V(ZDG(R)) = R \) and \( E(ZDG(R)) = \{ab / a, b \in R, either \ ab = 0 \ or \ ba = 0, a \neq b \} \).

3.6 Note: In case of commutative rings, the above concept coincides with the zero divisor graph defined in commutative rings by Beck [1].

3.7 Theorem: For an associative ring \( R \) we have that PG(R) is a subgraph of ZDG(R).

Proof: We know that \( V(PG(R)) = R = V(ZDG(R)) \).

Let \( \bar{uv} \in E(PG(R)) \). Then \( uRv = 0 \) or \( vRu = 0 \). Since \( 1 \in R \) we have that either \( uv = 0 \) or \( vu = 0 \).

By definition 3.5 we have that \( \bar{uv} \in E(ZDG(R)) \). This shows that PG(R) is a subgraph of ZDG(R).

3.8 Corollary: If \( R \) is a commutative ring then PG(R) = ZDG(R).

Proof: Let \( \bar{uv} \in E(ZDG(R)) \) with \( u, v \in V(ZDG(R)) = R \). Then \( uv = 0 \) or \( vu = 0 \). Suppose that \( uv = 0 \). This implies \( vxu = 0 \) for all \( x \in R \). Since \( R \) is commutative \( vxu = 0 \) for \( x \in R \) and so \( vRu = 0 \). This shows that \( \bar{uv} \in E(PG(R)) \).

Hence ZDG(R) is a subgraph of PG(R).

By theorem 3.7 we have that PG(R) = ZDG(R).

3.9 Remark: In case of associative rings which is not commutative, the converse of the theorem 3.7 need not be true. This was made clear by the example presented in the next section.

IV. An Example

In this section, we present an example of an associative ring for which PG(R) ≠ ZDG(R).

4.1 Example: Let \( F = \mathbb{Z}_2 \) be the field of integers modulo 2. Write \( R = \) set of \( 3 \times 3 \) matrices over the field \( F \). We know that \( R \) is an associative ring with respect to usual matrix addition and multiplication. Consider the two elements \( x, y \in R \) mentioned below

\[
x = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

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Since $xy = 0$ and $x \neq y$ we conclude that $\overline{xy} \in E(ZDG(R))$.

Consider $z = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \in R$.

Now $xzy = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $yzx = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, we have that $xzy \neq 0$.

Since $xzy \neq 0$ and $yzx \neq 0$, we have that $xy \notin E(PG(R))$.

Hence $E(ZDG(R)) \not\subseteq E(PG(R))$.

Thus we verified that for the ring of $3 \times 3$ matrices over $Z_2$, the two graphs: prime graph and zero divisor graph are not equal.

References


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