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On Scalar Weak Commutative Algebras

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Abstract: *The concept of scalar commutativity defined in an algebra A over a ring R is mixed with the concept of weak-commutativity defined in a Near-ring to coin the new concept of scalar weak commutativity in an algebra A over a ring R and many interesting results are obtained.*

I. Introduction

Let A be an algebra (not necessarily associative) over a commutative ring R.A is called scalar commutative if for each x,y ∈ A,there exists $\alpha \in R$ depending on x,y such that xy = αyx.Rich[8] proved that if A is scalar commutative over a field F,then A is either commutative or anti-commutative. KOH,LUH and PUTCHA [6] proved that if A is scalar commutative with 1 and if R is a principal ideal domain ,then A is commutative. A near-ring N is said to be weak-commutative if xyz = xzy for all x,y,z ∈ N (Definition 9.4, p.289, Pliz[7]. In this paper we define scalar weak commutativity in an algebra A over a commutative ring R and prove many interesting results analogous to Rich and LUH.

II. Preliminaries

In this section we give some basic definitions and well known results which we use in the sequel.

2.1 Definition [7]:
Let N be a near-ring.N is said to be weak commutative if xyz = xzy for all x,y,z ∈N.

2.2 Definition:
Let N be a near-ring.N is said to be anti-weak commutative if xyz = - xzy for all x,y,z∈N.

2.3 Definition [8]:
Let A be an algebra (not necessarily associative) over a commutative ring R.A is called scalar commutative if for each x,y ∈ A,there exists $\alpha = \alpha(x,y) \in R$ depending on x,y such that xy = αyx.A is called scalar anti-commutative if xy = -αyx.

2.4 Lemma[5]:
Let N be a distributive near-ring.If xyz = $\pm$ xzy for all x,y,z ∈ N,then N is either weak commutative or weak anti-commutative.

III. Main Results

3.1 Definition
Let A be an algebra (not necessarily associative) over a commutative ring R. A is called scalar weak-commutative if for every x,y,z ∈ A, there exists $\alpha = \alpha(x,y,z) \in R$ depending on x,y,z such that xyz = αxzy. A is called scalar anti-weak commutative if xyz = -αxzy.

3.2 Theorem:
Let A be an algebra (not necessarily associative) over a field F.If A is scalar weak commutative,then A is either weak commutative or anti- weak commutative.

Proof:

Suppose xyz = xzy for all x,y,z ∈ A,there is nothing to prove.
Suppose not we shall prove that xyz = - xzy for all x,y,z ∈ A.
We shall first prove that,if x,y,z ∈ A such that xyz ≠ xzy,then $x y^2 = x z^2 = 0$.
Let x,y,z ∈ A such that xyz ≠ xzy.
Since A is scalar weak commutative,there exists $\alpha = \alpha (x,y,z) \in F$ such that

$$xyz = \alpha\, xzy \quad \rightarrow (1)$$

Also there exists $\gamma = \gamma (x,y+z,z) \in F$ such that

$$x (y+z) z = \gamma\, xz (y+z) \quad \rightarrow (2)$$

(1) – (2) gives

$$xyz - xyz - x z^2 = \alpha\, xzy - \gamma\, xzy - \gamma\, x z^2.$$

$$\gamma\, x z^2 - x z^2 = (\alpha - \gamma)\, xzy.$$

$x z^2 - \gamma\, x z^2 = (\gamma - \alpha)\, xzy$ → (3)

Now, $xzy \neq 0$ for if $xzy = 0$, then from (1), we get $xyz = 0$ and so $xyz = xzy$; contradicting our assumption that $xyz \neq xzy$.

Also $\gamma \neq 1$, for if $\gamma = 1$, then from (3) we get

$$\alpha = \gamma = 1.$$

Then from (1) we get

xyz = xzy, again contradicting assumption that $xyz \neq xzy$.

