On the Convergence of a Polynomial

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Abstract: We have tested the convergence of a newly defined polynomial on the interval \([0,1 + \frac{r}{n}]\) for Lebesgue integral in \(L_1\) norm as

\[
U^\alpha_{nr}(f, x) = (n + r + 1) \sum_{k=0}^{n+r} \left( \frac{(k+1)/(n+r+1)}{x/(n+r+1)} \right) q_{nr,k}(x; \alpha)
\]

where

\[
q_{nr,k}(x; \alpha) = \binom{n + r}{k} x^{(x+ka)/(1-x)}(1-x(y+(n-r-k)\alpha))^{n+r-k-1}
\]

This result shows that all polynomials of Bernstein type as the modified polynomials, i.e., (1.3) gives an interesting property of Bernstein polynomials. Therefore, this result is also extended to the well-known Bernstein operator. Bernstein (1912-13) proved that if \(f(x)\) is continuous in closed interval \([0,1]\), then

\[
B_n^f(x) \to f(x)\text{ uniformly as } n \to \infty.
\]

This yields a simple constructive proof of Weierstrass’s approximation theorem.

Keywords: Bernstein Polynomials, Convergence, Bernstein type Polynomial, L_1 norm, Lebesgue integrable function

I. Introduction and Results

If \(f(x)\) is a function defined \([0,1]\), the Bernstein polynomial \(B_n^f(x)\) of \(f\) is given as

\[
B_n^f(x) = \sum_{k=0}^{n} f(k/n) p_{n,k}(x) \quad \ldots \ldots \quad (1.1)
\]

where

\[
p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \quad \ldots \ldots \quad (1.2)
\]

Schurer [6] introduced an operator

\[
S_{w, c} : [0, 1+\frac{w}{n}] \to [0, 1]
\]

defined by

\[
S_{w, c}(f, x) = \sum_{k=0}^{n+r} f\left( \frac{k}{n+r}\right) p_{n,k}(x)
\]

where

\[
p_{n,k}(x) = \binom{n + r}{k} x^{x+ka} (1-x)^{n+r-k} \quad \ldots \ldots \quad (1.4)
\]

and \(n\) is a non-negative integer. In case \(n = 0\), this reduces to the well-known Bernstein operator. Bernstein (1912-13) proved that if \(f(x)\) is continuous in closed interval \([0,1]\), then

\[
B_n^f(x) \to f(x)\text{ uniformly as } n \to \infty.
\]

This yields a simple constructive proof of Weierstrass’s approximation theorem.

A slight modification of Bernstein polynomials due to Kantorovich [8] makes it possible to approximate Lebesgue integrable function in \(L_1\) norm by the modified polynomials

\[
p_{n,k}(x) = \sum_{k=0}^{n} f\left( \frac{k/(n+1)}{x/(n+1)}\right) dt \prod_{n,k}(x) \quad \ldots \ldots \quad (1.5)
\]

where

\[
B_n^f(x) \to f(x)\text{ uniformly as } n \to \infty
\]

By Abel’s formula (see Jensen [7]) defined on \([0,1]\) is given by

\[
(x+y)(x+y+n\alpha)^{-1} = \sum_{k=0}^{n} \binom{n}{k} x^{x+ka} (1-x)^{n-r-k} y(y+(n-r-k)\alpha)^{n-r-k-1} \quad \ldots \ldots \quad (1.6)
\]

Which can be defined on \([0,1+\frac{r}{n}]\) as

\[
(x+y)(x+y+(n+r)\alpha)^{n+r-1} = \sum_{k=0}^{n+r} \binom{n+r}{k} x^{x+ka} (1-x)^{n+r-k} y(y+(n-r-k)\alpha)^{n+r-k-1} \quad \ldots \ldots \quad (1.7)
\]

If we put \(y = 1 - x\), we obtain (see Cheney and Sharma [5])

\[
1 = \sum_{k=0}^{n+r} \binom{n+r}{k} x^{x+ka} (1-x)^{1-x+\alpha} (1-x)^{1-x+\alpha} x^{x+ka} (1-x)^{1-x+\alpha} \quad \ldots \ldots \quad (1.8)
\]

Thus defining

\[
q_{nr,k}(x; \alpha) = \binom{n + r}{k} x^{x+ka} (1-x+\alpha) y(y+(n-r-k)\alpha)^{n+r-k-1} \quad \ldots \ldots \quad (1.9)
\]
we have
\[ \sum_{k=0}^{n+r} q_{nr,k}(x; \alpha) = 1. \] 

(1.10)

For a finite interval \([0, 1+\frac{r}{n}]\), we now define a Bernstein type Polynomials (see Anwar Habib [2])

\[ U_{n,c} : c [0,1+\frac{r}{n}] \rightarrow \mathbb{C}[0,1] \]

by
\[ U_{nr}^\alpha (f, x) = (n + r + 1) \sum_{k=0}^{n+r} \left\{ \int_{k/(n+r+1)}^{(k+1)/(n+r+1)} f(t) dt \right\} q_{nr,k}(x; \alpha) \] 

(1.11)

where \( q_{nr,k}(x; \alpha)\) same as (1.9) and \( r \) is a non-negative integer. When \( r=0 \) \& \( \alpha = 0 \), then (1.9) \\& (1.11) reduces to (1.2) \\& (1.5) respectively.

