Rank of Product of Certain Algebraic Classes

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Abstract: The properties of rank of a finite semigroup were presented by Howie in [1], and this properties is more general in algebraic system such as semigroup, or indeed even a group. We use this concept (properties) to compute the upper rank and the intermediate rank of direct product of a distinct monoid and the quotient group.

Keywords: Monoid, Independent set, Quotient group and Cyclic group.

I. Introduction And Preliminaries

Many authors have studied the rank properties in the context of general algebras since the work of Marczewski in [2]. This property is similar to the concept of dimension in linear algebra. Howie and Ribeiro in [1] considered the following definition of rank for a finite semigroup.

1. \( r_1(S) = \max \{ k : \text{every subset } U \text{ of cardinality } k \text{ in } S \text{ is independent} \} \), this is called the small rank
2. \( r_2(S) = \min \{ |U| : U \subseteq S, (U)=S \} \) This is called the lower rank
3. \( r_3(S) = \min \{ |U| : U \subseteq S, U \text{ is independent} \} \). This is called the intermediate rank
4. \( r_4(S) = \max \{ |U| : U \subseteq S, U \text{ is independent} \} \). This is called the upper rank
5. \( r_5(S) = \min \{ k : \text{every subset } U \text{ of cardinality } k \text{ in } S \text{ generate } S \} \). This is called the larger rank.

1.1 Definition (independent subset)

A subset \( U \) of a semigroup \( S \) is said to be independent if for all element \( a \) belonging to \( U \), \( a \) does not belong to the generating subset \( U \setminus \{ a \} \) of \( S \). That is

\[(\forall a \in U) \ a \notin (U \setminus \{a\})\]

1.2 All the five ranks coincide in certain semigroups. However, there exist semigroups for which all these ranks are distinct.

In this work, we adopt the notations and definition given in [1] and [3]. A monoid is a semigroup with identity element. We shall in section 2 compute the intermediate rank and the upper rank of the direct product of monoid. In section 3, we compute that of the quotient group. Throughout this work, our semigroup \( S \) is a monoid.

1.3 REMARK

As the definition of different rank implies, lower intermediate and upper ranks, it has been shown that \( r_2(S) \leq r_3(S) \leq r_4(S) \).

SECTION 2

We present in this section the result of the rank of the direct and subdirect product of the monoid. Our intermediate rank \( r_3(S) \) is denoted by \( r(S) \), and the upper rank \( r_4(S) \) by \( R(S) \), except otherwise stated.

Theorem 2.1

Let A, B be monoids, then \( R(A \times B) \geq R(A) + R(B) \)

Proof

If \( a_1, a_2, ..., a_k \) and \( b_1, b_2, ..., b_l \) are maximal sets in A and B, respectively, then

\((a_1, 1), (a_2, 1), ..., (a_k, 1), (1, b_1), (1, b_2), ..., (1, b_l)\)

are independent in \( A \times B \). Also ,

\{\( (a_1, 1), (a_2, 1), ..., (a_k, 1), (1, b_1), (1, b_2), ..., (1, b_l) \) \}

are independent in \( A \times B \). Then \( R(A \times B) \geq R(A) + R(B) \).

Similarly, for monoids A, B, C, we have that for independent sets \( (c_1, c_2, ..., c_t) \) in C and C is not a subset of A or B, we would have

\{\( (a_1, 1, 1), (a_2, 1, 1), ..., (a_k, 1, 1), (1, b_1, 1), (1, b_2, 1), ..., (1, b_l, 1) \) \}

is independent in \( A \times B \). Moreover, From product set, we have \( R(A \times B \times C) \geq R(A) + R(B) + R(C) \).

Corollary 2.2

If our monoids is distinct, then \( R(B \times A) > R(B) + R(A_{K+1}) \)

Proof
Let \( A_1, A_2, \ldots, A_m \) be distinct monoids, then a typical element in \( A_1 \times A_2 \times \ldots \times A_m \) is \((a_1, a_2, \ldots, a_m)\), \(a_i \in A_i\). An independent set in \( A_\nu \) is \((a_{\nu 1}, a_{\nu 2}, \ldots, a_{\nu r})\). Also, for the subdirect product
\[
\{(a_{\nu 1}, 1, \ldots, 1), (1, a_{\nu 2}, 1, \ldots, 1), \ldots, (1, 1, \ldots, 1, a_{\nu r})\}
\]
is independent set in \( (A_1 \times A_2 \times \ldots \times A_m) \).