Now from (3) we get

$$x z^2 = \frac{\gamma-\alpha}{1-\gamma}\ xzy.$$

i.e., $x z^2 = \beta\, xzy$ for some $\beta \in F$. → (4)

Similarly $x y^2 = \delta\, xzy$ for some $\delta \in F$ → (5)

Now corresponding to each choice of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ in F, there is an $\eta \in F$ such that

$$x(\alpha_1 y + \alpha_2 z)(\alpha_3 y + \alpha_4 z) = \eta\, x(\alpha_3 y + \alpha_4 z)(\alpha_1 y + \alpha_2 z)$$

$$x(\alpha_1\alpha_3 y^2 + \alpha_1\alpha_4 yz + \alpha_2\alpha_3 zy + \alpha_2\alpha_4 z^2)$$
$$= \eta\, x(\alpha_3\alpha_1 y^2 + \alpha_3\alpha_2 yz + \alpha_4\alpha_1 zy + \alpha_4\alpha_2 z^2)$$

$$\alpha_1\alpha_3 x y^2 + \alpha_1\alpha_4 x yz + \alpha_2\alpha_3\, xzy + \alpha_2\alpha_4\, xz^2$$
$$= \eta(\alpha_3\alpha_1\, xy^2 + \alpha_3\alpha 1\alpha_2\, xyz + \alpha_4\alpha_1 x zy + \alpha_4\alpha_2\, xz^2) \rightarrow (6)$$

Using (4) and (5) we get,

$$\alpha_1\alpha_3\delta\, xzy + \alpha_1\alpha_4\, x yz + \alpha_2\alpha_3\, xzy + \alpha_2\alpha_4\beta\, xzy$$
$$= \eta(\alpha_3\alpha_1\delta\, xzy + \alpha_3\alpha_2\, xyz + \alpha_4\alpha_1 x zy + \alpha_4\alpha_2\beta\, xzy).$$

Using (1) we get,

$$\alpha_1\alpha_3\delta\,\alpha^{-1} xyz + \alpha_1\alpha_4\, x yz + \alpha_2\alpha_3\,\alpha^{-1} xyz + \alpha_2\alpha_4\beta\,\alpha^{-1} xyz$$
$$= \eta(\alpha_3\alpha_1\delta\, xzy + \alpha_3\alpha_2\alpha\, xzy + \alpha_4\alpha_1 x zy + \alpha_4\alpha_2\beta\, xzy).$$

$$(\alpha_1\alpha_3\delta\,\alpha^{-1} + \alpha_1\alpha_4 + \alpha_2\alpha_3\,\alpha^{-1} + \alpha_2\alpha_4\beta\,\alpha^{-1})\ xyz$$
$$= \eta(\alpha_3\alpha_1\delta + \alpha_3\alpha_2\alpha + \alpha_4\alpha_1 + \alpha_4\alpha_2)\ xzy \rightarrow (7)$$

If in (7), we choose $\alpha_2 = 0$, $\alpha_3 = \alpha_1 = 1$, $\alpha_4 = -\delta$, the right hand side of (7) is zero

Whereas the left hand side of (7) is

$$(\delta\,\alpha^{-1} - \delta)\ xyz = 0.$$

i.e., $\delta(\alpha^{-1} - 1)\ xyz = 0.$

Since $xyz \neq 0$ and $\alpha \neq 1$, we get $\delta = 0$.

Hence from (5) we get $xy^2 = 0$.

Also, if in (7), we choose $\alpha_3 = 0$, $\alpha_4 = \alpha_2 = 1$ and $\alpha_1 = -\beta$, the right hand side of (7) is zero whereas the left hand side of (7) is

$$(-\beta + \beta\,\alpha^{-1})\ xyz = 0$$

i.e., $\beta(\alpha^{-1} - 1)\ xyz = 0.$

Since $xyz \neq 0$ and $\alpha \neq 1$, we get $\beta = 0$.

Hence from (4), we get $xz^2 = 0$.

Then (6) becomes

$$\alpha_1\alpha_4\, xyz + \alpha_2\alpha_3\, xzy = \eta(\alpha_3\alpha_2\, xyz + \alpha_4\alpha_1\, xzy).$$

$$\alpha_1\alpha_4\, xyz + \alpha_2\alpha_3\alpha^{-1}\, xyz = \eta(\alpha_3\alpha_2\, xyz + \alpha_4\alpha_1\alpha^{-1}\, xyz).$$

$$(\alpha_1\alpha_4 + \alpha_2\alpha_3\alpha^{-1})\ xyz = \eta(\alpha_3\alpha_2 + \alpha_4\alpha_1\alpha^{-1})\ xyz.$$

This is true for any choice of $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in F$.

