On Some Coefficient Estimates For Certain Subclass of Analytic And Multivalent Functions

M. Elumalai

Department of Mathematics, Presidency College, Chennai – 600 005, Tamilnadu, India.

Abstract: In this paper, motivated by the works of Jenkins [11], Leung [12] and Panigrahi and Murugusundaramoorthy [16] we defined a subclass of p - valent analytic functions using a generalized differential operator and compute coefficient differences. We also point out, as particular cases, the results obtained earlier by various authors.

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I. Introduction and Definition

Let \( A_p \) denote the class of analytic functions in the open unit disk \( U := \{ z \in \mathbb{D} : |z| < 1 \} \) of the form

\[
f(z) = z^n + \sum_{p=1}^{\infty} a_{n+p} z^{n+p} \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\})
\]

and let \( A = A_1 \).

Let \( S \) denote the subclass of \( A_p \) consisting of multivalent functions. A function \( f \in A_p \) given by (1.1) is said to be \( p \)-valently starlike if it satisfies the inequality

\[
\text{Re} \left( \frac{zf'(z)}{pf(z)} \right) > 0, \quad (z \in U).
\]

We denote this class of functions by \( S^*_p \). Note that the class \( S^*_p \) reduces to \( S^*_1 := S^* \), the class of starlike functions in \( U \), introduced by Robertson [17].

A function \( f \in A_p \) is said to be \( p \)-valently convex if it satisfies the condition

\[
\text{Re} \left( \frac{1}{p} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right) > 0, \quad (z \in U).
\]

We denote by \( C \) the familiar subclass of \( A_p \). In particular \( p = 1 \), \( C = C \) the class of convex functions in \( U \), introduced by Robertson [17] (also see [4]).

For \( n \geq 2 \), Hayman [9] showed the difference of successive coefficients is bounded by an absolute constant i.e.

\[
\|a_{n+1}|-|a_n\| \leq A.
\]

Using different technique, Milin [15] showed that \( A < 9 \). Ilinna [10] improved this to \( A < 4.26 \). Further, Grispan [8] restricted to \( A < 3.61 \). For starlike function \( S^* \), Leung [12] proved that the best possible bound is \( A = 1 \). On the other hand, it is known that for the class \( S \), \( A \) cannot be reduced to 1. When \( n = 2 \), Golusin [5,6], Jenkins [11] and Duren [4] showed that for \( f \in S \), \( -1 \leq |a_1|-|a_2| \leq 1.029 \) and that both upper and lower bounds in (1.1) are sharp. When \( n = 2 \) and \( n = 3 \), Panigrahi [16] showed that for \( f \in C \), \( |a_1|-|a_2| \leq 0.521 \) and \( |a_1|-|a_2| \leq 0.521 \). Also for \( f \in S^* \), \( |a_1|-|a_2| \leq 1.25 \) and \( |a_1|-|a_2| \leq 2 \) both the inequalities are sharp.

We now define the following differential operator \( D^{\mu}_{\delta,\gamma} : A_p \rightarrow A_p \) by

\[
D^{\mu}_{\delta,\gamma} f(z) = z^n + \sum_{p=1}^{\infty} (n+p)^\mu + (n+p-1)(n+p)^\mu \mu! C(\delta, n, p) a_{n+p} z^{n+p}
\]

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where

$$C(\delta, n, p) = \frac{\Gamma(n + p + \delta)}{\Gamma(n + 1)\Gamma(\delta + p)}.$$ 

and \( j, \alpha \in \mathbb{D}_o := \mathbb{D} \cup \{0\}, p \in \mathbb{D}, \mu, \delta \geq 0 \).

