Abstract: In this paper, we introduce and study sets called \( \rho \)-closed set and \( \rho \)-open set. Also we have investigated some of their basic properties.

Keywords: \( \rho \) – closed set, \( \rho \) – open set

I. Introduction

Levine \[6\], Mashhour et. al. \[9\] introduced semi-open sets, preopen sets in topological spaces respectively. The complement of a semi-open (resp. preopen) set is called a semi-closed (resp. preclosed) set. Levine \[5\] introduced generalized closed (briefly g-closed) sets. Maki et al. \[8\] introduced the concepts of generalized preclosed sets. In this paper, we define and study a set called \( \rho \) – closed set.

II. Preliminaries

Throughout this paper \((X, \tau)\) represents non-empty topological space on which no separation axioms are assumed unless otherwise mentioned. For a subset \(A\) of a space \((X, \tau)\), \(\text{cl}(A)\) and \(\text{int}(A)\) denote the closure of \(A\) and interior of \(A\) in \(X\) respectively.

2.1 Definition

Let \((X, \tau)\) be a topological space. A subset \(A\) of the space \(X\) is said to be

1. a semi-open set \[6\] if \(A \subseteq \text{cl}(\text{int}(A))\) and a semi-closed set if \(\text{int}(\text{cl}(A)) \subseteq A\).
2. a preopen set \[9\] if \(A \subseteq \text{int}(\text{cl}(A))\) and a preclosed set if \(\text{cl}(\text{int}(A)) \subseteq A\).
3. an \(\alpha\)-open set \[10\] if \(A \subseteq \text{cl}(\text{int}(\text{cl}(A)))\) and a \(\alpha\)-closed set if \(\text{cl}(\text{int}(\text{cl}(A))) \subseteq A\).
4. a semi-preopen set \[1\] (\(\beta\)-open) if \(A \subseteq \text{int}(\text{cl}(\text{int}(A)))\) and a semi-preclosed set (\(\beta\)-closed ) if \(\text{int}(\text{cl}(\text{int}(A))) \subseteq A\).

2.2 Definition \([9]\)

Let \((X, \tau)\) be a topological space and \(A \subseteq X\).
1. The Pre-closure of \(A\), denoted by \(\text{pcl}(A)\), is the intersection of all preclosed sets containing \(A\).
2. The Pre-interior of \(A\), denoted by \(\text{pint}(A)\), is the union of all preopen subsets of \(A\).

2.3 Lemma \([1]\)

For any subset \(A\) of \(X\), the following relations hold.
1. \(\text{Scl}(A) = A \cup \text{int}(\text{cl}(A))\)
2. \(\text{acl}(A) = A \cup \text{cl}(\text{int}(\text{cl}(A)))\)
3. \(\text{Pcl}(A) = A \cup \text{cl}(\text{int}(A))\)
4. \(\text{Spcl}(A) = A \cup \text{int}(\text{cl}(\text{int}(A)))\)

2.4 Definition

A subset \(A\) of a space \((X, \tau)\) is called

1. a generalized closed set \[5\] (g-closed) if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open,
2. a generalized semi-closed set \[2\] (gs-closed) if \(\text{sc}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open,
3. a generalized preclosed set \[8\] (gp-closed) if \(\text{pcl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open,
4. a generalized \(\alpha\)-closed set (g-\(\alpha\)-closed) \[7\] if \(\text{acl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\alpha\)-open,
5. a generalized semi-preclosed set \[4\] (gsp-closed) if \(\text{spcl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open.

The complements of the above mentioned sets are their respective open sets.

III. Basic Properties Of \( \rho \) – Closed Set

We introduce the following definition.
3.1 Definition
A subset A of a space (X, τ) is said to be \( \hat{\beta} \)-closed in (X, τ) if Pcl(A) \( \subseteq \) Int(U) whenever A \( \subseteq \) U and U is semi open in (X, τ).

3.2 Theorem Every \( \hat{\beta} \)-closed set is pre closed.
Proof: Let A be a \( \hat{\beta} \)-closed set in X. Then Pcl(A) \( \subseteq \) Int(U) whenever A \( \subseteq \) U. Now, A \( \cup \) cl(int(A)) \( \subseteq \) Int(U). Therefore, cl(int(A)) \( \subseteq \) A \( \cap \) Int(U). Since A \( \subseteq \) U, cl(int(A)) \( \subseteq \) A. Hence, A is pre closed.

3.2.1 Example Let X = \{a, b, c\} and \( \tau = \{\phi, X, \{a\}\}\). Then the set A = \{b\} is pre closed but not \( \hat{\beta} \)-closed in (X, \( \tau \)).

3.3 Theorem Every \( \hat{\beta} \)-closed set is semipre closed.
Proof: Let A be a \( \hat{\beta} \)-closed set in X. Since every pre closed set is semipre closed and by theorem 1, it follows that A is semipre closed.

3.3.1 Example Let X = \{a, b, c\} and \( \tau = \{\phi, \{a\}, \{a,b\}, X\}\). Then the set A = \{c\} is semipre closed but not \( \hat{\beta} \)-closed in (X, \( \tau \)).

3.4 Theorem Every \( \hat{\beta} \)-closed set is gp-closed.
Proof: Let A be a \( \hat{\beta} \)-closed set in X. Let A \( \subseteq \) U and U is open in X. Every open set is semi open and thus A is \( \hat{\beta} \)-closed. Therefore Pcl(A) \( \subseteq \) Int(U) = U. Hence A is gp-closed.

3.4.1 Example Let X = \{a, b, c\} and \( \tau = \{\phi, X, \{a\}\}\). Then the set A = \{a, b\} is gp-closed but not \( \hat{\beta} \)-closed in (X, \( \tau \)).

3.5 Remark \( \hat{\beta} \)-closed sets are independent concepts of semi-closed sets and \( \alpha \)-closed sets as we illustrate by means of the following examples.

3.5.1 Example
1. Let X = \{a, b, c\} and \( \tau = \{\phi, \{a\}, \{a,b\}, X\}\). Then the set A = \{c\} is both semi-closed and \( \alpha \)-closed but not \( \hat{\beta} \)-closed in (X, \( \tau \)).
2. Let X = \{a, b, c\} and \( \tau = \{\phi, X, \{a, b\}\}\). Then the set A = \{b, c\} is \( \hat{\beta} \)-closed but neither semi-closed nor \( \alpha \)-closed in (X, \( \tau \)).

3.6 Remark The union (intersection) of two \( \hat{\beta} \)-closed sets need not be \( \hat{\beta} \)-closed.

3.6.1 Example Let X = \{a, b, c, d\} and \( \tau = \{\phi, \{c\}, \{a,b\}, \{a, b, c\}\}\).
1. Let A = \{a\} and B = \{b\}. Here A and B are \( \hat{\beta} \)-closed sets. But A \( \cup \) B = \{a, b\} is not \( \hat{\beta} \)-closed.
2. Let A = \{b, c, d\} and B = \{a, c, d\}. Here A and B are \( \hat{\beta} \)-closed sets. But A \( \cap \) B = \{c, d\} is not \( \hat{\beta} \)-closed.

3.7 Theorem If a set A is \( \hat{\beta} \)-closed, then Pcl(A) – A contains no nonempty closed set.
Proof: Let F \( \subseteq \) Pcl(A) – A be a nonempty closed set. Then F \( \subseteq \) Pcl(A) and A \( \subseteq \) X – F. Since X – F is semi open, then A is \( \hat{\beta} \)-closed. Therefore Pcl(A) \( \subseteq \) Int(X – F) = X – cl(F), cl(F) \( \subseteq \) X – Pcl(A). And so F \( \subseteq \) X – Pcl(A), F \( \subseteq \) Pcl(A) \( \cap \) (X – Pcl(A)) = \{\phi\}. Hence Pcl(A) – A contains no nonempty closed set.

3.7.1 Example Let X = \{a, b, c\} and \( \tau = \{\phi, X, \{a\}\}\). Let A = \{b\}, then Pcl(A) – A contains no nonempty closed set. But A is not \( \hat{\beta} \)-closed in (X, \( \tau \)).

3.8 Theorem If a set A is \( \hat{\beta} \)-closed, then Pcl(A) – A contains no nonempty semi-closed set.
Proof: Let F be a nonempty semi-closed set such that F \( \subseteq \) Pcl(A) – A. Then F \( \subseteq \) Pcl(A) and A \( \subseteq \) X – F. We have Pcl(A) \( \subseteq \) Int(X – F), Pcl(A) \( \subseteq \) X – cl(F), cl(F) \( \subseteq \) X – Pcl(A). Therefore F \( \subseteq \) Pcl(A) \( \cap \) (X – Pcl(A)) = \{\phi\}. Hence Pcl(A) – A contains no nonempty semi-closed set.

The converse of the above theorem need not be true as it is seen from the following example.

DOI: 10.9790/5728-1205067981 www.iosrjournals.org 80 | Page
3.8.1 Example Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{a,b\}, X\}$. Let $A = \{c\}$, then $Pcl(A) - A$ contains no nonempty semi-closed set. But $A$ is not $\beta$-closed in $(X, \tau)$.

3.9 Theorem An open set $A$ of $(X, \tau)$ is gp-closed if and only if $A$ is $\beta$-closed. 

Proof: Let $A$ be an open and gp-closed set. Let $A \subseteq U$ and $U$ be semi-open in $X$. Since $A$ is open, $A = Int(A) \subseteq Int(U)$. Observe that $Int(U)$ is open and thus $A$ is gp-closed. Hence $Pcl(A) \subseteq Int(U)$ and $A$ is an $\beta$-closed set. Conversely, by Theorem 3, every $\beta$-closed set is gp-closed.

3.10 Remark From the above discussions and known results we have the following implications $A \rightarrow B$ represents $A$ implies $B$ but not conversely and $A \leftrightarrow B$ represents $A$ and $B$ are independent of each other. See Figure 1.

![Figure 1: Implications](image)

IV. $\beta$-Open Sets

4.1 Definition A subset $A$ of $(X, \tau)$ is said to be $\beta$-open in $(X, \tau)$ if its complement $X - A$ is $\beta$-open in $(X, \tau)$.

4.2 Theorem A subset $A$ of a topological space $(X, \tau)$ is a $\beta$-open set if and only if $cl(K) \subseteq pont(A)$ whenever $K \subseteq A$ and $K$ is semi-closed.

Proof: Necessity. Let $A$ be a $\beta$-open set in $(X, \tau)$. Let $K \subseteq A$ and $K$ be semi-closed. Then $X - A$ is $\beta$-closed and it is contained in the semi-open set $X - K$. Therefore $Pcl(X - A) \subseteq Int(X - K)$, $X - pont(A) \subseteq X - cl(K)$. Hence $cl(K) \subseteq pont(A)$.

Sufficiency. If $K$ is semi-closed set such that $cl(K) \subseteq pont(A)$ whenever $K \subseteq A$. It follows that $X - A \subseteq X - K$ and $X - pint(A) \subseteq X - cl(K)$. Therefore $Pcl(X - A) \subseteq Int(X - K)$. Hence $X - A$ is $\beta$-closed and $A$ becomes an $\beta$-open set.

4.3 Theorem If $A \subseteq K$ is $\beta$-closed the $Pcl(A) - A$ is $\beta$-open.

Proof: Let $A$ be an $\beta$-closed. Then by Theorem 5, $Pcl(A) - A$ contains no nonempty semi-closed set. Therefore $\emptyset = K \subseteq Pcl(A) - A$ and $\emptyset = K$ is semi-closed. Clearly, $cl(K) \subseteq pont(Pcl(A) - A)$. Hence by theorem 7, $Pcl(A) - A$ is $\beta$-open.

References