Choose $\alpha_1 = \alpha_3 = \alpha_4 = 1$ and $\alpha_2 = -\alpha^{-1}$.

We get $(1 - (\alpha^{-1})^2)\ xyz = 0$.

Since $xyz \neq 0$, $1 - (\alpha^{-1})^2 = 0$.

Hence $(\alpha^{-1})^2 = 1$.

i.e., $\alpha = \pm 1$.

Since $\alpha \neq 1$, we get $\alpha = -1$.

i.e., $xyz = -xzy$ for $x,y,z \in A$.

Thus A is either weak commutative or anti-weak commutative.

3.3 Lemma:

Let A be an algebra(not necessarily associative)over a commutative ring R.Suppose A is scalar weak commutative.Then for all $x,y,z \in A$, $\alpha \in R$, $\alpha\, xyz = 0$ if and only if $\alpha\, xzy = 0$. Also $xyz = 0$ if and only if $xzy = 0$.

Proof:
Let x,y,z $\in$ A and $\alpha \in$ R such that α xyz = 0.
Since A is scalar weak commutative,there exists $\beta = \beta$ (αx,z,y) $\in$ R such that α xzy = β α xyz = 0.
Similarly if α xzy = 0, then there exists $\gamma = \gamma$ (αx,y,z) $\in$ R such that

α xyz = γ αxzy = 0.

Thus α xyz = 0 iff α xzy = 0.
Assume xyz = 0.Since A is scalar commutative,there exists $\delta = \delta$ (x,y,z) $\in$ R such that xzy = δ xyz = 0.
Similarly if xzy = 0,there exists $\eta = \eta$ (x,y,z) $\in$ R such that xyz = η xzy =0.
Thus xyz = 0 if and only if xzy = 0.

3.4 Lemma:
Let A be an algebra over a commutative ring R.Suppose A is scalar weak commutative.
Let x,y,z,u $\in$ A, $\alpha, \beta \in$ R such that zu = uz, xzy = α xyz and x (y + u) z = β xz (y + u).
Then x (zu - αzu - βzu + α βzu) = 0.

Proof:

Given x (y + u) z = β xz (y + u) → (1)

xzy = α xyz → (2)

and zu = uz → (3)

From (1) we get

xyz + xuz = β xzy + β xzu.

xyz + xuz = $\beta\alpha$xzy + β xzu. (using (2))

x { yz + uz – $\alpha\beta yz$ – βzu } = 0.

x { yz + uz – $\alpha\beta yz$ – βuz } = 0. (using (3))

x (y + u – $\alpha\beta y$ – βu) z = 0.

By Lemma 3.3 we get

xz (y + u - $\alpha\beta y$ – βu) = 0.

i.e., xzy + xzu - $\alpha\beta xyz$ – βxzu = 0.

i.e., αxyz + xzu - $\alpha\beta xyz$ – βxzu = 0. using (2) → (4)

Now from (1) we get

xyz + xuz = β xzy + β xzu.

xyz - β xzy = βxzu – xuz.

Multiplying by α we get,

αxyz - $\alpha\beta$xzy = $\alpha\beta$xzu – αxuz. → (5)

From (4) and (5) we vget

xzu – βxzu + $\alpha\beta$xzu – αxuz = 0.

i.e., x { zu - βzu + $\alpha\beta$zu – αuz } = 0 (using (3))

x { zu -αzu - βzu + $\alpha\beta$uz } = 0.

3.5 Corollary:
Taking u = z,we get

x { z^2 - α z^2 – β z^2 + $\alpha\beta$ z^2 } = 0.

i.e., x (z (z – αz) – βz (z – αz)) = 0.

i.e., x (z – α z) (z – β z) = 0.

3.6 Theorem:
Let A be an algebra over a commutative ring R.Suppose A has no zero divisors.If A is scalar weak commutative,then A is weak commutative.