By specializing the parameters \( j, \alpha, \mu, \delta \) and \( p \) we obtain the following operators studied earlier by various researchers: Namely,

- If \( \alpha = p = 1, \mu = 0, \delta = 0 \) or \( \alpha = \delta = 0, \mu = p = 1 \), the operator \( D_{\mu, \alpha, p}^{i, 0} = D_{\mu, 0, 1}^{i, 0} = D_i \) is the popular Salagean operator [19];
- When \( j = 0, p = 1 \), then \( D_{\mu, \alpha}^{i, 0} \) which is the Ruscheweyh differential operator (see [18]);
- For \( \alpha = 0, \delta = 0, p = 1 \), then \( D_{\mu, 0}^{i, 0} = D_i \) which is the differential operator studied by Al-Oboudi (see [1]);
- If \( \alpha = 0 \) and \( p = 1 \) then \( D_{\mu, 0}^{i, 0} = D_{\mu, \delta}^{i, 0} \) has been studied by Darus and Ibrahim (see [2]);
- When \( p = 1 \), then \( D_{\mu, \delta}^{i, 0} = D_{\mu, \delta}^{i, 0} \) which is the generalized differential operator studied by Panigrahi and Murugu sandramoorthy (see [16]).

Motivated by the above concept, in this paper, making use of the differential operator \( D_{\mu, \alpha, p}^{i, 0} \) we introduce and investigate a new subclass of multivalent functions, as in

**Definition 1.1.** A function \( f \in A_p \) is said to be in the class \( M_{\mu, \alpha, p}^{i, 0}(\alpha) \) if it satisfies the inequality

$$\Re\left\{ \frac{(1-t)z(D_{\mu, \alpha, p}^{i, 0}f(z))^t + tz(D_{\mu, \alpha, p}^{i, 0}f(z))^t}{(1-t)D_{\mu, \alpha, p}^{i, 0}f(z) + tD_{\mu, \alpha, p}^{i, 0}f(z)} \right\} > 0, \quad (z \in U)$$

where \( 0 \leq t \leq 1, j, \alpha \in \mathbb{D}_o, p \in \mathbb{D}, \mu \) and \( \delta \geq 0 \).

Note that by taking \( t = j = \delta = 0 \) and \( t = \alpha = 1, j = \mu = \delta = 0 \) the class \( M_{\mu, \delta, p}^{i, 0}(\alpha) \), reduces the classes \( S_p^* \) and \( C_p \), respectively.

**Remark 1.1.** If \( t = j = \delta = 0, p = 1 \), then \( M_{\mu, \delta, p}^{i, 0}(\alpha) \) reduces to the well-known class of starlike functions in \( U \). Similarly, if we let \( t = \alpha = p = 1, j = \mu = \delta = 0 \) then \( M_{\mu, \delta, p}^{i, 0}(\alpha) \) reduces to the well-known class of convex functions in \( U \).

The purpose of the present study is to estimate the coefficient differences for the function class \( M_{\mu, \alpha, p}^{i, 0}(\alpha) \), when \( n = p + 1 \) and \( n = p + 2 \).

**II. Preliminary Results**

In order to derive our main results, we have to recall the following preliminary lemmas:

Let \( P \) be the family of all functions \( h \) analytic in \( U \), for which \( \Re\{h(z)\} > 0 \) and

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad \forall \ z \in U. \quad (2.1)$$

**Lemma 2.1.** [4] If \( h \in P \), then \( |c_k| \leq 2 \), for each \( k \geq 1 \).

**Lemma 2.2.** [7] The power series for \( h \) given in (2.1) converges in the unit disc \( U \) to a function in \( P \) if and only if the Toeplitz determinants

$$D_k = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_k \\ c_{-1} & 2 & c_1 & \cdots & c_{k-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{-k} & c_{-k+1} & c_{-k+2} & \cdots & 2 \end{vmatrix}, \quad k = 1, 2, 3, \ldots$$

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and \( c_{k} = \overline{c_{k}} \), are all non-negative. These are strictly positive except for \( h(z) = \sum_{k=1}^{\infty} \rho_{k} h_{k} e^{i\theta_{k}} \), \( \rho_{k} > 0 \), \( t_{k} \) real and \( t_{k} \neq t_{j} \) for \( k \neq j \), in this case \( D_{k} > 0 \) for \( k < (m-1) \) and \( D_{k} = 0 \) for \( k \geq m \).

