Properties (T) and (Gt) for f (T) Type Operators

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Abstract: in this paper we establish the relation with property (t) and (gt) introduced by M. H. M. RACHID in [1] and spectrume mapping theorem. We will special establish several sufficient and necessary conditions for which \(f(T)\) verify the (t) and (gt) as \(f\) an analytic function on the \(T\) spectrum and see the validity of these results for the semi-Browder operators. Analogously we ask question about the conditions for which the spectral theory hold for generalized a-weyl’s theorem and generalized a-browder’s theorem [18].

Keywords: generalized a-weyl’s theorem. Generalized a-browder’s theorem. Semi-Browder operators. Property (t). Property (gt). Spectral theory.

I. Introduction and Preliminary

Throughout this paper, \(X\) denote an infinite-dimensional complex space, \(L(X)\) the algebra of all bounded linear operators on \(X\). For \(T \in L(X)\), let \(T^*, \ker(T), R(T), \sigma(T), \sigma_a(T)\) and \(\sigma_r(T)\) denote the adjoint, the null space, the range, the spectrum, the approximate point spectrum and the surjectivity spectrum of \(T\) respectively. Let \(\alpha(T)\) and \(\beta(T)\) be the nullity and deficiency of \(T\) defined by \(\alpha(T) = \dim \ker(T)\) and \(\beta(T) = \dim \text{R}(T)\). Let \(SF_a(\lambda) = \{T \in L(X) : \alpha(T) < \infty \text{ and } R(T) \text{ is closed}\}\) and \(SF_r(\lambda) = \{T \in L(X) : \beta(T) < \infty\}\). Let \(SF_{a,r}(\lambda) = \{T \in L(X) : \alpha(T) + \beta(T) = \infty\}\) denote the semi-group of upper semi-Fredholm and lower semi-Fredholm operators on \(X\) respectively. If both \(\alpha(T)\) and \(\beta(T)\) are finite, then \(T\) is called Fredholm operator. If \(T\) is semi-Fredholm then the index of \(T\) is defined by \(\text{ind}(T) = \alpha(T) - \beta(T)\).

A bounded linear operator \(T\) acting on a Banach space \(X\) is Weyl if it is Fredholm of index zero and Browder if \(T\) is Fredholm of finite ascent and descent. Let \(\mathbb{C}\) denote the set of complex numbers. The Weyl spectrum and Browder spectrum of \(T\) are defined by \(\sigma_w(\lambda) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}\) and \(\sigma_B(\lambda) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\}\). For \(T \in L(X)\), \(SF_w(X) = \{T \in SF_a(X) : \text{ind}(T) \leq 0\}\). Then the upper Weyl spectrum of \(T\) is defined by \(\sigma_{SF_w}(\lambda) = \{ \lambda \in \mathbb{C} : T - \lambda \not\in SF_w(X)\}\). Let \(\Delta(T) = \sigma(T) \setminus \sigma_w(T)\) and \(\Delta_a(T) = \sigma_a(T) \setminus \sigma_{SF_w}(T)\). Following Cubrun [10], we say that Weyl’s theorem holds for \(T \in L(X)\) if \(\Delta(T) = E^0(T)\), where \(E^0(T) = \{ \lambda \in \sigma(T) : 0 < \alpha(T - \lambda) < \infty\}\). Here and elsewhere in this paper, for \(K \subset \mathbb{C}\), \(iso(K)\) is the set of isolated points of \(K\).

According to Rakocevic [20], an operator \(T \in L(X)\) is said to satisfy a-Weyl’s theorem if \(E^0_a(T) = \sigma_a(T) \setminus \sigma_{SF_w}(T)\), where \(E^0_a(T) = \{ \lambda \in \sigma_a(T) : 0 < \alpha(T - \lambda) < \infty\}\). It is known from [20] that an operator satisfying a-Weyl’s theorem satisfies Weyl’s theorem, but the converse does not hold in general. For \(T \in L(X)\) and a non negative integer \(n\) define \(T^{[n]}\) to be the restriction of \(T\) to \(R(T^n)\) viewed as a map for \(R(T^n) \rightarrow R(T^n)\) in particular \(T^{[0]} = T\). If for some integer \(n\) the range space \(R(T^n)\) is closed and \(T^{[n]}\) is an upper (resp., lower) semi-Fredholm operator, then \(T\) is called upper (resp., lower) semi-B-Fredholm operator.

