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On Certain Second Order Neutral Difference Inequality

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Abstract: *In this paper, some sufficient conditions that ensure second order delay difference inequality have no eventually positive solution are obtained.*
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I. Introduction

Very little work has been done on the topic the problem of difference inequalities with deviating arguments. For first order second order and n^{th} order difference inequalities. We can refer to [1-9] and their references. In this paper, we consider the following second order neutral difference inequality with deviating arguments

$$\Delta^2\left[x(n)+\sum_{i=1}^{m} c_i(n)x\left(\tau_i(n)\right)\right]+\sum_{n=a}^{b} p(n,\xi)f(x[g(n,\xi)]) \le 0. \qquad (1)$$

The aim of this paper is to obtain some sufficient conditions under which (1) have no eventually positive solution. We first assume the following condition throughout this paper that

(A_1) $\{c_i(n)\}_{i=1\ to\ n}$ is a positive real sequence

(A_2) $\{\tau_i(n)\}_{i=1\ to\ n}$ is a sequence of positive integers such that $\{\tau_i(n)\} \le n$ and $\lim_{n\to\infty}\tau_i(n)=\infty.$

(A_3) The functions $g(n,\xi),\ \xi\in[a,b]$ is a non-decreasing with respect to n and ξ respectively such that $g(n,\xi)\le n$ *and* $\lim_{n\to\infty} g(n,\xi)=\infty.$

(A_4) $p(n,\xi)$ is a non-decreasing sequence with respect to n and ξ.

II. Main Results

For convenience, we first give the following lemmas.

Lemma 2.1. Suppose that the following conditions holds

$$\sum_{i=1}^{m} c_i(n) \le 1, \qquad (2)$$

$$\frac{f(x)}{x} \ge \lambda > 0 \quad (x>0,\ \lambda \text{ is a constant}). \qquad (3)$$

If $x(n)$ is an eventually positive solution of inequality (1), and let

$$y(n) = x(n)+\sum_{i=1}^{m} c_i x(\tau_i(n)) \qquad (4)$$

then there exists a $n_1 \ge 0$ such that

$$y(n)>0,\quad \Delta^2 y(n)>0 \quad and \quad \Delta y(n)>0. \qquad (5)$$

Proof. Since x(n) is an eventually positive solution of (1), and from (A_3), there exists a $n_1\ge 0$ such that

$$x(n)>0,\quad x\left(\tau_i(n)\right)>0 \quad and \quad x\left[g(n,\xi)\right]>0,\ \ n\ge n_1,\ \ \xi\in[a,b].$$

Noting (2) we have $y(n)>0,\ \ n\ge n_1$ and from (2), we have

$$\Delta^2 y(n)\le -\sum_a^b p\left(n,\xi\right) f\left(x\left[g(n,\xi)\right]\right)\le 0, \tag{6}$$

then $\Delta y(n)$ is a monotonic decreasing and we can further prove $\Delta y(n)>0,\quad n\ge n_1.$

In fact, if there is a $n_2\ge n_1$ with $\Delta y(n_2)=0.$ Then from (6), we have $\Delta y(n)\le \Delta y(n_3)<\Delta y(n_2)=0,\ \ n>n_3,$ then

$$y(n)-y(n_3)\le \sum_{n_3}^{n-1}\Delta y(s)\le \sum_{n_3}^{n-1}\Delta y(n_3)<0,\quad n\ge n_3,$$

therefore $\lim_{n\to\infty} y(n)=-\infty.$ This contradicts the assumption that $y(n)>0,\ n\ge n_1.$
This complete the proof of Lemma 2.1. ■

Lemma 2.2. Suppose that x(n) is an eventually positive solution of inequality (1), then there exists a n_2 for any $\gamma\in(0,1)$ such that

$$y(n)\ge \gamma n\Delta y(n). \tag{7}$$

Proof. Since x(n) is an eventually positive solution of (1) by Lemma 2.1, there exists a $n_1\ge 0$ such that (5) holds, and it is easily seen that there exists a η such that

$$y(n)-y(n_1)=\Delta y(\eta)\left(n-n_1\right). \tag{8}$$

From (5), for any $\gamma\in(0,1),$ we have

$$y(n)\ge \Delta y(\eta)\left(n-n_1\right).$$

Let $w=\dfrac{1}{1-\gamma}$, then $\gamma=1-\dfrac{1}{w},$ and

$$n-n_1\ge n-\frac{n}{w}=n\left(1-\frac{1}{w}\right)=\gamma n,\quad n\ge wn_1\le n_2. \tag{9}$$

Form (8) and (9), we can get (7).
This completes the proof of Lemma 2.2. ■

Lemma 2.3. Suppose $Q(n,\xi)$ is real positive sequence $\xi\in[a,b]$ and

(H$_1$) there exists a function $h(n,\xi)$ such that $h\left(h(n,\xi)\right)=g(n,\xi)$; $h(n,\xi)$ is non-decreasing function with respect to n and ξ and $n\ge h(n,\xi)\ge g(n,\xi),$

(H$_2$) $\liminf_{n\to\infty}\sum_{g(n,b)}^{n-1}\sum_a^b Q(s,\xi)>\dfrac{1}{e},$

(H$_3$) $\liminf_{n\to\infty}\sum_{h(n,b)}^{n-1}\sum_a^b Q(s,\xi)>0,$ then the first order retarded difference inequality

$$\Delta x(n)+\sum_a^b Q(s,\xi)x\left[g(n,\xi)\right]\le 0, \tag{10}$$

have no eventually positive solution.