This necessary and sufficient condition due to Caratheodory and Toeplitz can be found in [7].

We may assume without restriction that \( c_{k} > 0 \) and on using [Lemma 2.2], for \( k = 2 \) and \( k = 3 \) respectively, we get

\[
D_{k} = \begin{vmatrix} 2 & c_{1} & c_{2} \\ c_{1} & 2 & c_{1} \\ c_{2} & c_{1} & 2 \end{vmatrix} = \begin{bmatrix} 8 + 2 \Re(c_{1}c_{2}) - 2 |c_{2}|^2 - 4c_{1}^2 \end{bmatrix} \geq 0,
\]

which is equivalent to

\[
2c_{2} = \left\{ c_{1}^2 + x(4 - c_{1}^2) \right\}, \text{ for some } x, \ |x| \leq 1.
\] (2.2)

Then \( D_{k} \geq 0 \) is equivalent to

\[
(4c_{3} - 4c_{1}c_{2} + c_{1}^3)(4 - c_{1}^2) + c_{1}(2c_{2} - c_{1}^2)^2 \leq 2(4 - c_{1}^2)^2 - 2 |2c_{2} - c_{1}^2|^2.
\] (2.3)

From the relations (2.2) and (2.3), after simplifying, we get

\[
4c_{3} = \left\{ c_{1}^3 + 2c_{1}(4 - c_{1}^2)x - c_{1}(4 - c_{1}^2)x^2 + 2(4 - c_{1}^2)(1 - |x|^2)z \right\}
\]

for some real value of \( z \), with \( |z| \leq 1 \). (2.4)

**III. Main Results**

In this section, we prove to estimate the coefficient differences for the function class \( \mathcal{M}_{\mu, \delta, p}(\alpha) \).

**Theorem 3.1.** Let \( f \) given by (1.1) be in the class \( \mathcal{M}_{\mu, \delta, p}(\alpha) \). If \( \frac{(p+2)}{(p+3)} A_{1} \leq A_{2} \leq \frac{(p+2)}{(p+1)} A_{1} \), then

\[
\|a_{p, 2}|-|a_{p, 1}\| \leq \frac{8pA_{1}^2 + (p+1)A_{1}^2}{4p(p+1)A_{1}^2},
\] (3.1)

and

\[
\|a_{p, 3}|-|a_{p, 2}\| \leq \frac{(3p+1)^2 A_{1}^2 + 8p(p+1)A_{1}^2}{4p(p+1)^2 A_{1}^2}.
\] (3.2)

where

\[
A_{1} = (p+1)^{\alpha}(1 + p\mu)^{i(\delta + p)[1 + ((p+1)^{\alpha}(1 + p\mu) - 1)]},
\]

\[
A_{2} = (p+2)^{\alpha}(1 + (p+1)\mu)^{i(\delta + p)[(p+1)^{\alpha}(1 + (p+1)\mu) - 1)]},
\]

and

\[
A_{3} = (p+3)^{\alpha}(1 + (p+2)\mu)^{i(\delta + p)[(p+2)^{\alpha}(1 + (p+2)\mu) - 1)]}.
\]

**Proof:** Let the function \( f(z) \) represented by (1.1) be in the class \( \mathcal{M}_{\mu, \delta, p}(\alpha) \). By geometric interpretation, there exists a function \( h \in \mathcal{P} \) given by (2.1) such that

\[
\frac{(1-t)x(D_{p, \mu, \delta, p}^{1/\alpha}f(z))' + tx(D_{p, \mu, \delta, p}^{1/\alpha}f(z))'}{1-t} = h(z),
\] (3.3)
Replacing \( D_{\mu, \delta, p}^{\alpha} f(z), D_{\mu, \delta, p}^{\alpha + 1} f(z), (D_{\mu, \delta, p}^{\alpha} f(z)')' \) and \( D_{\mu, \delta, p}^{\alpha + 1} f(z) \) by their equivalent expressions and the equivalent expression for \( h(z) \) in series (3.3), we have

\[
(1-t) z (D_{\mu, \delta, p}^{\alpha} f(z))' + tz (D_{\mu, \delta, p}^{\alpha + 1} f(z))' = h(z) \{(1-t) D_{\mu, \delta, p}^{\alpha} f(z) + t D_{\mu, \delta, p}^{\alpha + 1} f(z)\},
\]

(1-t)\[
\begin{aligned}
&z p^{n-1} + \sum_{n=1}^{\infty} \left[(n+p)(n+p)\mu(n+p-1)(n+p-\mu)\mu' C(\delta, n, p)a_{n+p}\right] + \left[(n+p)(n+p-1)(n+p-\mu)\mu' C(\delta, n, p)a_{n+p}\right] \\
&= (1-t) \left[z p^{n-1} + \sum_{n=1}^{\infty} \left[(n+p)\mu(n+p-1)(n+p-\mu)\mu' C(\delta, n, p)a_{n+p}\right] + \left[(n+p)(n+p-1)(n+p-\mu)\mu' C(\delta, n, p)a_{n+p}\right] \times \left[1 + \sum_{n=1}^{\infty} c_{n} p^{n}\right]
\end{aligned}
\] (3.4)

Equating the coefficients of like power of \( z^{p+1}, z^{p+2} \) and \( z^{p+3} \) respectively on both sides of (3.4), we have

\[
\begin{align*}
(p+1)A_{p+1} a_{p+1} &= c_{1}, \\
(p+2)A_{p+2} a_{p+2} &= c_{2} + c_{1} A_{p+1}, \\
(p+3)A_{p+3} a_{p+3} &= c_{3} + c_{2} A_{p+2} + c_{1} A_{p+1},
\end{align*}
\]

where \( A_{1}, A_{2} \) and \( A_{3} \) are given in the statement of theorem.

After simplifying, we get

\[
|a_{p+1}| = c_{1} / (p+1) A_{p}, a_{p+2} = c_{2} / (p+2) A_{p+1} + c_{1} A_{p+1}, a_{p+3} = c_{3} / (p+3) A_{p+2} + c_{2} A_{p+2} + c_{1} A_{p+1}.
\] (3.5)

and

\[
a_{p+1} = c_{1} / (p+1) A_{p}, a_{p+2} = c_{2} / (p+2) A_{p+1} + c_{1} A_{p+1}, a_{p+3} = c_{3} / (p+3) A_{p+2} + c_{2} A_{p+2} + c_{1} A_{p+1}.
\]

Since,

\[
|a_{p+1} - |a_{p+2}| \leq |a_{p+1} - a_{p+2}|
\]

we need to consider \( |a_{p+2} - a_{p+1}| \) and \( |a_{p+3} - a_{p+2}| \).

Taking into account (3.5) and (2.2) we obtain

\[
|a_{p+2} - a_{p+1}| = \left| c_{2} / (p+1) A_{p} + c_{1} A_{p+1} - c_{1} / p A_{1} \right|
\] (3.6)

We can assume without loss of generality that \( c_{1} > 0. \) For convenience of notation, we take \( c_{1} = c \) \((c \in [0, 2])\) (see Lemma 2.1). Applying triangle inequality and replacing \( |x| \) by \( \eta \) in the right hand side of (3.6) and using the inequality \( A_{1} \leq (p+2) / (p+1) A_{1} \) it reduces to

\[
|a_{p+2} - a_{p+1}| \leq \left| c / p A_{1} - (p+2)c^{2} / 2(p+1) A_{2} + 4c^{2} / 2(p+1) A_{2} \right| \eta
\] (3.7)

where

\[
\chi(c, \eta) = \left| c / p A_{1} - (p+2)c^{2} / 2(p+1) A_{2} + 4c^{2} / 2(p+1) A_{2} \right| \eta.
\] (3.8)
We assume that the upper bound for (3.7) occurs at an interior point of the \( \{(\eta, c) : \eta \in [0,1] \} \) and \( c \in [0,2] \). Differentiating (3.8) partially with respect to \( \eta \), we get

\[
\frac{\partial \chi}{\partial \eta} = \frac{4-c^2}{2(p+1)A_2}.
\]  

