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Green's Function Approach to Solve a Nonlinear Second Order Four Point Directional Boundary Value Problem

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Abstract: *In this article a four point boundary value problem associated with a second order differential equation involving directional derivative boundary conditions is proposed. Then its solution is developed with the help of the Green's function associated with the homogeneous equation. Using this idea and an Iteration method is proposed to solve the corresponding nonlinear problem.*
Key words: *Green's function, Schauder fixed point theorem, Vitali's convergence theorem.*

I. Introduction

Non local boundary value problems raise much attention because of its ability to accommodate more boundary points than their corresponding order of differential equations [5], [8]. Considerable studies were made by Bai and Fag [2], Gupta [4] and Web [9]. This research article is concerned with the existence and uniqueness of solutions for the second order four point boundary value problem with directional type boundary conditions

$$u''+f(t,u)=0, \quad t\in[a,b], \tag{1.1}$$

$$u(a)=k_1u'(\eta_1), \qquad u(b)=k_2u'(\eta_2); \tag{1.2}$$

where $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ is a given function $a<\eta_1<\eta_2<b$ and $k_1, k_2\in R$

The Green's function plays an important role in solving boundary value problems of differential equations. The exact expressions of the solutions for some linear ODEs boundary value problems can be expressed by the corresponding Green's functions of the problems. The Green's function method will be used to obtain an initial estimate for shooting method. The Greens function method for solving the boundary value problem is an effect tools in numerical experiments. Some BVPs for nonlinear integral equations the kernels of which are the Green's functions of corresponding linear differential equations. The undetermined parametric method we use in this paper is a universal method, the Green's functions of many boundary value problems for ODEs can be obtained by similar method.

In (2008), Zhao discussed the solutions and Green's functions for non local linear second-order Three-point boundary value problems.

$u''+f(t)=0, \quad t\in[a,b]$

subject to one of the following boundary value conditions:

i. $u(a)=ku(\eta), \quad u(b)=0$ ii. $u(a)=0, \quad u(b)=ku(\eta)$ iii. $u(a)=ku'(\eta), \quad u(b)=0$

iv. $u(a)=0, \ u(b)=ku'(\eta)$ where k was the given number and $\eta\in(a,b)$ is a given point.

In (2013), Mohamed investigate the positive solutions to a singular second order boundary value problem with more generalized boundary conditions. He consider the Sturm-Liouville boundary value problem

$u''+\lambda g(t)f(t)=0, \quad t\in[0,1]$ with the boundary conditions

$\propto u(0)-\beta u'(0)=0, \ \gamma u(1)+\delta u'(1)=0$

where $\propto>0, \ \beta>0, \ \gamma>0 \ and \ \delta>0$ are all constants, λ is a positive parameter and $f(.)$ is singular at $u=0$.

Also the existence of positive solutions of singular boundary value problems of ordinary differential equations has been studied by many researchers such as **Agarwal and Stanek** established the existence criteria for positive solutions singular boundary value problems for nonlinear second order ordinary and delay differential equations using the Vitali's convergence theorem. **Gatical et al** proved the existence of positive solution of the problem

$u''+f(t)=0, \quad t\in[0,1]$ with the boundary conditions

$\propto u(0)-\beta u'(0)=0, \ \gamma u(1)+\delta u'(1)=0$

using the iterative technique and fixed point theorem for cone for decreasing mappings.

Recently **Goteti V.R.L. Sarma** et al., studied the solvability of a four point boundary value problem with ordinary boundary conditions $u''+f(t)=0, \quad t\in[a,b]$ satisfying the boundary conditions

$u(a)=k_1u(\eta_1), \ u(b)=k_2u(\eta_2);$ where $a<\eta_1<\eta_2<b \ and \ k_1 \ and \ k_2$ are real constants.

This article is organized as follows: In section 2 we construct the Green's function to the homogeneous BVP corresponding to (1.1) satisfying (1.2) and then using this we proved the existence and uniqueness of the solution of the boundary value problem (1.1) satisfying the condition (1.2). In section 3 we present the iterative method of solution to the corresponding non linear boundary value problem. We illustrated our results by constructing a suitable example.