In this case index of \(T\) is defined as the index of semi-Fredholm operator \(T^{[n]}\). Moreover, if \(T^{[n]}\) is a Fredholm operator then \(T\) is a semi-Fredholm operator. An operator \(T\) is said to be B-Weyl operator if it is a B-Fredholm operator of index zero. Let \(\sigma_{BW}(\lambda) = \{ \lambda \in \mathbb{C} : T - \lambda \not\in B - Weyl\}\). Recall that the ascent, \(\alpha(T)\), of an operator \(T \in L(X)\) is the smallest non negative integer \(p\) such that \(\ker(T^p) = \ker(T^{p+1})\) and if such integer does not exist we put \(\alpha(T) = \infty\). Analogously the descent, \(\delta(T)\), of an operator \(T \in L(X)\) is the smallest non negative integer \(s\) such that \(R(T^s) = \text{R}(T^{s+1})\) and if such integer does exist we put \(\delta(T) = \infty\). According to Berkani [3], an operator \(T \in L(X)\) is said to be Drazin invertible if it has finite ascent and descent. The Drazin spectrum of \(T\) is defined by \(\sigma_D(T) = \{ \lambda \in \mathbb{C} : T - \lambda \not\in D - \text{invertible}\}\). Define the set \(LD(X) = \{T \in L(X) : \alpha(T) < \infty \text{ and } R(T^{\sigma_a(T)+1}) \text{ is closed}\}\) and \(\sigma_{LD}(T) = \{ \lambda \in \mathbb{C} : \lambda \not\in LD(X)\}\). Following [4], an operator \(T \in L(X)\) is said to be left Drazin invertible if \(T \in LD(X)\). We say that \(\lambda \in \sigma_a(T)\) is a left pole of \(T\) if \(T - \lambda \in LD(X)\), and that \(\lambda \in \sigma_r(T)\) is a left pole of \(T\) of finite rank if \(\lambda\) is a left pole of \(T\) and \(\alpha(T - \lambda) < \infty\) [4, Definition 2.6]. Let \(\pi_a(T)\) denote the set of all left poles of \(T\) and let \(\pi_{LD}(T)\) denote set of all left poles of finite rank. It follows from [4, Theorem 2.8] that if \(T \in L(X)\) is left Drazin invertible, then \(T\) is upper semi-B-Fredholm operator.
Fredholm of index less than or equal to 0. We say that Browder’s theorem holds for \( T \in L(X) \) if \( \Delta(T) = \pi^0(T) \), where \( \pi^0(T) \) is the set of all poles of \( T \) of finite rank and that \( a \)-Browder’s theorem holds for \( T \) if \( \Delta_a(T) = \pi_a^0(T) \). Let \( \Delta^a(T) = \sigma(T) \setminus \sigma_{gf}(T) \). Following [3], we say that generalized Weyl’s theorem holds for \( T \in L(X) \) if \( \Delta^a(T) = E(T) \), where \( E(T) \) is the set of all eigenvalues of \( T \) which are isolated in \( \sigma(T) \), and that generalized Browder’s Theorem holds for \( T \) if \( \Delta^a(T) = \pi(T) \), where \( \pi(T) \) is the set of poles of \( T \). It is proved in [8, theorem 2.1] that generalized Browder’s theorem is equivalent to Browder’s theorem.

Let \( SBF^+_a(X) \) denote the class of all upper semi-B-Fredholm operators such that \( \text{ind}(T) \leq 0 \). The upper B-Weyl spectrum of \( T \) is defined by \( \sigma_{SBF}^+(T) = \{ \lambda \in \mathbb{C} : T - \lambda \notin SBF^+_a(X) \} \). We say that \( T \in L(X) \) satisfies generalized a-Weyl’s theorem, if \( \sigma_a(T) \setminus \sigma_{SBF}^+(T) \), where \( \sigma_a(T) \) is the set of all eigenvalues of \( T \) which are isolated in \( \sigma_a(T) \) and that \( T \in L(X) \) satisfies generalized a-Browder’s theorem if \( \Delta_a^g(T) = \pi_a(T) \) [4, Definition 2.13]. It is proved in [8, theorem 2.2] that generalized a-Browder’s theorem is equivalent to a-Browder’s theorem.