In this paper we shall test the convergence of the polynomial (1.11) for Lebesgueintegrable function in \( L_1 \) norm.

In fact our result is as follows

**Theorem:** If \( f(x) \) is continuous Lebesgueintegrable function \([0, 1+\frac{r}{n}]\) then \( \alpha = \alpha_{nr} = \alpha \left( \frac{1}{n+r} \right) \)

\[ \lim_{(nr)^+ \rightarrow \infty} U_{nr}^\alpha (f, x) = f(x) \]

holds uniformly on \([0, 1 + \frac{r}{n}]\)

**II. Lemmas**

In order to prove our result we need the following lemmas (see Anwar Habib [2])

**Lemma 2.1:** For all values of \( x \)

\[ \sum_{k=0}^{n+r} k q_{nr,k}(x; \alpha) \leq \frac{1 + (n + r)\alpha}{1 + \alpha} \left( n + r \right) x - \frac{1 + (n + r)\alpha}{1 + \alpha} \left( n + r \right) (n + r - 1) x \alpha + \frac{1}{1 + 2\alpha} \]

**Lemma 2.2:** For all values of \( x \)

\[ \sum_{k=0}^{n+r} k (k-1) q_{nr,k}(x; \alpha) \leq (n + r)(n + r - 1)(x + 2\alpha)(1 + (n + r)\alpha)^2 - \frac{(n + r - 2)\alpha}{1 + 3\alpha} \]

\[ + (n + r - 2)\alpha^2 \left( \frac{(n + r - 2)\alpha}{1 + 3\alpha} \right) \]

**Lemma 2.3:** For all values of \( x \) of \([ 0, 1+\frac{r}{n}]\) and for \( \alpha = \alpha_{nr} = 0 \left( \frac{1}{n+r} \right) \), we have

\[ (n + r + 1) \sum_{k=0}^{n+r} \left\{ \int_{k/(n+r+1)}^{(k+1)/(n+r+1)} ((t-x)\alpha)q_{nr,k}(x; \alpha) \leq \frac{x(1-x)}{n+r} \]

**III. Proof of the theorem**

\[ |U_{nr}^\alpha (f, x) - f(x)| \leq (n + r + 1) \sum_{k=0}^{n+r} \left\{ \int_{k/(n+r+1)}^{(k+1)/(n+r+1)} |f(t) - f(x)| dt \right\} q_{nr,k}(x; \alpha) \]

Now splitting above inequality into two parts corresponding to those values of \( t \) for which \( |t-x| \leq \delta \) and those for which \( |t-x| \geq \delta \), we get

\[ \leq (n + r + 1) \sum_{|t-x| \leq \delta} \left\{ \int_{k/(n+r+1)}^{(k+1)/(n+r+1)} |f(t) - f(x)| dt \right\} q_{nr,k}(x; \alpha) \]

\[ + (n + r + 1) \sum_{|t-x| \geq \delta} \left\{ \int_{k/(n+r+1)}^{(k+1)/(n+r+1)} |f(t) - f(x)| dt \right\} q_{nr,k}(x; \alpha) \]

\[ = I_1 + I_2 \] (say) 

(3.1)

If the function \( f(x) \) is bounded say \( |f(x)| \leq M \) in \( 0 \leq x \leq 1 + \frac{r}{n} \) then \( x \) is a point of continuity, for a given \( \varepsilon > 0 \), \( \exists \) a number \( \delta > 0 \), \( \exists \): \( |x_2 - x_1| < \delta \) implies \( |f(x_2) - f(x_1)| \leq \varepsilon \)

and therefore
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\[ I_1 \leq \frac{\varepsilon}{2} (n + r + 1) \sum_{k=0}^{n+r} \left( \int_{\alpha}^{1} (k+1)/(n+r+1) dt \right) q_{nr,k}(x; \alpha) = \frac{\varepsilon}{2} \]

\[ I_2 \leq 2M(n + r + 1) \sum_{|r|=0}^{n+r+1} (k+1)/(n+r+1) \left( \int_{k/(n+r+1)}^{(k+1)/(n+r+1)} dt \right) q_{nr,k}(x; \alpha) \]

\[ \leq \frac{2M}{\delta^2} (n + r + 1) \sum_{k=0}^{n+r} \left( \int_{k/(n+r+1)}^{(k+1)/(n+r+1)} (t - x)^2 dt \right) q_{nr,k}(x; \alpha) \]

\[ \leq \frac{2M}{4(n+r)\delta^2} \]

by lemma 2.3 and the fact \( x(1 - x) \leq \frac{1}{4} \) on \( [0, 1 + \frac{r}{n}] \) for large \( n \)

On substituting the values of \( I_1 \) & \( I_2 \) in (3.1) we get

\[ \left| U_{\alpha}^{\alpha} (f, x) - f(x) \right| \leq \frac{\varepsilon}{2} + \frac{2M}{4(n+r)\delta^2} \]

For \( \delta = \left( \frac{M}{n+r} \right)^{1/2} \), we get

\[ \left| U_{\alpha}^{\alpha} (f, x) - f(x) \right| < \varepsilon \]

Hence the proof of the theorem.

IV. Conclusion

The result of Bernstein has been extended for Lebesgue integrable function in \( L_1 \)-norm by our newly defined Polynomials \( U_{\alpha}^{\alpha} (f, x) \) on the interval \( [0, 1 + \frac{r}{n}] \).

References