(3.9)

From (3.9) we observe that \( \frac{\partial \chi}{\partial \eta} > 0 \) for \( 0 < \eta < 1 \) and for fixed \( c \) with \( 0 < c < 2 \). Therefore \( F(c, \eta) \) is an increasing function of \( \eta \), which contradicts our assumption that the maximum value of \( \chi \) occurs at an interior point of the set \( \{(\eta, c) : \eta \in [0,1] \} \) and \( c \in [0,2] \). So, fixed \( \eta \) with \( 0 < \eta < 1 \) and for fixed \( c \) with \( 0 < c < 2 \), we have

\[
\max \chi(c, \eta) = \chi(c, 1) = \tau(c) \ (\text{say}).
\]

Therefore replacing \( \mu \) by 1 in (3.8), we obtain

\[
\begin{align*}
\tau(c) &= \frac{c}{pA_1} + \frac{2p - (p+1)c^2}{p(p+1)A_2} , \\
\tau'(c) &= \frac{1}{pA_1} - \frac{2c}{pA_2} , \\
\tau''(c) &= -\frac{2}{pA_2} < 0 .
\end{align*}
\]  

(3.10)

(3.11)

(3.12)

For optimum value of \( \tau(c) \), consider \( \tau'(c) = 0 \). It implies that \( c = \frac{A_1}{2A_1} \). Therefore, the maximum value of \( \tau(c) \) is

\[
\frac{8pA_1^2 + (p+1)A_2^2}{4p(p+1)A_1A_2}
\]

which occurs at \( c = \frac{A_1}{2A_1} \), from the expression (3.10), we get

\[
\tau_{\max} = \tau\left(\frac{A_1}{2A_1}\right) = \frac{8pA_1^2 + (p+1)A_2^2}{4p(p+1)A_1A_2}.
\]  

(3.12)

From (3.7) and (3.12), we have

\[
|a_{p+3} - a_{p+2}| \leq \frac{8pA_1^2 + (p+1)A_2^2}{4p(p+1)A_1A_2},
\]

which proves the assertion (3.1) of Theorem 3.1. Using the same technique, we will prove (3.2). From (3.5) and an application of (2.4) we have

\[
|a_{p+3} - a_{p+2}| = \left| \frac{c_1}{(p+2)A_1} \right| + \frac{(2p+1)c_1c_2}{p(p+1)(p+2)A_1} + \frac{c_1^3}{p(p+1)(p+2)A_1} - \frac{c_2}{(p+1)A_2} - \frac{c_1^2}{p(p+1)A_2} \\
+ \frac{1}{4(p+2)A_1} \left[ c_1^2 + 2(4-c_1^2)cx - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z \right] \\
+ \frac{1}{2(p+1)A_1} \left[ c_1^2 + x(4-c_1^2) \right] + \frac{c_1^3}{p(p+1)(p+2)A_1} - \frac{c_1^2}{p(p+1)A_1} \\
+ \frac{1}{2(p+1)A_1} \left[ c_1^2 + x(4-c_1^2) \right] - \frac{c_1^2}{p(p+1)A_1} \\
= \frac{(p+3)c_1^3}{4(p+2)A_1} - \frac{(p+2)c_1^2}{2p(p+1)A_1} + \frac{(p^2 + 3p + 1)c_1}{2p(p+1)(p+2)A_1}(4-c_1^2) \xi \\
- \frac{c_1(4-c_1^2)x^2}{4(p+2)A_1} + \frac{1}{2(p+2)A_1} (4-c_1^2)(1-|x|^2)z \\
- \frac{1}{2(p+1)A_2} (4-c_1^2)x \\
\]  