Proof. Refer Conjecture A in [10]. ■

Now we give the main results of this paper.

Theorem 2.1. Suppose that (2) and (3) hold. Assume further that $\varphi(n)$ is a decreasing positive real sequence such that

$$\sum_{n_0}^{\infty}\left[\lambda\varphi(s)\sum_a^b p(s,\xi)\left\{1-\sum_{i=1}^m c_i\left[g(s,\xi)\right]\right\}-\frac{\left(\Delta\varphi(s)\right)^2}{4\varphi(s)}\right]=\infty, \tag{11}$$

then inequality (1) has no eventually positive solution.

Proof. Suppose that x(n) is an eventually positive solution of (1). Then there exists a $n_1\ge 0$ such that $x(n)>0,\ x\left(\xi_i(n)\right)>0$ and $x\left[g(n,\xi)\right]>0,\ \ n\ge n_1,\ \xi\in[a,b].$ From (4) we have

$$x\left[g(n,\xi)\right]=y\left[g(n,\xi)\right]-\sum_{i=1}^{m}c_i\left[g(n,\xi)\right]x\left(\tau_i\left[g(n,\xi)\right]\right),$$

and from (3), we have $f\left(x\left[g(n,\xi)\right]\right)\geq \lambda x\left[g(n,\xi)\right]>0.$ Thus

$$0\geq \Delta^2 y(n)-\sum_{a}^{b}p(n,\xi)f\left(x\left[g(n,\xi)\right]\right)$$

$$\geq \Delta^2 y(n)+\lambda\sum_{a}^{b}p(n,\xi)\left\{y\left[g(n,\xi)\right]-\sum_{i=1}^{m}c_i\left[g(n,\xi)\right]\right\}x\left[\tau_i(n)\right] \quad (12)$$

From Lemma 2.1, $\Delta y(n)\geq 0,$ and noting $y(n)\geq x(n),\ \ n\geq n_1,$ we have

$$\Delta^2 y(n)+\lambda\sum_{a}^{b}p(n,\xi)\left\{1-\sum_{i=1}^{m}c_i\left[g(n,\xi)\right]\right\}y\left[g\left(n,\xi\right)\right]\leq 0. \quad (13)$$

Using (A_3), $g(n,\xi)$ is non-decreasing in ξ, we have $g(n,a)\leq g(n,\xi),\ \ \xi\in[a,b].$ Therefore, we have

$$\Delta^2 y(n)+\lambda y\left[g(n,a)\right]\sum_{a}^{b}p(n,\xi)\left\{1-\sum_{i=1}^{m}c_i\left[g(n,\xi)\right]\right\}\leq 0. \quad (14)$$

Set

$$W(n)=\varphi\frac{\Delta y(n)}{y\left[g(n,a)\right]}, \quad (15)$$

then $W(n)\geq 0.$ Using the conditions $g(n,\xi)\leq n,\ \ \xi\in[a,b],\ \Delta y(n)\leq \Delta y[g(n,x)],$ then

$$\begin{aligned}\Delta W(n) &= \frac{\Delta\varphi\,\Delta y(n)}{y\left[g(n,a)\right]}+\varphi(n+1)\left[\frac{\Delta^2 y(n)y[g(n,a)]-\Delta y(n)\Delta y[g(n,a)]}{y^2\left[g(n,a)\right]}\right]\\ &\leq \frac{\Delta\varphi\,\Delta y(n)}{y\left[g(n,a)\right]}+\varphi\frac{\Delta^2 y(n)}{y\left[g(n,a)\right]}-\frac{\varphi(n)\Delta^2 y(n)}{y^2\left[g(n,a)\right]}\\ &= \frac{\varphi(n)\Delta^2 y(n)}{y\left[g(n,a)\right]}+\frac{\left(\Delta\varphi(n)\right)^2}{4\varphi(n)}-\left[\sqrt{\varphi(n)}\frac{\Delta y(n)}{y\left[g(n,a)\right]}-\frac{\Delta\varphi(n)}{2\sqrt{\varphi(n)}}\right]^2.\end{aligned}$$

From (14), we have

$$\Delta W(n)\leq -\left[\lambda\varphi(n)\sum_{a}^{b}p(n,\xi)\left\{1-\sum_{i=1}^{m}c_i\left[g(n,\xi)\right]\right\}-\frac{\left(\Delta\varphi(n)\right)^2}{4\varphi(n)}\right].$$

Summing both sides of the last inequality above from n_1 *to* $n-1\ (n>n_1),$ we have

$$W(n)\leq -W(n_1)-\sum_{n_1}^{n-1}\left[\lambda\varphi(n)\sum_{a}^{b}p(n,\xi)\left\{1-\sum_{i=1}^{m}c_i\left[g(n,\xi)\right]\right\}-\frac{\left(\Delta\varphi(n)\right)^2}{4\varphi(n)}\right]. \quad (16)$$