Let \( m=k \)
\[
R(A_1 \times A_2 \times \ldots \times A_k) \geq R(A_1) + R(A_2) + \ldots + R(A_k)
\]
For \( m=k+1 \)
\[
R(A_1 \times A_2 \times \ldots \times A_{k+1}) \geq R(A_1) + R(A_2) + \ldots + R(A_{k+1})
\]
Let \( A_1 \times A_2 \times \ldots \times A_k \) be \( B \)
Then \( R(B \times A) \geq R(B) + R(A_{k+1}) \)

**Theorem 2.3**

Then intermediate rank \( \rho \) is given by
\[
R(B \times A) \geq R(B) + R(A_{k+1})
\]

**Proof**
The proof is straightforward from theorem (2.2) for the same distinct monoid.

**SECTION 3**

The collection of all cosets of normal subgroups form a group usually referred to as quotient group. We now compute the rank of this quotient group in this section.

**REMARK 3.1**

The notion of a quotient group is fundamental for group theory and indeed is one of the most important concepts in mathematics. We therefore repeat some of the relevant points:

1. The elements of \( G/N \) \((G \) is a group and \( N \) is a normal subgroup) are the distinct coset of \( N \), the law of composition being multiplication of subset (or addition of cosets when \( G \) is written additively.)

2. The identity (neutral) element in the group \( N \), regarded as one of the cosets.

3. It is immaterial whether we use right or left coset since \( Nt=\operatorname{Nt} \), because \( N \) is normal for \( t \in G \).

4. Recall that the representative of a particular coset is not unique.

**Theorem 3.2**

For any quotient group \( G/N \) \((the \ group \ G \ is \ finite) \) where \( G_1 \) and \( G_2 \) are distinct in \( G/N \),
\[
R(G_1 \times G_2) \geq R(G_1) \times R(G_2)
\]
and
\[
\rho(G_1 \times G_2) \geq \rho(G_1) \times \rho(G_2)
\]

**Proof**

Let \( G \) be a group and \( N \subseteq G \), then \( G/N \) is a group,
\[
G/N = \{N_{x_0}, N_{x_1}, \ldots, N_{x_t}\}
\]

Put \( N_{x_0} \equiv N \) \((i.e \ x_{0} \equiv \epsilon)\)
\[
G/N \cong G^* \{g_{0}, g_{1}, \ldots, g_{t}\}.
\]

Let \( t \) be the minimum rank of independent set of \( G^* \)
By Lagranges theorem, It is well known that
\[
[G^*] = \frac{|G|}{|N|}
\]

Thus, the rank of any quotient group \( G^* < \text{rank of } G \).

If for \( G^*_1, G^*_2, \ldots, G^*_t \) is a set of respective quotients group of \( G \) modulo \( N_1, \ldots, N_t \) respectively, we have
\[
G^*_1 \times G^*_2 \times \ldots \times G^*_t = \frac{G_1}{N_1} \times \frac{G_2}{N_2} \times \ldots \times \frac{G_t}{N_t}
\]
\[
R(G^*_1 \times G^*_2 \times \ldots \times G^*_t) = R\left(\frac{G_1}{N_1} \times \frac{G_2}{N_2} \times \ldots \times \frac{G_t}{N_t}\right)
\]
\[
\rho(G^*_1 \times G^*_2 \times \ldots \times G^*_t) \leq R(G^*_1 \times G^*_2 \times \ldots \times G^*_t)
\]

Each \( G^* = H(G) \) \( \cong Q(G) \)
Rewriting this for intermediate rank we have
\[
\rho(H(G)) \leq \rho(G)
\]

**The notions of independence in Abelian group \( G \) is compare to that of subsemigroup of a group \( G \). We make use of the definition in \([6]\)**

**Corollary 3.3**

For a cyclic group \( H \) of order \( \prod q_i \)
The rank \( R(K_{n_1}) \geq R(H_1) + R(H_2) + \ldots + R(H_k) \)

**Proof**

Let \( n = p_1^{a_1} \cdots p_k^{a_k} = n_1 n_2 \cdots n_k \). Where \( n_i = p_i^{a_i} \) and \( q_i = \frac{a_i}{p_i^{a_i}} \)

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$K_a = H_1 \times H_2 \times \ldots \times H_k = \langle a^{q_i} \rangle$

$H_i = \langle a^{q_i} \rangle$ is a cyclic group of order $P_i^{\alpha_i}$.

$$R(K_a) = R(H_1 \times H_2 \times \ldots \times H_k) \geq R(H_1) + R(H_2) + \ldots + R(H_k)$$

Rank $r(H_i) = r(\langle a^{q_i} \rangle) = R(K_a) \geq R(H_1) + R(H_2) + \ldots + R(H_k) \geq r(H_1) + r(H_2) + \ldots + r(H_k) = K$.

$R(H_i)$ is the lower rank of the cyclic group $H$.

**REMARK 3.4**

1. For any commutative semigroup $S$ and $T$, the rank $R(S \times T) = \text{rank}(S) + \text{rank}(T)$ [5]
2. The rank of the direct product of any algebraic classes is computed based on the defining structure as shown above.

**References**