(3.13)
As earlier, we assume without loss of generality that \( c_i = c \) with \( 0 \leq c \leq 2 \). Applying triangle inequality and replacing \( |x| \) by \( \eta \) in the right hand side of (3.13) and using the fact that \( A_i \leq \frac{p+3}{p+2} A_i \), it reduces to

\[
|a_{p+3} - a_{p+2}| \leq \frac{(p+3)c^2}{4(p+2)A_i} - \frac{(p+2)}{2p(p+1)A_i} c^2 + \frac{(p^2+3p+1)c}{2p(p+1)(p+2)A_i} (4-c^2)\eta
\]

\[
- \frac{c(4-c^2)\eta^2}{4(p+2)A_i} + \frac{1}{2(p+2)A_i} (4-c^2)(1-\eta^2)z
\]

(3.14)

\[
= \xi(c, \eta),
\]

where

\[
\xi(c, \eta) = \frac{(p+3)c^2}{4(p+2)A_i} - \frac{(p+2)}{2p(p+1)A_i} c^2 + \frac{(p^2+3p+1)c}{2p(p+1)(p+2)A_i} (4-c^2)\eta
\]

\[
- \frac{c(4-c^2)\eta^2}{4(p+2)A_i} + \frac{1}{2(p+2)A_i} (4-c^2)(1-\eta^2)z
\]

(3.15)

Suppose that \( \xi(c, \eta) \) in (3.15) attains its maximum at an interior point \((c, \eta)\) of \([0, 2] \times [0, 1]\). Differentiating (3.15) partially with respect to \( \eta \), we have

\[
\frac{\partial \xi}{\partial \eta} = \frac{(p^2+3p+1)c(4-c^2)}{2p(p+1)(p+2)A_i} + \frac{c(4-c^2)\eta}{2(p+2)A_i} - \frac{(4-c^2)\eta}{p(p+1)A_i} + \frac{(4-c^2)}{2(p+1)A_i}
\]

\[
= - \frac{(c^2-4)}{2p(p+1)(p+2)A_i} \left[ c(p^2+3p+1) + p(p+1)\eta - 2p(p+1)\eta + \frac{(p+2)A_i}{A_i} \right]
\]

Now \( \frac{\partial \xi}{\partial \eta} = 0 \) which implies

\[
c = \frac{2p(p+1)}{p(p+1)\eta + p^2 + 3p+1} < 0 \quad (0 < \eta < 1),
\]

which is false since \( c > 0 \). Thus \( \xi(c, \eta) \) attains its maximum on the boundary of \([0, 2] \times [0, 1]\). Thus for fixed \( c \), we have

\[
\max_{(c, \eta)} \xi(c, \eta) = \xi(c, 1) = \psi(c) \quad (\text{say})
\]

Therefore, replacing \( \eta \) by 1 in (3.15) and simplifying we get

\[
\psi(c) = \frac{(3p+1)c}{2p(p+1)(p+2)A_i} + \frac{2}{pA_i} - \frac{c^2}{pA_i}
\]

(3.16)

\[
\psi'(c) = \frac{(3p+1) - 2c}{p(p+1)A_i} \quad \text{and} \quad \psi''(c) = - \frac{2}{pA_i}<0.
\]

(3.17)

For an optimum value of \( \psi(c) \), consider \( \psi'(c) = 0 \) which implies \( c = \frac{(3p+1)A_i}{2(p+1)A_i} \). Therefore, the maximum value of \( \psi(c) \) occurs at \( c = \frac{(3p+1)A_i}{2(p+1)A_i} \). From the expression (3.16) we obtain

\[
\psi_{\text{max}} = \psi \left( \frac{(3p+1)A_i}{2(p+1)A_i} \right) = \frac{(p+3)A_i}{2(p+1)A_i} + 8p(p+1)A_i^2 \]

(3.18)

From (3.14) and (3.18), we have
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\[ |a_{p+1} - a_{p+2}| = \frac{(3p+1)^2 A_2^2 + 8p(p+1)A_2^2}{4p(p+1)^2 A_2 A_3}. \]

The proof of Theorem 3.1 is thus completed.