Following [20], we say that \( T \in L(X) \) satisfies property (w) if \( \Delta_a(T) = E^0(T) \). The property (w) has been studied in [1, 5, 19]. In Theorem 2.8 of [5], it is shown that property (w) implies Weyl’s theorem, but the converse is not true in general. We say that \( T \in L(X) \) satisfies property (gw) if \( \Delta_a^g(T) = E(T) \). Property (gw) extends (w) to the context of B-Fredholm theory, and it is proved in [9] that an operator possessing property (gw) satisfies property (w) but the converse is not true in general. According to [16], an operator \( T \in L(X) \) is said to possess property (gb) if \( \Delta_a^g(T) = \pi(T) \), and is said to possess property (b) if \( \Delta_a(T) = \pi_a(T) \). It is shown in Theorem 2.3 of [16] that an operator possessing property (gb) satisfies property (b) but the converse is not true in general. Following [10], we say an operator \( T \in L(X) \) is said to be satisfies property (R) if \( \pi_{\Delta_a^g}(T) = E^0(T) \). In Theorem 2.4 of [8], it is shown that \( T \) satisfies property (w) if and only if \( T \) satisfies a-Browder’s theorem and \( T \) satisfies property (R).

The single valued extension property plays an important role in local spectral theory, see the recent monograph of Laursen and Neumann [21] and Aiena [1]. In this article we shall consider the following local version of this property, which has been studied in recent papers. [5,20] and previously by Finch [19]. Following [19] we say that \( T \in L(X) \) has the single valued extension property (SVEP) at point \( \lambda \in \mathbb{C} \), if for every open neighborhood \( U_0 \) of \( \lambda \), the only analytic function \( f: U_0 \rightarrow X \) which satisfies the equation \( (T - \mu)f(\mu) = 0 \) is the constant function \( f \equiv 0 \). It is well-known that \( T \in L(X) \) has SVEP at every point of the resolvent \( \rho(T) = \mathbb{C} \setminus \sigma(T) \). Moreover, from the identity Theorem for analytic function it easily follow that \( T \in L(X) \) has SVEP at every point of the boundary \( \partial \sigma(T) \) of the spectrum. In particular, \( T \) has SVEP at every isolated point of \( \sigma(T) \). In [20, Proposition 1.8], Laursen proved that if \( T \) is of finite ascent, then \( T \) has SVEP.

**Theorem 1.1** [18, Theorem 1.3] if \( T \in SF^+_a(X) \) the following statements are equivalents:

(i) \( T \) has SVEP at \( \lambda_0 \).

(ii) \( \sigma(T - \lambda_0 I) < \infty \).

(iii) \( \sigma_a(T) \) does not cluster at \( \lambda_0 \).

(iv) \( H_0(T - \lambda_0 I) \) is finite dimensional.

By duality we have

**Theorem 1.2** if \( T \in SF_a(X) \) the following statements are equivalents:

(i) \( T^* \) has SVEP at \( \lambda_0 \).

(ii) \( d(T - \lambda_0 I) < \infty \).

(iii) \( \sigma_a(T) \) does not cluster at \( \lambda_0 \).

According to M. H. M. Rashid, we say that \( T \in L(X) \) satisfies property (t) if \( \Delta_a(T) = E^0(T) \), where \( \Delta_a(T) = \sigma(T) \setminus \sigma_{SBF}^+(T) \), and \( T \) satisfies property (gt) if \( \Delta_a^g(T) = E(T) \), where \( \Delta_a^g(T) = \sigma(T) \setminus \sigma_{SBF}^+(T) \) [18, Definition 2.1]. It is shown in [18, Theorem 2.2] that if \( T \) satisfies property (gt), then \( T \) satisfies property (t). The converse of this Theorem does not hold in general. In seem reference it is proved that if \( T \) satisfies property (t), then \( T \) satisfies property (w) and we have equivalence if \( \sigma(T) = \sigma_a(T) \). Analogously it is shown that if \( T \) satisfies property (gt), then \( T \) satisfies property (gw) and the equivalence hold is \( \sigma(T) = \sigma_a(T) \). As a consequence in [2] it is proved that if \( T \) possesses property (t), then \( T \) satisfies Weyl’s Theorem also \( T \) satisfies a-Browder’s Theorem and \( \pi_a^0(T) = E^0(T) \). More general if \( T \) possesses property (gt), then \( T \) satisfies generalized Weyl’s Theorem also \( T \) satisfies generalized a-Browder’s Theorem and \( \pi_a(T) = E(T) \).