Proof:
Let x,y,z $\in$ A.Since A is scalar weak commutative,there exist scalars $\alpha = \alpha$ (x,z,y) $\in$ R and $\beta = \beta$ (x,y + z,z) $\in R$ such that

xzy = α xy → (1)

and x (y + z) z = β xz (y + z) → (2)

Then by the above corollary,we get

x (z - α z) (z – β z) = 0.

Since A has no zero divisors

z = α z or z = βz.

If z = α z,then from (1) we get

xzy = xyz

If z = βz ,then from (2) we get

x (y + z) z = x z (y + z)

$xyz + xz^2 = xzy + xz^2$

i.e., xyz = xzy.

Thus A is weak commutative.

3.7 Definition:

Let R be any ring and x,y,z ∈ R.We define xyz – xzy as the weak commutator of x,y,z .i.e., xyz – xzy = x [y,z] is called the weak commutator of x,y,z.

3.8 Theorem:

Let A be an algebra over a commutative ring R.Let A be scalar weak commutative.If A has an identity,then the square of every weak commutator is zero.

i.e., $(xyz – xzy)^2 = 0$ for all x,y,z ∈ A.

Proof:

Let x,y,z ∈ A.Since A is scalar weak commutative,there exist scalars $\alpha = \alpha$ (x,y,z) ∈ R and $\beta = \beta$ (x, y+ 1,z) ∈ R such that

$xzy = \alpha\, xyz$ → (1)

$x (y + 1)z = \beta\, xz (y + 1)$ → (2)

From (2) we get

$xyz + xz - \beta\, xzy - \beta\, xz = 0$

$xyz + xz - \beta\alpha\, xyz - \beta\, xz = 0$ (using (1))

$xyz + xz - \alpha\beta\, xyz - \beta\, xz = 0$

i.e., $x (y + 1- \alpha\beta y - \beta) z = 0$

Using Lemma 3.3 we get

$xz (y + 1- \alpha\beta y - \beta) z = 0$

$x z y + xz - \alpha\beta xzy - \beta xz = 0$

$\alpha x yz + xz - \alpha\beta xzy - \beta xz = 0$ (using (1)) → (3)

Also from (2) we get

$xyz + xz = \beta\, xzy + \beta\, xz$

Multiplying by α we get

$\alpha\, xyz + \alpha\, xz = \alpha\, \beta\, xzy + \alpha\beta\, xz$

i.e., $\alpha\, xyz - \alpha\, \beta\, xzy = \alpha\beta\, xz - \alpha\, xz.$ → (4)

From (3) and (4) we get

$xz – \beta\, xz + \alpha\, \beta\, xz – \alpha\, xz = 0.$

i.e., $xz – \alpha\, xz - \beta\, xz + \alpha\, \beta\, xz = 0.$

i.e., $x (z – \alpha z) = x (\beta z – \alpha\beta\, z)$

Multiplying by y+1 on the right we get

$x \{ z (y + 1) – \alpha z (y + 1) \} = x \{ \beta z (y + 1) – \alpha\beta\, z (y + 1) \}$

$= \beta xz (y + 1) – \alpha\beta xz (y + 1)$

$= x (y + 1) z – \alpha x (y + 1) z$ (using (2))

$= x \{ (y + 1) z - \alpha (y + 1) z \}$

i.e., $x \{ z (y + 1) – \alpha z (y + 1) \} = x \{ (y + 1) z - \alpha (y + 1) z \}$

i.e., $x \{ z (y + 1) – (y + 1) z \} = x \{ \alpha\, z (y + 1) - \alpha (y + 1) z \}$

i.e., $x \{ zy + z – yz – z \} = \alpha\, x \{ zy + z – yz – z \}$

$x \{ zy – yz \} = \alpha\, x \{ zy – yz \}$

i.e., $x \{zy – \alpha\, zy \} = x \{ yz – \alpha\, yz \}$

i.e., $xyz – \alpha\, xyz = xzy – \alpha\, xzy$

$= \alpha\, xyz - \alpha\, \alpha\, xyz$

i.e., $xyz - 2\, \alpha\, xyz + \alpha^2\, xyz = 0$

i.e., $x (y - 2\, \alpha\, y + \alpha^2\, y) z = 0$ → (5)