Taking \( t = \alpha = 1; \mu = \delta = j = 0 \) in Theorem 3.1 we get

**Corollary 3.2.** Let \( f \) given by (1.1) be in the class \( C \). then

\[ \|a_{p+2} \| \|a_{p+1}\| \leq \frac{2p + (p+1)(p+2)}{2p^2(p+1)^2(p+2)} \]

and

\[ \|a_{p+3} \| \|a_{p+2}\| \leq \frac{9(3p+1)^2 + 8p(p+1)(p+3)^2}{2p^2(p+1)^2(p+2)^2} \]

Both the inequalities are sharp.

Putting \( t = j = \delta = 0 \) in Theorem 3.1 we get

**Corollary 3.3.** Let \( f \) given by (1.1) be in the class \( S^* \). Then

\[ \|a_{p+2} \| \|a_{p+1}\| \leq \frac{2p + (p+1)^3}{2p^2(p+1)^2} \]

and

\[ \|a_{p+3} \| \|a_{p+2}\| \leq \frac{9(3p+1)^2 + 8p(p+1)(p+2)^2}{2p^2(p+1)^2(p+2)^2} \]

Both the inequalities are sharp.

For \( p = 1 \), Theorem 3.1 reduces to the results obtained in

**Corollary 3.4.** [16] Let \( f \) given by (1.1) be in the class \( M^{j\mu} (\alpha) \). If \( \frac{3A_0}{4} \leq A_2 \leq \frac{3A_0}{2} \), then

\[ \|a_{p+2} \| \|a_{p+1}\| \leq \frac{A_2^2 + A_0^2}{4A_0^2 A_2}, \]

and

\[ \|a_{p+3} \| \|a_{p+2}\| \leq \frac{A_2^2 + A_0^2}{A_2 A_3}, \]

where

\[ A_2 = 2^\alpha (1 + \mu)^j (\delta + 1)(1 + (2^\alpha (1 + \mu) - 1)t), \]

\[ A_2 = 3^\alpha (1 + 2\mu)^j \frac{(\delta + 1)(\delta + 2)}{2}[1 + (3^\alpha (1 + 2\mu) - 1)t], \]

and

\[ A_0 = 4^\alpha (1 + 3\mu)^j \frac{(\delta + 1)(\delta + 2)(\delta + 3)}{6}[1 + (4^\alpha (1 + 3\mu) - 1)t]. \]

**Remark 3.1.** Here we remark that the results obtained in (corollary 1, [16]) is computationally wrong. The estimates \( \|a_{p+1} \| \|a_{p+2}\| \leq \frac{25}{38} \) must be \( \|a_{p+1} \| \|a_{p+2}\| \leq \frac{25}{48} \) and \( \|a_{p+1} \| \|a_{p+2}\| \leq \frac{25}{48} \).

Taking \( t = \alpha = p = 1; \mu = \delta = j = 0 \) in Theorem 3.1 we get following

**Corollary 3.5.** [16] Let \( f \) given by (1.1) be in the class \( C \). Then

\[ \|a_{p+1} \| \|a_{p+2}\| \leq \frac{25}{48} \] and \( \|a_{p+1} \| \|a_{p+2}\| \leq \frac{25}{48} \)

Both the inequalities are sharp.

Putting \( t = j = \delta = 0 \) and \( p = 1 \) in Theorem 3.1 we get following
Corollary 3.6. [16] Let \( f \) given by (1.1) be in the class \( S' \). Then
\[
\| a_3 - |a_2| \| \leq \frac{5}{4} \text{ and } \| a_4 - |a_3| \| \leq 2
\]
Both the inequalities are sharp.

References


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