**II. About generalized a-Weyl’s and a-Browder’s Theorems and SVEP**

We start this part by this consequence

**Corollary 2.1** If \( T \) obey generalized a-Browder’s Theorem then \( T \) has the SVEP at \( \lambda \notin \sigma_{SBF}^+(T) \)

*Proof*. If \( T \) obey generalized a-Browder’s Theorem, then \( \Delta_a^g(T) = \pi_a(T) \). Let \( \lambda \in \pi_a(T) \) then \( \lambda \) is isolated in \( \sigma_a(T) \), so \( T - \lambda \) has SVEP en 0. (ie) \( T \) has SVEP at \( \lambda \notin \sigma_{SBF}^+(T) \) [19].

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Recall that for each $T \in L(X)$ and $f \in \mathcal{H}(\sigma(T))$, then we have $f\left(\sigma_{SBF^c}(T)\right) = \sigma_{SBF^c}(f(T))$.

We show now the relationship between the spectral mapping Theorem and generalized a-Weyl’s Theorem.

**Proposition 2.2.** Let $T \in L(X)$ obey to generalized $-$ Weyl’s theorem then, for each $f \in \mathcal{H}(\sigma(T))$: $\sigma_a(f(T))\setminus E_a(f(T)) = f(\sigma_a(T)\setminus E_a(T))$.

**Proof.** For the direct inclusion we use the argument in [18, lemma 3.89].

To prove the inverse inclusion, let $\lambda_0 \in f(\sigma_a(T)\setminus E_a(T))$. From the equality $f(\sigma_a(T)) = \sigma_a(f(T))$ we know that $\lambda_0 \in \sigma_a(f(T))$. Suppose that $\lambda_0 \in E_a(T)$, so $\lambda_0$ is isolated in $\sigma_a(f(T))$. Now we can write $\lambda_0 - f(T) = p(T)g(T)$ with $g(T)$ invertible and

$$p(T) = \prod_{i=1}^k(\lambda_i - T)^{n_i}.$$  

From the equality (1) and the fact that $T$ satisfies generalized a-Weyl’s Theorem it follows any of $\lambda_1, ... , \lambda_k$ must be an isolated point in $\sigma_a(T)$, hence an eigenvalue of $T$. Moreover, since $\lambda_0$ is an eigenvalue isolated in $\sigma_a(f(T))$ any $\lambda_i$ must also be an eigenvalue isolated an $\sigma_a(T)$, so $\lambda_i \notin E_a(T)$. This contradicts $\lambda_0 \notin f(\sigma_a(T)\setminus E_a(T))$. Therefore $\lambda_0 \notin E_a(T)$, so we have the equality $\sigma_a(f(T))\setminus E_a(f(T)) = f(\sigma_a(T)\setminus E_a(T))$.

**Corollary 2.3.** If $T \in L(X)$ satisfies generalized a-Weyl’s Theorem then $\sigma_{SBF^c}(f(T)) = \sigma_a(f(T))\setminus E_a(f(T))$, $\forall f \in \mathcal{H}(\sigma(T))$.

**Proof.** It’s immediate from the preview result and the fact that $f\left(\sigma_{SBF^c}(T)\right) = \sigma_{SBF^c}(f(T))$.

We should recall some results due to [19]

**Theorem 2.4.** [Aiena, Theorem 1.62] If $T \in SF(X)$ then $T$ is essentially semi-regular.

**Theorem 2.5.** [Aiena, Theorem 1.83] Suppose that $T \in L(X)$ is upper semi B-Fredholm. Then there exists an open disc $D(0,\varepsilon)$ centered at $0$ such that $\lambda I - T$ is upper semi-Fredholm for all $\lambda \in D(0,\varepsilon)\setminus \{0\}$.

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**References**


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Books: