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Some Properties of a Class of Modified New Bernstein Type Operators

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Abstract: *In this paper we study some properties of the modified new Bernstein type operators introduced by M. A. Siddiqui et. al. in 2014. We show that some properties of the original function such as shape preserving properties, smoothness properties, etc. are preserved by these modified operators. We obtain an estimate on rate of convergence of these operators in terms of modulus of continuity. We also study pointwise approximation by these operators with use of Peetre's K-functional and Ditzian-Totik moduli of smoothness and obtain a direct theorem for these operators.*
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I. Introduction

If f is a continuous function defined on [0, 1], the Bernstein operators B_n ,$n \in \mathbb{N}$, are defined as

$$(B_nf)(x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0,1]$$

Where

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \qquad k = 0,1,2, \dots \dots \dots n.$$

(B_nf) is a polynomial of degree atmost n , $n \in \mathbb{N}$, which converges to f uniformly on [0,1]. In the year 2008, N. deo et al. [2] introduced the class of new Bernstein type operators as

$$(V_nf)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right) \qquad (1)$$

where $f \in C\left[0, \frac{n}{n+1}\right]$ and

$$p_{n,k}(x) = \left(\frac{1+n}{n}\right)^n \binom{n}{k} x^k \left(\frac{n}{n+1} - x\right)^{n-k} \qquad (2)$$

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In 2014, M. A. Siddiqui et al. [9] introduced a class of modified new Bernstein type operators as

$$V_n^*(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n+1}\right) \qquad (3)$$

where $f \in C\left[0, \frac{n}{n+1}\right]$ and $p_{n,k}(x)$ is same as in equation (2). Note that these operators are defined on the mobile interval $\left[0, \frac{n}{n+1}\right]$, $n \in \mathbb{N}$. and for sufficiently large n, they convert in the classical Bernstein operators.

It was proved in [9] that $V_n^* f$ converge uniformly to f on $\left[0, \frac{n}{n+1}\right]$ and that they are more close to the classical Bernstein operators in comparison with the operators in (1). In the present paper we study some other properties of the operators V_n^* .

II. Basic Results

In [9] following results were proved.

Lemma 2.1. *For each* $x \in \left[0, \frac{n}{n+1}\right], n \in \mathbb{N}$, *we have*

$$V_n^*(e_0;x) = 1$$
$$V_n^*(e_1;x) = x$$
$$V_n^*(e_2;x) = x^2 + \frac{x}{n}\left(\frac{n}{n+1} - x\right)$$
$$V_n^*(e_3;x) = x^3\frac{(n-1)(n-2)}{n^2} + 3x^2\frac{(n-1)}{n(n+1)} + \frac{x}{(n+1)^2}$$
$$V_n^*(e_4;x) = x^4\frac{(n-1)(n-2)(n-3)}{n^3} + 6x^3\frac{(n-1)(n-2)}{n^2(n+1)} + 7x^2\frac{(n-1)}{n(n+1)^2} + \frac{x}{(n+1)^3}$$

where $e_j(t) = t^j, j = 0,1,2,3,4.$

Lemma 2.2.

$$V_n^*((t-x);x) = 0$$
$$V_n^*((t-x)^2;x) = \frac{x}{n}\left(\frac{n}{n+1} - x\right)$$
$$V_n^*((t-x)^3;x) = \frac{2x^3}{n^2} - \frac{3x^2}{n(n+1)} + \frac{x}{(n+1)^2}$$
$$V_n^*((t-x)^4;x) = \frac{(3n-6)}{n^3}x^4 + \frac{6x^3}{n^2(n+1)} + \frac{x^2}{(n+1)^2}\frac{(3n-7)}{n} + \frac{x}{(n+1)^3}$$

III. Shape Preserving Properties Of V_n^*

In [7] certain shape preserving properties of Bernstein polynomials were established in the form of following theorem.

Theorem (A). *Let* $f : [0,1] \to \mathbb{R}$. *Then*

$$B_n^m(f;x) = \frac{n!}{(n-m)!}\sum_{k=0}^{n-m}\binom{n-m}{k}x^k(1-x)^{n-m-k}\Delta_{\frac{1}{n}}^m f\left(\frac{k}{n}\right), m = 0,1,\ldots,n$$

In particular if f is m-convex , then so is $B_n(f)$.