By taking $n\to\infty$ and noticing (14), we have $W(n)\to-\infty,$ which contradicts $W(n)>0.$
This completes the proof of Theorem 2.1. ■

Theorem 2.2. Suppose that (2)-(3) and (H_1) hold, and that

(H_4) $\liminf_{n\to\infty}\sum_{g(n,b)}^{n-1}\sum_{a}^{b}\left(\frac{1}{2}\right)g(n,\xi)Q(n,\xi)\left\{1-\sum_{i=1}^{m}c_i g\left[g(s,\xi)\right]\right\}>\frac{1}{e},$

(H_5) $\liminf_{n\to\infty}\sum_{h(n,b)}^{n-1}\sum_{a}^{b}\left(\frac{1}{2}\right)g(n,\xi)Q(n,\xi)\left\{1-\sum_{i=1}^{m}c_i g\left[g(s,\xi)\right]\right\}>0.$

Then the inequality (1) has no eventually positive solutions.

Proof. Suppose that x(n) is an eventually positive solution, by Lemma 2.1, we know that there exists a $n_3\geq n_1$ such that $x(\tau_i(n))>0,\ x\left[g(n,\xi)\right]>0,$ and $\Delta x\left[g(n,\xi)\right]>0,\ \ n\geq n_3,\ \xi\in[a,b].$

Noticing $y(n)\geq x(n)$, we have

$$y(n)\leq x(n)+\sum_{i=1}^{m}c_i(n)y\left(\tau_i(n)\right)\leq x(n)+\sum_{i=1}^{m}c_i(n)y(n),\quad n\geq n_3,$$

then

$$\left[1-\sum_{i=1}^{m}c_i(n)\right]y(n)\le x(n). \tag{17}$$

Using Lemma 2.2, (13) and (17), we have

$$0\ge \Delta^2 y(n)+\lambda\sum_a^b p(n,\xi)\left\{1-\sum_{i=1}^{m}c_i g\left[g(n,\xi)\right]\right\}y\left[g(n,\xi)\right]$$

$$\ge \Delta^2 y(n)+\lambda\sum_a^b p(n,\xi)\left\{1-\sum_{i=1}^{m}c_i g\left[g(s,\xi)\right]\right\}\gamma g(n,\xi)y\left[g(n,\xi)\right].$$

Choosing $\gamma=\frac{1}{2}\in(0,1)$, we have

$$\Delta^2 y(n)+\frac{\lambda}{2}\sum_a^b p(n,\xi)\left\{1-\sum_{i=1}^{m}c_i g\left[g(s,\xi)\right]\right\}g(n,\xi)\Delta y\left[g\left(n,\xi\right)\right]\le 0,\ n\ge n_3,$$

let $z(n)=\Delta y(n)$

$$\Delta^2 y(n)+\lambda\sum_a^b \frac{1}{2}p(n,\xi)\left\{1-\sum_{i=1}^{m}c_i g\left[g(s,\xi)\right]\right\}g(n,\xi)z\left[g\left(n,\xi\right)\right]\le 0,\ n\ge n_4. \tag{18}$$

Choosing

$$Q(n,\xi)=\frac{1}{2}p(n,\xi)\left\{1-\sum_{i=1}^{m}c_i g\left[g(s,\xi)\right]\right\}g(n,\xi),\quad n\ge n_3,$$

then we have

$$\liminf_{n\to\infty}\sum_{g(n,b)}^{n-1}\sum_a^b Q(s,\xi)>\frac{1}{e},$$

$$\liminf_{n\to\infty}\sum_{h(n,b)}^{n-1}\sum_a^b Q(s,\xi)>0,$$

$$\Delta z(n)+\sum_a^b Q(s,\xi)z[g(n,\xi)]\le 0. \tag{19}$$

Then if follows from Lemma 2.3 that inequality (19) has no eventually positive solutions, which contradicts the fact that $z(n)=\Delta y(n)>0$ is a solution of (18).
This completes the proof of Theorem 2.2. ■

Remark 2.1. Similar to the above results on equation (1), we can consider the following second order delay difference inequality

$$\Delta^2\left[x(n)+\sum_{i=1}^{m}c_i(n)x\left(\tau_i(n)\right)\right]+\sum_a^b p\left(s,\xi\right)f\left(x\left[g(n,\xi)\right]\right)\ge 0 \tag{1'}$$

and obtain sufficient conditions that ensure two inequality has no eventually negative solutions.

For the second order delay difference equation

$$\left[x(n)+\sum_{i=1}^{m}c_i(n)x\left(\tau_i(n)\right)\right]+\sum_a^b p\left(s,\xi\right)f\left(x\left[g(n,\xi)\right]\right)=0. \tag{20}$$

We have the following results.

Theorem 2.3. Suppose that the conditions of Theorem 2.1 hold. Then every solution of equation (20) is oscillatory.

Theorem 2.4. Suppose that the conditions of Theorem 2.2 hold. Then every solution of equation (20) is oscillatory.

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