II. The Green's function:

We have the following conclusions:

Theorem 2.1 Assume $b-k_2 \neq a-k_1$. Then the Green's function for the second-order four-point linear boundary value problem (1.1), (1.2) is given by

$$G(t,s)=K(t,s)+\frac{k_1\left(b-k_2-t\right)K_t(\eta_1,s)+k_2\left(k_1-a+t\right)K_t(\eta_2,s)}{b-k_2-a+k_1} \tag{2.1}$$

where

$$K(t,s)=\begin{cases}\dfrac{(s-a)(b-t)}{b-a}, & a\le s\le t\le b\\[2ex] \dfrac{(b-s)(t-a)}{b-a}, & a\le t\le s\le b\end{cases} \tag{2.2}$$

Proof:

It is well known that the Green's function is K(t, s) as in (2.2) for the second-order two-point linear boundary value problem

$$\begin{cases}u''+f(t)=0, & t\in[a,b],\\ u(a)=0, & u(b)=0\end{cases} \tag{2.3}$$

And the solution of (2.3) is given by

$$w(t)=\int_a^b K(t,s)f(s)ds, \tag{2.4}$$

and

$$w(a)=0, \qquad w(b)=0, \qquad w(\eta)=\int_a^b K(\eta,s)f(s)ds. \tag{2.5}$$

The four-point boundary value problem (1.1), (1.2) can be obtained from replacing *u(a) = 0, u(b) = 0* by $u(a)=k_1u'(\eta_1)$ and $u(b)=k_2u'(\eta_2)$ in (2.3). Thus, we suppose the solution of the four-point boundary value problem (1.1), (1.2) can be expressed by

$$u(t)=w(t)+\left(c+dt\right)\left[w'(\eta_1)+w'(\eta_2)\right] \tag{2.6}$$

where *c* and *d* are constants that will be determined.

From (2.5), (2.6) we know that

$$u(a)=\left(c+da\right)\left[w'(\eta_1)+w'(\eta_2)\right]$$

$$u(b)=\left(c+db\right)\left[w'(\eta_1)+w'(\eta_2)\right]$$

$$u'(\eta_1)=w'(\eta_1)+\left(c+d\eta_1\right)\left[w'(\eta_1)+w'(\eta_2)\right]$$

$$u'(\eta_2)=w'(\eta_2)+\left(c+d\eta_2\right)\left[w'(\eta_1)+w'(\eta_2)\right]$$

Putting this into (1.2) yields

$$\begin{cases}c\left[w'(\eta_1)+w'(\eta_2)\right]+d\left(a-k_1\right)\left[w'(\eta_1)+w'(\eta_2)\right]=k_1w'(\eta_1)\\ c\left[w'(\eta_1)+w'(\eta_2)\right]+d\left(b-k_2\right)\left[w'(\eta_1)+w'(\eta_2)\right]=k_2w'(\eta_2)\end{cases}$$

Since $b-k_2 \neq a-k_1$, solving the system of linear equations on the unknown numbers *c*, *d*, using Cramer's rule we obtain

$$\begin{cases} c = \dfrac{\left(k_1 w'(\eta_1)\right)\left(b-k_2\right)-\left(k_2 w'(\eta_2)\right)\left(a-k_1\right)}{\left(b-k_2-a+k_1\right)\left[w'(\eta_1)+w'(\eta_2)\right]} \\ d = \dfrac{\left(-k_1 w'(\eta_1)\right)+k_2 w(\eta_2)}{\left(b-k_2-a+k_1\right)\left[w'(\eta_1)+w'(\eta_2)\right]} \end{cases}$$

Hence, the solution of (1.1) with the boundary condition (1.2) is

$$u(t) = w(t) + (c+dt)\left[w'(\eta_1)+w'(\eta_2)\right]$$

$$= w(t) + \frac{k_1(b-k_2-t)w'(\eta_1)+k_2(k_1-a+t)w'(\eta_2)}{b-k_2-a+k_1}$$

This together with (2.4) implies that

$$u(t) = \int_a^b K(t,s) f(s)ds \ + \frac{k_1(b-k_2-t)}{b-k_2-a+k_1}\int_a^b K_t(\eta_1,s) f(s)ds \ + \frac{k_2(k_1-a+t)}{b-k_2-a+k_1}\int_a^b K_t(\eta_2,s) f(s)ds$$

Consequently, the Green's function G(t, s) for the boundary value problem (1.1), (1.2) is as described in (2.1).

From Theorem 2.1 we obtain the following corollary.