In this section we shall establish similar properties for the operators $V_n^*(f)$ involving forward differences of the function. The forward differences of any function $f : [0,1] \to \mathbb{R}$ are defined as,

$$\Delta_h f(x) = f(x+h) - f(x), x, h \in [0,1],$$
$$\Delta_h^r f(x) = \Delta_h\left(\Delta_h^{r-1} f(x)\right), r \geq 2.$$

Thus f is monotonically increasing if $\Delta_h f(x) \geq 0$ and convex if $\Delta_h^2 f(x) \geq 0$ for all $x \in [0,1]$. More generally, f is m-convex if $\Delta_h^m f(x) \geq 0$ for all $x \in [0,1]$.

Theorem 3.1. *Let* $f : \left[0, \frac{n}{n+1}\right] \to \mathbb{R}$. *Then*

$$[V_n^*(f;x)]^m = \frac{n!}{(n-m)!}\left(\frac{1+n}{n}\right)^n \sum_{k=0}^{n-m}\binom{n-m}{k}x^k\left(\frac{n}{n+1} - x\right)^{n-m-k}$$

$$\times \Delta^m_{\frac{1}{n+1}} f\left(\frac{k}{n+1}\right) \qquad (4)$$

where m = 0, 1, . . . , *n. In particular if f is m-convex, then so is* $V_n^*(f)$.

Proof. Differentiating both sides of equation (3) with respect to *x* we obtain,

$$[V_n^*(f;x)]'$$

$$= \left(\frac{1+n}{n}\right)^n \sum_{k=0}^{n} \binom{n}{k} \left[kx^{k-1}\left(\frac{n}{n+1}-x\right)^{n-k} - (n-k)x^k\left(\frac{n}{n+1}-x\right)^{n-k-1}\right]$$

$$\times f\left(\frac{k}{n+1}\right)$$

$$= \left(\frac{1+n}{n}\right)^n \sum_{k=1}^{n} \frac{n!}{(k-1)!\,(n-k)!} x^{k-1}\left(\frac{n}{n+1}-x\right)^{n-k} f\left(\frac{k}{n+1}\right)$$

$$- \left(\frac{1+n}{n}\right)^n \sum_{k=0}^{n-1} \frac{n!}{(k)!\,(n-k-1)!} x^{k}\left(\frac{n}{n+1}-x\right)^{n-k-1} f\left(\frac{k}{n+1}\right)$$

$$= \left(\frac{1+n}{n}\right)^n \sum_{k=0}^{n-1} \frac{n(n-1)!}{k!\,(n-k-1)!} x^{k}\left(\frac{n}{n+1}-x\right)^{n-k-1} f\left(\frac{k+1}{n+1}\right)$$

$$- \left(\frac{1+n}{n}\right)^n \sum_{k=0}^{n-1} \frac{n(n-1)!}{k!\,(n-k-1)!} x^{k}\left(\frac{n}{n+1}-x\right)^{n-k-1} f\left(\frac{k}{n+1}\right)$$

$$= \left(\frac{1+n}{n}\right)^n \sum_{k=0}^{n-1} n\binom{n-1}{k} x^{k}\left(\frac{n}{n+1}-x\right)^{n-1-k} \left[f\left(\frac{k+1}{n+1}\right) - f\left(\frac{k}{n+1}\right)\right] \qquad (5)$$

$$= \frac{n!}{(n-1)!}\left(\frac{1+n}{n}\right)^n \sum_{k=0}^{n-1} \binom{n-1}{k} x^{k}\left(\frac{n}{n+1}-x\right)^{n-1-k} \Delta_{\frac{1}{n+1}} f\left(\frac{k}{n+1}\right)$$

Again differentiating

$$[V_n^*(f;x)]''$$

$$= \left(\frac{1+n}{n}\right)^n \sum_{k=0}^{n-1} \binom{n-1}{k} [kx^{k-1}\left(\frac{n}{n+1}-x\right)^{n-k-1}$$

$$-(n-k-1)x^k\left(\frac{n}{n+1}-x\right)^{n-k-2}]\Delta f\left(\frac{k}{n+1}\right)$$

$$= n\left(\frac{1+n}{n}\right)^n \sum_{k=1}^{n-1} \frac{(n-1)!}{(k-1)!\,(n-k-1)!} x^{k-1}\left(\frac{n}{n+1}-x\right)^{n-k-1} \Delta f\left(\frac{k}{n+1}\right)$$

$$-n\left(\frac{1+n}{n}\right)^n \sum_{k=0}^{n-2} \frac{(n-1)!}{(k)!\,(n-k-2)!} x^{k}\left(\frac{n}{n+1}-x\right)^{n-k-2} \Delta f\left(\frac{k}{n+1}\right)$$

$$= n\left(\frac{1+n}{n}\right)^n \sum_{k=0}^{n-2} \frac{(n-1)(n-2)!}{k!\,(n-k-2)!} x^{k}\left(\frac{n}{n+1}-x\right)^{n-k-2} \Delta f\left(\frac{k+1}{n+1}\right)$$

$$-n\left(\frac{1+n}{n}\right)^n \sum_{k=0}^{n-2} \frac{(n-1)(n-2)!}{k!\,(n-k-2)!} x^{k}\left(\frac{n}{n+1}-x\right)^{n-k-2} \Delta f\left(\frac{k}{n+1}\right)$$

$$= n(n-1)\left(\frac{1+n}{n}\right)^n \sum_{k=0}^{n-2} \binom{n-2}{k} x^k \left(\frac{n}{n+1}-x\right)^{n-2-k} \left[\Delta f\left(\frac{k+1}{n+1}\right)\right.$$

$$\left.-\Delta f\left(\frac{k}{n+1}\right)\right]$$

$$= \frac{n!}{(n-2)!}\left(\frac{1+n}{n}\right)^n \sum_{k=0}^{n-2} \binom{n-2}{k} x^k \left(\frac{n}{n+1}-x\right)^{n-2-k} \Delta^2_{\frac{1}{n+1}} f\left(\frac{k}{n+1}\right)$$

So by induction we have

$$[V_n^*(f;x)]^m = \frac{n!}{(n-m)!}\left(\frac{1+n}{n}\right)^n \sum_{k=0}^{n-m} \binom{n-m}{k} x^k \left(\frac{n}{n+1}-x\right)^{n-m-k} \times \Delta^m_{\frac{1}{n+1}} f\left(\frac{k}{n+1}\right)$$

m=0,1,...,n. Therefore if *f* is *m*-convex , then $\Delta^m f(x) \geq 0$ and hence $V_n^*(f)$ is also *m*-convex.

□

Next theorem is a direct consequence of previous theorem.

Theorem 3.2. *If f(x) is non–decreasing on* $\left[0,\frac{n}{n+1}\right]$ *, then for each* $n \in \mathbb{N}$, $V_n^*(f;x)$ *are also non-decreasing .*

Proof. If *f*(*x*) is non–decreasing on $\left[0,\frac{n}{n+1}\right]$, then from equation (5) it follows that $[V_n^*(f;x)]' \geq 0$. And hence for each $n \in \mathbb{N}$, $V_n^*(f;x)$ are also nondecreasing.

□

In [6] it was proved that Bernstein polynomials preserve starshapedness of the function. A function $f:[0,1] \to \mathbb{R}$ is called starshaped on [0,1] if, $f(\lambda x) \leq \lambda f(x), \forall \lambda \in [0,1], x \in [0,1]$. If there exist $f'(x)$ on [0,1], *f*(0) = 0, $f(x) \geq 0$, $x \in [0,1]$ then the starshapedness can be expressed by the differential inequality ,

$$xf'(x) - f(x) \geq 0, \forall x \in [0,1]$$

Now we show that operators V_n^* also preserve starshapedness on $\left[0,\frac{n}{n+1}\right], \forall n \in \mathbb{N}$.

Theorem 3.3. *If* $f : \left[0,\frac{n}{n+1}\right] \to \mathbb{R}$ *satisfy f*(0) = 0, $f(x) \geq 0, x \in \left[0,\frac{n}{n+1}\right]$ *and f is starshaped on* $\left[0,\frac{n}{n+1}\right]$*, then* $(V_n^* f)(0) = 0, (V_n^* f)(x) \geq 0,\ x \in \left[0,\frac{n}{n+1}\right]$ *and* $(V_n^* f)$ *is starshaped on* $\left[0,\frac{n}{n+1}\right], \forall n \in \mathbb{N}$.

Proof. From equation (3) we have

$$(V_n^* f;x) = \left(\frac{1+n}{n}\right)^n \left[\binom{n}{0}\left(\frac{n}{n+1}-x\right)^n f(0) + \sum_{k=1}^{n} \binom{n}{k} x^k \left(\frac{n}{n+1}-x\right)^{n-k} \times f\left(\frac{k}{n+1}\right)\right] \quad (6)$$

If *f*(0) = 0 then clearly from equation (6) , we see that $(V_n^* f)(0) = 0$. Also it is evident from definition of the operator (3) that, if $f(x) \geq 0, x \in \left[0,\frac{n}{n+1}\right]$ then $(V_n^* f)(x) \geq 0, \forall x \in \left[0,\frac{n}{n+1}\right], \forall n \in \mathbb{N}$.

Now from equation (5) we have

$$[V_n^*(f;x)]' = \left(\frac{1+n}{n}\right)^n \sum_{k=0}^{n-1} n\binom{n-1}{k} x^k \left(\frac{n}{n+1}-x\right)^{n-1-k} \left[f\left(\frac{k+1}{n+1}\right) - f\left(\frac{k}{n+1}\right)\right] \quad (7)$$

By definition of the operator we have

$$\frac{V_n^*(f;x)}{x} = \left(\frac{1+n}{n}\right)^n \sum_{k=1}^{n} \frac{n!}{k!\,(n-k)!} x^{k-1}\left(\frac{n}{n+1}-x\right)^{n-k} f\left(\frac{k}{n+1}\right)$$

$$= \left(\frac{1+n}{n}\right)^n \sum_{k=0}^{n-1} \frac{n(n-1)!}{(k+1)k!\,(n-1-k)!} x^{k}\left(\frac{n}{n+1}-x\right)^{n-1-k} f\left(\frac{k+1}{n+1}\right)$$

$$= \left(\frac{1+n}{n}\right)^n \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{n}{(k+1)} x^{k}\left(\frac{n}{n+1}-x\right)^{n-1-k} f\left(\frac{k+1}{n+1}\right) \qquad (8)$$

So that from eqs. (7) and (8) we have

$$[V_n^*(f;x)]' - \frac{V_n^*(f;x)}{x} = \left(\frac{1+n}{n}\right)^n \sum_{k=0}^{n-1} n\binom{n-1}{k} x^{k}\left(\frac{n}{n+1}-x\right)^{n-1-k}$$

$$\times\left[\left(\frac{k}{k+1}\right) f\left(\frac{k+1}{n+1}\right) - f\left(\frac{k}{n+1}\right)\right]$$

$$= \left(\frac{1+n}{n}\right)^n \sum_{k=0}^{n-1} n\binom{n-1}{k} x^{k}\left(\frac{n}{n+1}-x\right)^{n-1-k} A_{n,k}(f) \qquad (9)$$

If f is starshaped then $f(\lambda x) \leq \lambda f(x), 0 \leq \lambda \leq 1$.Then

$$\left(\frac{k}{k+1}\right) f\left(\frac{k+1}{n+1}\right) \geq f\left(\frac{k}{k+1}.\frac{k+1}{n+1}\right) = f\left(\frac{k}{n+1}\right) \qquad (10)$$

So $A_{n,k}(f) = \left(\frac{k}{k+1}\right) f\left(\frac{k+1}{n+1}\right) - f\left(\frac{k}{n+1}\right) \geq 0$. Hence from equation (9) we have

$$[V_n^*(f;x)]' - \frac{V_n^*(f;x)}{x} \geq 0$$

Hence $V_n^*(f)$ is starshaped on $\left[0,\frac{n}{n+1}\right], \forall n \in \mathbb{N}$. □

IV. Some Other Properties of V_n^*

In this section firstly we prove that like Bernstein operators [See [5]] , operators V_n^* are also variation diminishing. Recall that a variation diminishing operator L_n has the property

$$V[L_n f] \leq V[f]$$

where V[f] is the total variation of f as x varies across its domain.

Theorem 4.1. *For each $n \in \mathbb{N}$ and $f \in C\left[0,\frac{n}{n+1}\right]$, V_n^* is variation diminishing.*

Proof. By definition of variation of a function and equation (5)

$$V[V_n^* f] = \int_0^{\frac{n}{n+1}} |[V_n^*(f;x)]'|dx$$

$$\leq n\left(\frac{1+n}{n}\right)^n \int_0^{\frac{n}{n+1}} \sum_{k=0}^{n-1} \binom{n-1}{k} x^{k}\left(\frac{n}{n+1}-x\right)^{n-1-k} \left|f\left(\frac{k+1}{n+1}\right) - f\left(\frac{k}{n+1}\right)\right| dx$$

On substituting $= \left(\frac{n}{n+1}\right)t$, the previous inequality takes the following form

$$V[V_n^* f] \leq n \sum_{k=0}^{n-1} \binom{n-1}{k} \left|f\left(\frac{k+1}{n+1}\right) - f\left(\frac{k}{n+1}\right)\right| \int_0^1 t^k(1-t)^{n-1-k}\,dt$$

$$= n \sum_{k=0}^{n-1} \binom{n-1}{k} B(k+1, n-k) \left|f\left(\frac{k+1}{n+1}\right) - f\left(\frac{k}{n+1}\right)\right|$$

$$= n\sum_{k=0}^{n-1}\frac{(n-1)!}{k!\,(n-1-k)!}\cdot\frac{k!\,(n-1-k)!}{n!}\left|f\left(\frac{k+1}{n+1}\right)-f\left(\frac{k}{n+1}\right)\right|$$
$$=\sum_{k=0}^{n-1}\left|f\left(\frac{k+1}{n+1}\right)-f\left(\frac{k}{n+1}\right)\right|$$
$$= V[f]$$

Thus the result follows. □

Now we give two more properties(see [10]) of these modified new Bernstein type operators.

Theorem 4.2. *If* $f(x)$ *is a non-negative function such that* $x^{-1}f(x)$ *is non-increasing on* $\left(0,\frac{n}{n+1}\right]$ *, then for each* $n \geq 1$ *,* $x^{-1}V_n^* f(x)$ *is also non-increasing.*

*Proof***.** Suppose $f(x)$ is a non-negative function such that $x^{-1}f(x)$ is non-increasing on $\left(0,\frac{n}{n+1}\right]$. Then for $n \geq 1$

$$\frac{d}{dx}[x^{-1}V_n^*(f;x)\ ]$$
$$=\frac{d}{dx}\left[x^{-1}\sum_{k=0}^{n}p_{n,k}(x)f\left(\frac{k}{n+1}\right)\right]$$
$$=\left(\frac{1+n}{n}\right)^n\left[\sum_{k=1}^{n}\binom{n}{k}f\left(\frac{k}{n+1}\right)\frac{d}{dx}\left\{x^{k-1}\left(\frac{n}{n+1}-x\right)^{n-k}\right\}\right.$$
$$\left.+\frac{d}{dx}\left\{x^{-1}\left(\frac{n}{n+1}-x\right)^{n}\right\}f(0)\right]$$
$$=\left(\frac{1+n}{n}\right)^n\sum_{k=1}^{n}\frac{n!}{k!\,(n-k)!}f\left(\frac{k}{n+1}\right)\left\{(k-1)x^{k-2}\left(\frac{n}{n+1}-x\right)^{n-k}\right.$$
$$\left.-x^{k-1}(n-k)\left(\frac{n}{n+1}-x\right)^{n-k-1}\right\}-\left(\frac{1+n}{n}\right)^n\frac{f(0)}{x^2}\left(\frac{n}{n+1}-x\right)^{n}$$
$$-\left(\frac{1+n}{n}\right)^n\frac{nf(0)}{x}\left(\frac{n}{n+1}-x\right)^{n-1}$$
$$=\left(\frac{1+n}{n}\right)^n\sum_{k=1}^{n-1}\left(\frac{n}{n+1}\right)\binom{n-1}{k}\frac{k(n+1)}{k+1}\,f\left(\frac{k+1}{n+1}\right)x^{k-1}\left(\frac{n}{n+1}-x\right)^{n-k-1}$$
$$-\left(\frac{1+n}{n}\right)^n\sum_{k=1}^{n-1}n\binom{n-1}{k}\,f\left(\frac{k}{n+1}\right)x^{k-1}\left(\frac{n}{n+1}-x\right)^{n-k-1}$$
$$-\left(\frac{1+n}{n}\right)^n\frac{f(0)}{x}\left(\frac{n}{n+1}-x\right)^{n-1}\left\{\left(\frac{n}{n+1}-x\right)\frac{1}{x}+n\right\}$$
$$=\left(\frac{1+n}{n}\right)^{n-1}\sum_{k=1}^{n-1}\left[\left(\frac{k+1}{n+1}\right)^{-1}f\left(\frac{k+1}{n+1}\right)-\sum_{k=1}^{n-1}\left(\frac{k}{n+1}\right)^{-1}f\left(\frac{k}{n+1}\right)\right]k\binom{n-1}{k}$$
$$\times x^{k-1}\left(\frac{n}{n+1}-x\right)^{n-k-1}-\left(\frac{1+n}{n}\right)^n\frac{f(0)}{x}\left(\frac{n}{n+1}-x\right)^{n-1}\{n$$
$$+\left(\frac{n}{n+1}-x\right)\frac{1}{x}\}$$

This is non-positive by assumption. Hence $x^{-1}f(x)$ is non-increasing. □

Recall that a function ω(t) on [0, 1] is called a modulus of continuity if ω(t) is continuous, non- decreasing, semi-additive, and $\lim_{t\to 0+} \omega(t) = \omega(0) = 0$.

Theorem 4.3. *If ω(t) is a modulus of continuity then for each $n \geq 1, V_n^*(\omega; t)$ is also a modulus of continuity.*

*Proof***.** For any modulus of continuity ω(t) and $n \geq 1$, we see that

$$\lim_{t\to 0} V_n^*(\omega; t) = V_n^*(\omega; 0) = \omega(0) = 0$$

and $V_n^*(\omega; t)$ is continuous and non-decreassing. Let $x_1 \leq x_2$ be any two points in $\left[0, \frac{n}{n+1}\right]$ where $\frac{n}{n+1} \geq \max\{x_1, x_2\}$. Then following [1] we have from equation (3)

$$V_n^*(f; x_2) = \sum_{j=0}^{n} \left(\frac{1+n}{n}\right)^n \binom{n}{j} \left(x_1 + (x_2 - x_1)\right)^j \left(\frac{n}{n+1} - x_2\right)^{n-j} f\left(\frac{j}{n+1}\right)$$

$$= \sum_{j=0}^{n} \left(\frac{1+n}{n}\right)^n \binom{n}{j} \left(\frac{n}{n+1} - x_2\right)^{n-j} f\left(\frac{j}{n+1}\right) \times \left\{\sum_{k=0}^{j} \binom{j}{k} x_1^k (x_2 - x_1)^{j-k}\right\}$$

$$= \sum_{j=0}^{n} \sum_{k=0}^{j} \left(\frac{1+n}{n}\right)^n \frac{n!\, x_1^k\, (x_2 - x_1)^{j-k}}{k!\,(n-j)!\,(j-k)!} \left(\frac{n}{n+1} - x_2\right)^{n-j} f\left(\frac{j}{n+1}\right)$$

On inverting the order of summation and writing $k + l = j$, then

$$V_n^*(f; x_2) = \sum_{k=0}^{n} \sum_{l=0}^{n-k} \left(\frac{1+n}{n}\right)^n \frac{n!\, x_1^k\, (x_2 - x_1)^l}{k!\, l!\,(n-k-l)!} \left(\frac{n}{n+1} - x_2\right)^{n-k-l} f\left(\frac{k+l}{n+1}\right)$$

(11)

Again from (3)

$$V_n^*(f; x_1) = \sum_{k=0}^{n} \left(\frac{1+n}{n}\right)^n \binom{n}{k} x_1^k \left(\frac{n}{n+1} - x_1\right)^{n-k} f\left(\frac{k}{n+1}\right)$$

$$= \sum_{k=0}^{n} \left(\frac{1+n}{n}\right)^n \binom{n}{k} x_1^k \left(x_2 - x_1 + \left(\frac{n}{n+1} - x_2\right)\right)^{n-k} f\left(\frac{k}{n+1}\right)$$

$$= \sum_{k=0}^{n} \sum_{l=0}^{n-k} \left(\frac{1+n}{n}\right)^n \binom{n}{k} \binom{n-k}{l} x_1^k\, (x_2 - x_1)^l \left(\frac{n}{n+1} - x_2\right)^{n-k-l} \times f\left(\frac{k}{n+1}\right)$$

$$= \sum_{k=0}^{n} \sum_{l=0}^{n-k} \left(\frac{1+n}{n}\right)^n \frac{n!\, x_1^k\, (x_2 - x_1)^l}{k!\, l!\,(n-k-l)!} \left(\frac{n}{n+1} - x_2\right)^{n-k-l} f\left(\frac{k}{n+1}\right)$$

(12)

From (11) and (12) using semi-additivity of $\omega(t)$ we have, for $0 \leq t_1 < t_2 \leq \frac{n}{n+1}$ and $t_1 + t_2 \leq \frac{n}{n+1}$

$$V_n^*(\omega; t_1) - V_n^*(\omega; t_2)$$

$$= \sum_{k=0}^{n} \sum_{l=0}^{n-k} \left(\frac{1+n}{n}\right)^n \frac{n!\, t_1^k\, (t_2 - t_1)^l}{k!\, l!\,(n-k-l)!} \left(\frac{n}{n+1} - t_2\right)^{n-k-l} \left[\omega\left(\frac{k+l}{n+1}\right) - \omega\left(\frac{k}{n+1}\right)\right]$$

$$\leq \sum_{k=0}^{n}\sum_{l=0}^{n-k}\left(\frac{1+n}{n}\right)^{n}\frac{n!\,t_1^k\,(t_2-t_1)^l}{k!\,l!\,(n-k-l)!}\left(\frac{n}{n+1}-t_2\right)^{n-k-l}\left[\omega\left(\frac{l}{n+1}\right)\right]$$

$$=\sum_{l=0}^{n}\left(\frac{1+n}{n}\right)^{n}\frac{(t_2-t_1)^l n!}{l!\,(n-l)!}\,\omega\left(\frac{l}{n+1}\right)\left[\sum_{k=0}^{n-l}\binom{n-l}{k}t_1^k\left(\frac{n}{n+1}-t_2\right)^{n-l-k}\right]$$

$$=\sum_{l=0}^{n}\left(\frac{1+n}{n}\right)^{n}\binom{n}{l}(t_2-t_1)^l\left(\frac{n}{n+1}-t_2+t_1\right)^{n-l}\omega\left(\frac{l}{n+1}\right)$$

$$=V_n^*(\omega;t_2-t_1)$$

This shows that $V_n^*(\omega;t)$ is semi-additive and hence $V_n^*(\omega;t)$ is a modulus of continuity. □

V. Rate of Convergence of $V_n^*(f)$

In this section we compute rates of convergence of the operators $V_n^* f$ by the means of first and second modulus of continuities. Let $f\in C[0,a]$. The modulus of continuity of *f* denoted by $\omega(f,\delta)$ is defined to be

$$\omega(f,\delta)=\sup_{|s-x|<\delta, x\in[0,a]}|f(s)-f(x)|$$

The modulus of continuity of the function f in C[0, *a*] gives the maximum oscillation of f in any interval of length not exceeding $\delta > 0$. It is well known that a necessary and sufficient condition for a function f to be in C[0, *a*] is

$$\lim_{\delta\to 0}\omega(f,\delta)=0$$

It is also well-known that for any $\delta > 0$ we have

$$|f(s)-f(x)|\leq \omega(f,\delta)\left(\frac{|s-x|}{\delta}+1\right) \qquad (13)$$

Theorem 5.1. *Let* $f\in C\left[0,\frac{n}{n+1}\right]$, $V_n^*(f;x)$ *be given by* (3), *then*

$$||V_n^*(f;\cdot)-f||\leq\frac{3}{2}\,\omega(f,\delta_n)$$

where $\delta_n=n^{-1/2}$.

Proof. For the proof we use similar technique of Popoviciu [8]. Let $\omega(f,\delta)$ denote the modulus of continuity of function *f* , then we can write for any $\delta > 0$

$$|f(t)-f(x)|\leq \omega(f,\delta)\left(\frac{|t-x|}{\delta}+1\right) \qquad (14)$$

Now using linearity and positivity of $V_n^*(f;x)$ and (14), for $n\in\mathbb{N}$ and $x\in\left[0,\frac{n}{n+1}\right]$ we have

$$|V_n^*(f;x)-f(x)|\leq V_n^*(|f(t)-f(x)|;x)$$

$$\leq\omega(f,\delta)\left(\frac{V_n^*(|t-x|;x)}{\delta}+1\right)$$

$$\leq\omega(f,\delta)\left(\frac{(V_n^*((t-x)^2;x))^{1/2}}{\delta}+1\right)$$

$$=\omega(f,\delta)\left(\frac{1}{\delta}\sqrt{\frac{x}{n}\left(\frac{n}{n+1}-x\right)}+1\right)$$

We see that the maximum value of $\frac{x}{n}\left(\frac{n}{n+1}-x\right)$ in the interval $\left[0,\frac{n}{n+1}\right]$ is $\frac{n}{4(n+1)^2}$, so we have

$$\|V_n^*(f;\cdot)-f(x)\|\leq\omega(f,\delta)\left(\frac{1}{\delta}\frac{\sqrt{n}}{2(n+1)}+1\right)$$

$$= \omega(f,\delta)\left(\frac{n}{2\delta n^{1/2}(n+1)} + 1\right)$$
$$\leq \omega(f,\delta)\left(\frac{1}{2\delta n^{1/2}} + 1\right)$$

If we choose $\delta = \delta_n = n^{-1/2}$, then

$$||V_n^*(f;\cdot) - f|| \leq \frac{3}{2}\,\omega(f,\delta_n). \qquad \square$$

Ditzian in [3] gave a direct estimate for the Bernstien polynomials using Ditzian - Totik moduli of smoothness. Motivated from it we give a direct theorem
for the operators in equation (3). For this we give some notations. Let $C\left[0,\frac{n}{n+1}\right]$ be the set of continuous and bounded functions on $\left[0,\frac{n}{n+1}\right]$. Ditzian-Totik moduli of smoothness is given by

$$\omega_{\varphi^\lambda}^2(f,t) = \sup_{0,h\leq t}\ \sup_{x\pm h\varphi^\lambda(x)\in\left[0,\frac{n}{n+1}\right]} \left|f\left(x - h\varphi^\lambda(x)\right) - 2f(x) + f\left(x + h\varphi^\lambda(x)\right)\right|$$

(15)

where $\varphi(x)^2 = x\left(\frac{n}{n+1} - x\right)$ and K-functional

$$K_{\varphi^\lambda}(f,t^2) = \inf\left\{\|f-g\|_{C\left[0,\frac{n}{n+1}\right]} + t^2\left\|\varphi^{2\lambda}g''\right\|_{C\left[0,\frac{n}{n+1}\right]}\right\} \qquad (16)$$

where infimum is taken on functions satisfying $g, g' \in A.C_{loc}$.

It is well known (see [4], Theorem 3.1.2) that $\omega_{\varphi^\lambda}^2(f,t)$ is equivalent to $K_{\varphi^\lambda}(f,t^2)$. That means there exists C > 0 such that

$$C^{-1}K_{\varphi^\lambda}(f,t^2) \leq \omega_{\varphi^\lambda}^2(f,t) \leq CK_{\varphi^\lambda}(f,t^2) \qquad (17)$$

Theorem 5.2. *If* $f \in C\left[0,\frac{n}{n+1}\right], 0 \leq \lambda \leq 1$, *then we have*

$$|V_n^*(f;x) - f(x)| \leq C\omega_{\varphi^\lambda}^2(f, n^{-\frac{1}{2}}\varphi(x)^{1-\lambda}) \qquad (18)$$

where $\varphi(x)^2 = x\left(\frac{n}{n+1} - x\right)$.

Proof. Using (16) and (17), we may choose $g = g_{n,x,\lambda}$ for a fixed *x* and λ such that

$$\|f-g\| \leq A\omega_{\varphi^\lambda}^2\left(f, n^{-\frac{1}{2}}\varphi(x)^{1-\lambda}\right) \qquad (19)$$

$$n^{-1}\varphi(x)^{2-2\lambda}\left\|\varphi^{2\lambda}g''\right\| \leq B\omega_{\varphi^\lambda}^2\left(f, n^{-\frac{1}{2}}\varphi(x)^{1-\lambda}\right) \qquad (20)$$

So by linearity of $V_n^*(f)$ and using (19),(20) we have

$$|V_n^*(f;x) - f(x)| \leq |V_n^*(f-g;x)| + |f(x) - g(x)| + |V_n^*(g;x) - g(x)|$$
$$\leq 2\|f-g\| + |V_n^*(g;x) - g(x)|$$
$$\leq 2A\omega_{\varphi^\lambda}^2\left(f, n^{-\frac{1}{2}}\varphi(x)^{1-\lambda}\right) + \left|V_n^*(g;x) - g(x)\right| \qquad (21)$$

Using Taylor expansion with integral remainder

$$f(t) = f(x) + f'(x)(t-x) + \int_x^t (t-u)f''(u)du \qquad (22)$$

From([4],p.141),for $t < u < x$, we have

$$\frac{|t-u|}{\varphi(u)^{2\lambda}} \leq \frac{|t-x|}{\varphi(x)^{2\lambda}} \qquad (23)$$

Hence from Lemmas 2.1 and 2.2 and (22) , (23) we get

$$
\begin{aligned}
|V_n^*(g;x)-g(x)| &\le \left|g'(x)\right|\left|V_n^*(t-x;x)\right| + \left|V_n^*\left(\int_x^t (t-u)g''(u)du\,;x\right)\right| \\
&\le V_n^*\left(\frac{|t-x|}{\varphi(x)^{2\lambda}}\left|\int_x^t \varphi(u)^{2\lambda} g''(u)du\right|;x\right) \\
&\le \left\|\varphi^{2\lambda}g''\right\| V_n^*\left(\frac{(t-x)^2}{\varphi(x)^{2\lambda}};x\right) \\
&= \left\|\varphi^{2\lambda}g''\right\| \frac{n^{-1}\varphi(x)^2}{\varphi(x)^{2\lambda}} \\
&= n^{-1}\varphi(x)^{2-2\lambda}\left\|\varphi^{2\lambda}g''\right\| \\
&\le B\omega_{\varphi^\lambda}^2\left(f,n^{-\frac{1}{2}}\varphi(x)^{1-\lambda}\right) \qquad (24)
\end{aligned}
$$

From (21) and (24) we get the desired result. □

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