Now, $(xyz – xzy)^2 = (xyz – \alpha\, xyz)^2$ (using (1))

$= (xyz – \alpha\, xyz) (xyz – \alpha\, xyz)$

$= xyz\, xyz – \alpha\, xyz\, xyz – \alpha\, xyz\, xyz + \alpha^2\, xyz\, xyz$

$= xyz\, xyz – 2\, \alpha\, xyz\, xyz + \alpha^2\, xyz\, xyz$

$= x (y - 2\alpha y + \alpha^2\, y) zxyz$

$= 0.\, xyz$ (using (5))

$= 0.$

Thus $(xyz – xzy)^2 = 0.$

i.e., Square of every weak commutator is zero.

3.9 Definition:
Let R be a P.I.D (principal ideal domain) and A be an algebra over R.Let a ∈ A.
Then the order of a,denoted an O (a) is defined to be the generator of the ideal $I = \{ \alpha \in R \mid \alpha a = 0 \}$.
O(a) is unique upto associates and O(a) = 1 if and only if a = 0.

3.10 Lemma:
Let A be an algebra with unity over a principal ideal domain R.If A is scalar weak commutative, z ∈ A such that O(z) = 0,then xyz = xzy for all x,y,z ∈ A.

Proof:

Let z ∈ A with O(z) = 0.

For x,y ∈ A,there exists scalars $\alpha = \alpha$ (x,y,z) ∈ R and $\beta = \beta$ (x, y+1,z) ∈ R such that

$$xzy = \alpha \, xyz \quad \rightarrow (1)$$

$$x(y+1)z = \beta \, xz\,(y+1) \quad \rightarrow (2)$$

From (2) we get

$$xyz + xz - \beta xzy - \beta\, xz = 0$$

$$xyz + xz - \alpha\beta xyz - \beta\, xz = 0$$

$$x\,(y+1 - \alpha\beta y - \beta.1)\, z = 0$$

Using Lemma 3.3 we get

$$xz\,(y+1 - \alpha\beta y - \beta.1) = 0$$

$$xzy + xz - \alpha\beta xzy - \beta xz) = 0$$

$$\alpha xzy + xz - \alpha\beta xzy - \beta xz = 0 \quad \text{(using (1))} \quad \rightarrow (3)$$

From (2) we get

$$xyz + xz = \beta xzy + \beta\, xz$$

Multiplying by α we get

$$\alpha\, xyz + \alpha\, xz = \alpha\beta xzy + \alpha\beta xz$$

$$\text{i.e., } \alpha\, xyz - \alpha\beta xzy = \alpha\beta xz - \alpha\, xz \quad \rightarrow (4)$$

From (3) and (4) we get

$$xz - \beta xz + - \alpha\beta xz - \alpha\, xz = 0$$

$$(1 - \alpha)\,(1 - \beta)\, xz = 0 \quad \forall\, x \in A.$$

Then there exist scalars $\gamma \in R$, $\delta \in R$ such that $\gamma\, xz = 0 \quad \rightarrow (6)$

and

$$\delta\,(x+ 1)\, z = 0 \quad \rightarrow (7)$$

From (7)

$$\delta\, xz + \delta\, z = 0$$

Multiply by γ

$$\gamma\,\delta\, xz + \gamma\,\delta\, z = 0 \quad \rightarrow (8)$$

From (6) we get

$$\gamma\,\delta\, xz = 0 \quad \rightarrow (9)$$

From (8) and (9) we get

$$\gamma\,\delta\, z = 0$$

Since O(z) = 0 we get $\gamma = 0$ and $\delta = 0$.

Then from $1 - \alpha = 0$ or $1 - \beta$.

If $\alpha = 1$,from (1) we get xzy = xyz.

If $\beta = 1$,from (2) we get

$$x\,(y+1)\, z = xz\,(y+1)$$

$$xyz + xz = xzy + xz$$

$$xyz = xzy$$

3.10 (a) Lemma:
Let A be an algebra with idemtity over Principal ideal domain R.If A is scalar weak commutative, y ∈ R with O(y) = 0, then y is in the center of A.

Proof:

Let y ∈ A with O(y) = 0.

For any x ∈ A,there exist scalars $\alpha = \alpha$ (1,x,y) ∈ R and $\beta = \beta$ (1,y,x+1) ∈ R such that

$$\text{(i.e)} \quad 1.\, xy = \alpha\, .\, 1.\, yx.$$

$$xy = \alpha\, yx \quad \rightarrow (1)$$

and 1. y (x+1) = β .1.(x+1)y
(i.e)., y (x+1) = β (x+1)y → (2)
From (2) we get
yx + y = β xy + β y
yx + y = $\alpha\beta$ xy + β y (using (1))
yx + y - $\alpha\beta$ xy - β y = 0.
1.y (x+ 1 - $\alpha\beta$ x - β .1) = 0.
By Lemma 3.3
1. (x+ 1 - $\alpha\beta$ x - β .1) y = 0

xy + y - $\alpha\beta$ xy - β y = 0 → (3)

Also from (2)
yx + y - β xy - β y = 0.
Multiply by α
α yx + α y - $\alpha\beta$ xy - $\alpha\beta$ y = 0
xy + α y - $\alpha\beta$ xy - $\alpha\beta$ y = 0 (using (1)) → (4)
From (3) and (4) we get
y – βy – αy + $\alpha\beta$y = 0
(y – βy) – α (y – βy) = 0
(1 – α) (y – βy) = 0
(1 – α) (1 – β) y = 0
Since O(y) = 0,we get α = 1 or β = 1.
If α = 1, from (1) we get xy = yx.
If β = 2, from (2) we get
y (x+1) = (x+1) y
i.e., yx + y = xy + y
yx = xy
i.e., y commutes with x.
As x ∈ A is arbitrary, y is in the center.

3.11 Lemma:
Let A be an algebra with identity over a P.I.D R.Suppose that A is scalar weak commutative.
Assume further that there exists a prime p ∈ R and positive integer m ∈ z^+ such that p^m A = 0.Then A is Weak commutative.

Proof:
Let O(xy) = p^k for some k ∈ Z^+ .
We prove by induction on k that uxy = uyx for all u ∈ A.
If k = 0,then O(xy) = p^0 = 1 and so xy = 0.
So uxy = 0.Also by Lemma 3.3 uyx = 0.
Hence uxy = uyx for all u ∈ A.So,assume that k > 0 and that the statement is true for l > k.
We first prove that for any u ∈ A, uxy – uyx ≠ 0 implies ω (uy) x – ωx (uy) = 0 for all ω ∈ A.
So, let uxy – uyx ≠ 0.
Since A is scalar weak commutative,there exist scalars α = α (u,x,y) ∈ R and β = β (u,x+1,y) ∈ R such that
uxy = α uyx → (1)
and u (x+1) y = β uy (x+1) → (2)
From (2) we get
uxy + uy = βuyx + βuy.
α uyx + uy = βuyx + βuy (using (1))
(α –) uyx = (β - 1) uy → (3)
If (α – β) uyx = 0 then (β - 1) uy = 0 and so βuy = uy.So from (2) we get
u (x+1) y = uy (x+1)
i.e., uxy + uy = uyx + uy.
i.e., uxy – uyx = 0,contradicting our assumption that uxy – uyx ≠ 0.
So
(– β) uyx ≠ 0.In particular α – β ≠ 0.
Let α – β = p^t δ for some t ∈ Z^+ and δ ∈ R with (δ , p) = 1.If t ≥ k,then since O(xy) = p^k ,we would

get $(\alpha - \beta)$ uxy = 0, a contradiction.Hence t < k.
Now, since p^k uxy = 0,by Lemma 3.3,we have

$$p^k \text{ uyx} = 0.$$

So from (3), $p^{k-t} (\beta - 1)$ uy = $p^{k-t} (\alpha - \beta)$ uyx
= $p^{k-t} p^t \delta$ uyx.
= $p^k \delta$ uyx = 0.
Let O(uy) = p^i for some $i \in Z^+$.
If i< k then by induction hypothesis uxy = uyx , contradiction to our assumption that uxy – uyx ≠ 0.
So i ≥ k.
Hence

$$P^k \mid P^i \mid p^{k-t} (\beta - 1).$$

Thus $p^t \mid \beta - 1$ and let $\beta - 1 = p^t \gamma$ for some $\gamma \in R$.
From (3) we get

$$(\alpha - \beta)\text{uyx} = (\beta - 1)\text{ uy}.$$

$$p^t \delta \text{ uyx} = p^t \gamma \text{ uy} \quad \text{(using (4) and (5))}$$

i.e., $p^t ((uy) (\delta x - \gamma.1)) = 0$. Hence by induction hypothesis

$\omega (uy)(\delta x - \gamma.1) = \omega (\delta x - \gamma.1)(uy) \quad \forall \omega \epsilon A$
$\omega (uy) \delta x - \omega (uy) \gamma.1 = \omega \delta x (uy) - \omega \gamma.1 (uy)$
i.e., $\omega (uy) \delta x - \gamma.\omega (uy) = \omega \delta x (uy) - \gamma \omega (uy)$
$\delta \{ (uy) x - \omega x (uy) \} = 0 \quad \rightarrow (6).$
Since $(\delta, p) = 1$, there exist $,\gamma \in R$ such that $\mu p^m + \gamma\delta = 1$.

$\therefore \mu p^m \{ \omega (uy) x - \omega x (uy) \} + \gamma\delta \{ \omega (uy) x - \omega x (uy) \}$
$= \{ (uy) x - \omega x (uy) \}$
$0 + 0 = \omega (uy) x - \omega x (uy) \quad (\because p^m A = 0)$
i.e., $\omega (uy) x = \omega x (uy)$
i.e., uyx ≠ uxy implies $\omega (uy) x = \omega x (uy)$ for all $\omega \in A \quad \rightarrow (7)$
Now, we proceed to show that

$$\text{uxy = uyx for all } u \in A.$$

Suppose not there exist u ∈ A such that uyx ≠ uxy → (8)
Then we also have (u+1) yx ≠ (u+1) xy → (9)
From (7) and (8) we get

$\omega (uy) x = \omega x (uy)$ for all $\omega \in A \quad \rightarrow (10)$
$\omega (u+1) yx \neq (u+1) xy$ for all $\omega \in A \quad \rightarrow (11)$

From (11) we get

$\omega (uy) x + \omega yx = \omega x(uy) + \omega xy$ for all $\omega \in A$.
i.e., $\omega yx = \omega xy$ for all $\omega \in A$ (using (10))
a contradiction.

This contradiction prove that

$$\text{uxy = uyx for all } u \in A.$$

Thus A is vweak commutative.

3.12 Lemma:
Let A be an algebra with identity over a principal ideal domain R.If A is scalar weak commutative,then A is weak commutative.

Proof:
Suppose A is not weak commutative, there exists z ∈ A such that xyz ≠ xzy for all x,y ∈ A.
Also xy (z+1) ≠ x (z+1) y.
Hence by Lemma 3.9, O (z) ≠ 0 and O (z+1) ≠ 0.
Hence O(1) ≠ 0. Let O(1) = d ≠ 0.Then d is not a unit and hence $d = p_1^{i_1} p_2^{i_2} p_3^{i_3} \ldots\ldots\ldots\ldots p_k^{i_k}$ for Some primes $p_1, p_2, p_3 \ldots\ldots\ldots\ldots\ldots\ldots P_k \in A$ some positive integers $i_1, i_2, \ldots\ldots\ldots\ldots\ldots i_k$.
Let $A_j = \{ a \in A \mid p_j^{i_j a} = 0 \}$.Then each A_j is a non zero subalgebra of A and

$$A = A_1 \oplus A_2 \ldots\ldots\ldots\ldots\ldots \oplus A_k .$$

Being subalgebras of A,each A_i is scalar weak commutative.Being homomorphic image of A,all the Ai' s have identity elements.By Lemma 3.10 each A_i is weak commutative and hence A is weak commutative, a contradiction.Then contradiction proves that A is weak commutative.

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