Corollary 2.1. If $b-k_2 \neq a-k_1$, then the second-order four-point linear boundary value problem

$$\begin{cases} u'' + f(t) = 0, \quad t \in [a,b], \\ u(a) = k_1 u'(\eta_1), \qquad u(b) = k_2 u'(\eta_2) \end{cases}$$

has a unique solution $u(t) = \int_a^b G(t,s) f(s)ds$ where G(t, s) is as in (2.1).

Proof: Assume that the second-order four-point linear boundary value problem (2.7) has two solutions u(t) and v(t), that is

$$\begin{cases} u'' + f(t) = 0, \quad t \in [a,b], \\ u(a) = k_1 u'(\eta_1), \qquad u(b) = k_2 u'(\eta_2) \end{cases} \qquad (2.7) \qquad \text{and}$$

$$\begin{cases} v'' + f(t) = 0, \quad t \in [a,b], \\ v(a) = k_1 v'(\eta_1), \qquad v(b) = k_2 v'(\eta_2) \end{cases} \qquad (2.8)$$

Let

$$z(t) = v(t) - u(t), \quad t \in \left[a,b\right] \qquad (2.9)$$

Then (2.7) and (2.8)

$$z''(t) = v''(t) - u''(t) = 0, \quad t \in \left[a,b\right],$$

therefore

$$z(t) = C_1 t + C_2, \quad and \qquad z'(t) = C_1. \qquad (2.10)$$

where C_1 and C_2 are undetermined constants. From (2.7), (2.8) and (2.9) we have

$$z(a) = v(a) - u(a) = k_1 z'(\eta_1), \qquad (2.11)$$

$$z(b) = v(b) - u(b) = k_2 z'(\eta_2). \qquad (2.12)$$

Using (2.10) we obtain

$$z(a) = C_1 a + C_2, \qquad (2.13)$$

$$z(b) = C_1 b + C_2, \qquad (2.14)$$

$$z'(\eta_1) = C_1, \qquad (2.15)$$

$$z'(\eta_2) = C_1, \qquad (2.16)$$

From (2.11), (2.13) and (2.15) we know that

$$C_1\left(a-k_1\right)+C_2=0, \tag{2.17}$$

and from (2.12), (2.14) and (2.16) we know that

$$C_1\left(b-k_2\right)+C_2=0, \tag{2.18}$$

Solving the system of equations (2.17) and (2.18), we get $C_1=0$ and $C_2=0$.

Therefore $z(t)=0, t\in\left[a,b\right]$, so $u(t)=v(t),\ t\in\left[a,b\right]$, that is uniqueness of the solution.

Corollary 2.2. Suppose the nonlinear function g(t, u) is continuous on $\left[a,b\right]\times\mathbb{R}$, then if $b-k_2\neq a-k_1$, the nonlinear four-point boundary value problem

$$\begin{cases} u''+g(t,u)=0, & t\in[a,b],\\ u(a)=k_1u'(\eta_1), & u(b)=k_2u'(\eta_2)\end{cases}$$

is equivalent to the nonlinear integral equation $u(t)=\int_a^b G(t,s)g\left(s,u(s)\right)ds$ where G(t, s) as in (2.1)

If the endpoints of the interval are $a=0, b=1$ in the boundary condition, from Theorem 2.1, Corollaries 2.1 and 2.2 we obtain the following corollary.

Corollary 2.3. If $1-k_2\neq -k_1$ then the Green's function for the second-order four-point linear boundary value problem

$$\begin{cases} u''+f(t)=0, & t\in[0,1],\\ u(0)=k_1u'(\eta_1), & u(1)=k_2u'(\eta_2)\end{cases} \tag{2.19}$$

is

$$G(t,s)=B(t,s)+\frac{k_1\left(1-k_2-t\right)B_t(\eta_1,s)+k_2\left(k_1+t\right)B_t(\eta_2,s)}{1-k_2+k_1} \tag{2.20}$$

where

$$B(t,s)=\begin{cases} s(1-t), & 0\le s\le t\le 1\\ (1-s)t, & 0\le t\le s\le 1\end{cases} \tag{2.21}$$

Hence the problem (2.19) has a unique solution $u(t)=\int_0^1 G(t,s)f\left(s\right)ds.$

If g(t, u) is continuous on $\left[0,1\right]\times\mathbb{R}$, then the nonlinear four-point boundary value problem

$$\begin{cases} u''+g(t,u)=0, & t\in[0,1],\\ u(0)=k_1u'(\eta_1), & u(1)=k_2u'(\eta_2)\end{cases}$$

is equivalent to the nonlinear integral equation $u(t)=\int_0^1 G(t,s)g\left(s,u(s)\right)ds.$

Example: Consider the second-order four-point boundary value problem:

$$\begin{cases} u''+\cos t=0, & t\in[0,1],\\ u(0)=\frac{1}{2}u'\left(\frac{1}{6}\right), & u(1)=\frac{-1}{3}u'\left(\frac{1}{5}\right)\end{cases}$$

Since $1-k_2=1+\frac{1}{3}=\frac{4}{3}$ and $0-k_1=\frac{-1}{2}$ are not equal, from (2.20), the Green's function is:

$$G(t,s)=B(t,s)+\frac{k_1\left(1-k_2-t\right)B_t(\eta_1,s)+k_2\left(k_1+t\right)B_t(\eta_2,s)}{1-k_2+k_1}$$

where $k_1=\frac{1}{2},\qquad \eta_1=\frac{1}{6},\qquad k_2=\frac{-1}{3},\qquad \eta_2=\frac{1}{5}$

$$G(t,s)=B(t,s)+\frac{\left(4-3t\right)}{11}B_t\left(\frac{1}{6},s\right)+\frac{\left(2t+1\right)}{11}B_t\left(\frac{1}{5},s\right)$$ *where*

$$B(t,s)=\begin{cases}\left(1-t\right)s,\ 0\le s\le t\le 1\\ (1-s)t,\ 0\le t\le s\le 1\end{cases}$$

$$B_t(t,s)=\begin{cases}-s,\ 0\le s\le t\le 1\\ (1-s),\ 0\le t\le s\le 1\end{cases}$$ which implies

$$B_t(\eta_1,s)=\begin{cases}-s,\ 0\le s\le \eta_1\le 1\\ 1-s,\ 0\le \eta_1\le s\le 1\end{cases}$$ and $$B_t(\eta_2,s)=\begin{cases}-s\ ,\ 0\le s\le \eta_2\le 1\\ 1-s,\ 0\le \eta_2\le s\le 1\end{cases}$$

Hence the solution of second-order four-point boundary value problem is:

$$u(t)=\int_0^1 G(t,s)f\left(s\right)ds=\int_0^1\left\{B(t,s)+\left(4-3t\right)B\left(\frac{1}{6},s\right)+\left(2t+1\right)B\left(\frac{1}{5},s\right)\right\}f\left(s\right)ds$$

$$=\cos(t)-\frac{3(2t+1)}{11}\left[\cos(1)\right]+\frac{3t-4}{11}\sin\left(\frac{1}{6}\right)+\frac{2t+1}{11}\sin\left(\frac{1}{5}\right)+\frac{6t-8}{11}$$

III. Application to nonlinear problem:

In this section, we study the iterative solutions for the following nonlinear four-point boundary value problem

$$\begin{cases}u''+f(t,u)=0,\quad t\in(0,1),\\ u(0)=k_1u'(\eta_1),\qquad u(1)=k_2u'(\eta_2)\end{cases}$$

(3.1)

with $\eta_1,\eta_2\in\left(0,1\right),\ .k_2<1+k_1$

Let $J=\left(0,1\right),\quad I=\left[0,1\right],\quad \mathbb{R}^+=\left[0,\infty\right),$

$D=\left\{x\in C(I)\middle|\exists M_x\ge m_x>0,\text{ such that } m_x\left(1-t\right)\le x(t)\le M_x\left(1-t\right),t\in I\right\}.$

Concerning the function f we impose the following hypotheses:

$$\begin{cases}f(t,u)\text{ is nonnegative continuous on } J\times\mathbb{R}^+,\\ f(t,u)\text{ is monotone increasing on u, for fixed t}\in\text{J},\\ \exists q\in(0,1)\text{ such that } f(t,ru)\ge r^q f(t,u),\forall 0<r<1,(t,u)\in J\times\mathbb{R}^+.\end{cases}$$

(3.2)

Obviously, from (3.2) we obtain

$$f(t,\lambda u)\ge\lambda^q f(t,u),\quad \forall\lambda>1,\quad (t,u)\in J\times\mathbb{R}^+.$$

(3.3)

We can see that if $0<\alpha_i<1,\ a_i(t)$ are nonnegative continuous on J, for i = 0,1,2,…,m, then

$f(t,u)=\sum_{i=1}^{m}a_i(t)u^{\alpha_i}$ satisfy the condition (3.2).

Concerning the boundary value problem (3.1), we have following conclusions.

Theorem 3.1. Suppose the function f(t,u) satisfy the condition (3.2), it may be singular at t=0 and/or t=1, and

$$0 < \int_0^1 f(t, 1-t)dt < \infty. \tag{3.4}$$

Then nonlinear singular boundary value problem (3.1) has a unique solution w(t) in $C(I) \cap C^2(J)$. Constructing successively the sequence of functions

$$h_n(t) = \int_0^1 G(t,s) f(s, h_{n-1}(s))ds, \qquad n = 1, 2, ... \tag{3.5}$$

for any initial function $h_0(t) \geq 0 (\not\equiv 0), t \in I$ then $\{h_n(t)\}$ must converge to w(t) uniformly on I and the rate of convergence is

$$\max_{t \in I} |h_n(t) - w(t)| = O\left(1 - N^{q^n}\right), \tag{3.6}$$

where 0< N <1 , which depends on the initial function $h_0(t), G(t,s)$ as in (2.20).

Proof. Let

$$P = \left\{x(t) \middle| x(t) \in C(I), x(t) \geq 0\right\},$$

$$Fx(t) = \int_0^1 G(t,s) f(s, x(s))ds, \quad \forall x(t) \in D. \tag{3.7}$$

It is easy that the operator $F : D \to P$ is increasing; From Corollary 2.3 we know that if $u \in D$ satisfies

$$u(t) = Fu(t), \quad t \in I, \tag{3.8}$$

then $u \in C^1(I) \cap C^2(J)$ is a solution of (3.1).

For any $x \in D$, there exist positive numbers $0 < m_x < 1 < M_x$ such that

$$m_x(1-s) \leq x(s) \leq M_x(1-s), \quad s \in I,$$

$$(m_x)^q f(s, 1-s) \leq f(s, x(s)) \leq (M_x)^q f(s, 1-s), \quad s \in J. \tag{3.9}$$

By (2.20) and (2.21) we have $G(t,s) = B(t,s) + \dfrac{k_1(1-k_2-t)B_t(\eta_1,s) + k_2(k_1+t)B_t(\eta_2,s)}{1-k_2+k_1}$

$$G(t,s) = B(t,s) + \frac{k_1(1-k_2-t)B_t(\eta_1,s) + k_2(k_1+t)B_t(\eta_2,s)}{1-k_2+k_1}$$

$$\Rightarrow \quad G(t,s) \geq (1-t)\frac{k_1 B_t(\eta_1,s) + \left[\dfrac{k_2(k_1+t)}{1-t}\right]B_t(\eta_2,s)}{1-k_2+k_1} \tag{3.10}$$

$$G(t,s) \leq t(1-t) + \frac{-k_1 B_t(\eta_1,s) + k_2(k_1+t)B_t(\eta_2,s)}{1-k_2+k_1}$$

$$\Rightarrow \quad G(t,s) \leq (1-t)\left[1 + \frac{k_1 B_t(\eta_1,s) + \left[\dfrac{k_2(k_1+t)}{1-t}\right]B_t(\eta_2,s)}{1-k_2+k_1}\right] \tag{3.11}$$

Using (3.7), (3.3) and (3.9)-(3.11) and the conditions (3.2), we obtain

$$Fx(t)=\int_0^1 G(t,s)f(s,x(s))ds$$

$$\geq\int_0^1(1-t)\frac{k_1B_t(\eta_1,s)+\left[\frac{k_2(k_1+t)}{1-t}\right]B_t(\eta_2,s)}{1-k_2+k_1}\Big((m_x)^q\ f(s,1-s)\Big)ds$$

$$\geq(1-t)(m_x)^q\frac{k_1+\left[\frac{k_2(k_1+t)}{1-t}\right]}{1-k_2+k_1}\left[\int_0^1 B_t(\eta_1,s)\big(f(s,1-s)\big)ds+\int_0^1 B_t(\eta_2,s)\big(f(s,1-s)\big)ds\right],t\in I$$

(3.12)

$$Fx(t)=\int_0^1 G(t,s)f(s,x(s))ds$$

$$\leq\int_0^1(1-t)\left[1+\frac{k_1B_t(\eta_1,s)+\left[\frac{k_2(k_1+t)}{1-t}\right]B_t(\eta_2,s)}{1-k_2+k_1}\right]\Big((M_x)^q\ f(s,1-s)\Big)ds$$

$$\leq(1-t)(M_x)^q\int_0^1\left(1+\frac{[k_1+k_1k_2]B(\eta_1,s)+\left[\frac{k_2}{1-t}-k_2-k_1k_2\right]B(\eta_2,s)}{(1-k_1)(1-k_2\eta_2)+(1-k_2)(k_1\eta_1)}\right)\big(f(s,1-s)\big)ds,\quad t\in I$$

(3.13)

By (3.4), (3.12) and (3.13) we obtain

$$F:D\to D.$$

For any $h_o\in D$, we let

$$l_{h_o}=\sup\left\{l>0\,\middle|\,lh_o(t)\leq(Fh_o)(t),t\in I\right\},$$

$$L_{h_o}=\inf\left\{L>0\,\middle|\,Lh_o(t)\geq(Fh_o)(t),t\in I\right\},\qquad(3.14)$$

$$m=\min\left\{1,\left(l_{h_o}\right)^{\frac{1}{1-q}}\right\},\qquad M=\max\left\{1,\left(L_{h_o}\right)^{\frac{1}{1-q}}\right\}$$

$$u_0(t)=mh_0(t),\qquad u_n(t)=Fu_{n-1}(t)$$

$$v_0(t)=Mh_0(t),\qquad v_n(t)=Fv_{n-1}(t),\qquad n=0,1,2,...$$

(3.15)

Since the operator F is increasing, from (3.2), (3.14) and (3.15) we know that

$$u_0(t)\leq u_1(t)\leq\cdots\leq u_n(t)\cdots\leq v_n(t)\leq\cdots\leq v_1(t)\leq v_0(t),\qquad t\in I.\qquad(3.16)$$

For $t_0=\dfrac{m}{M}$, from (3.2), (3.7) and (3.15), it can obtained by induction that

$$u_n(t)\geq(t_0)^{q^n}\ v_n(t),\qquad t\in I,n=0,1,2,\cdots\qquad(3.17)$$

From (3.16) and (3.17) we know that

$$0\leq u_{n+p}(t)-u_n(t)\leq v_n(t)-u_n(t)\leq\Big(1-(t_0)^{q^n}\ Mh_0(t)\Big),\qquad\forall n,p\qquad(3.18)$$

so that there exist function $w(t) \in D$ such that

$$u_n(t) \to w(t), \qquad v_n(t) \to w(t), \quad \text{(uniformly on I)}, \tag{3.19}$$

and

$$u_n(t) \le w(t) \le v_n(t), \quad t \in I, n = 0,1,2,\cdots \tag{3.20}$$

From the operator F is increasing and (3.15) we have

$$u_{n+1}(t) = Fu_n(t) \le Fw(t) \le Fv_n(t) = v_{n+1}(t), \quad n = 0,1,2,\cdots$$

This together with (3.19) and uniqueness of the limit imply that $w(t)$ satisfy (3.8), hence $w(t) \in C^1(I) \cap C^2(J)$ is a solution of (3.1).

From (3.5) and (3.15) and the operator F is increasing, we obtain

$$u_n(t) \le h_n(t) \le v_n(t), \quad t \in I, n = 0,1,2,\cdots \tag{3.21}$$

thus, from (3.18), (3.20) and (3.21) we know

$$|h_n(t) - w(t)| \le |h_n(t) - u_n(t)| + |u_n(t) - w(t)|$$

$$\le 2|v_n(t) - u_n(t)| \le \left(1 - (t_0)^{q^n}\right) M\, |h_0(t)|,$$

so that $\max_{t \in I} |h_n(t) - w(t)| \le \left(1 - (t_0)^{q^n}\right) M \max_{t \in I} |h_0(t)|.$

So that (3.6) holds.

From $h_0(t)$ which is arbitrary in D we know that *w(t)* is the unique solution of the Eq. (3.8) in D. Suppose $w_1(t)$ is a $C^1(I) \cap C^2(J)$ solution of boundary value problem (3.1). Let

$$z(t) = w_1(t) - Fw(t), t \in I.$$

Similar to the proof of (2.9) in section 2 we obtain $w_1(t) = w(t),$ hence $w(t)$ is the unique solution of Eq. (3.1) in $C^1(I) \cap C^2(J)$.

Remark: If $f(t,u)$ is continuous on I X R^+ then it is quite evident that the condition (3.4) holds. Hence the unique solution $w(t) \in C^2(J).$

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